# THE PROBLEM OF INCOMPRESSIBLE JETS WITH CURVILINEAR WALLS

*Anniversary Session* ***"Celebrating 100 year of the first jet aircraft invented by Henri Coanda"***, *organized by INCAS, COMOTI and Henri Coanda Association, 14 December 2010, Bucharest, Romania*

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DOI: 10.13111/2066-8201.2010.2.4.10

***Abstract:*** *The jet flow problem concerning the discharge of a fluid (from an orifice in a container) into the atmosphere is studied herein in the framework of the Helmholtz-Kirchhoff model. The problem is reduced to the study of a system of nonlinear equations. Using Leray-Schauder's fixed point theorem we prove that the system of functional equations has at least one solution. Then we present a semi-inverse method which gives us the possibility to calculate numerically the unknown free lines for symmetric jets whose walls consist of semi-infinite straight lines and arcs of circle and for non-symmetric jets whose walls consist of semi-infinite straight lines.*

*Key Words: jet flow problem, nonlinear equations, curvilinear walls, semi-inverse method.*

## 1. INTRODUCTION

The jet flow problem is concerned with the discharge of a fluid from an orifice (in a fixed vessel or container) into an atmosphere at constant pressure. For purposes of theory the convenient idealization assumes that the vessel has two semi-infinite walls $\varpi_1$ and $\varpi_2$ (extending to infinity upstream) each of them consisting of a semi-infinite straight portion and a finite curvilinear portion nearby the orifice (figure 1). We use herein the Helmholtz-Kirchhoff theory as it is presented in [1], [2], [3], [4], [5], [8].

In the present paper we extend the results obtained in [3], [4] for symmetric jets, taking also into consideration the case of non-symmetric jets with straight walls.

We assume that the wall $\varpi_1$ consists of an arc of circle having the radius $R$ and the length $\nu_1 \pi R$ and a semi-infinite straight line, stretching to infinity upstream ($x \to -\infty$) and making with the $Ox$ - axis the angle $(1-\mu)\pi$ where $0 \le \nu_1 \le \mu \le \frac{1}{2}$.

We also assume that the wall $\varpi_2$ consists of an arc of circle having the radius $R$ and the length $\nu_2 \pi R$ and a semi-infinite straight line, stretching to infinity upstream ($x \to -\infty$) and making with the $Ox$ - axis the angle $\mu\pi$ where $0 \le \nu_2 \le \mu \le \frac{1}{2}$. Let $A$ and $B$ be the edges of the orifice of the jet (i.e. the endpoints of the walls $\varpi_1$ and $\varpi_2$) and let $L = |z_A - z_B|$ be the length of the jet orifice.

Two free lines $\lambda_1$ and $\lambda_2$ detach from the edges of the orifice (named the *detachment points*) and extend to infinity downstream. The domain bounded by the walls of the vessel and by the free lines is the *flow domain*. We neglect the gravity and we consider that the jet emerges because of the difference of the pressures inside and outside the vessel. We consider that the fluid is ideal, incompressible and the fluid flow is plane, steady and irrotational. We denote by $\mathbf{v} = (u, v)$ the velocity, by $\varphi$ the potential of the velocity and by $\psi(x, y)$ the stream function.

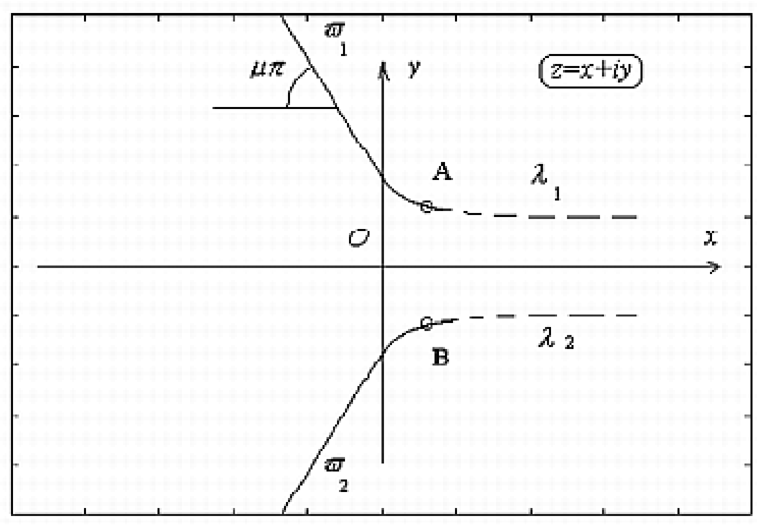


Fig. 1 Flow domain

The function $f(z) = \varphi(x, y) + i\,\psi(x, y)$ (named the complex potential) is holomorphic, and denoting by $w(z) = u(x, y) - i\,v(x, y)$ the complex velocity, we have $\dfrac{df}{dz} = w$.

The walls of the vessel and the free lines are stream-lines i.e.

$$\psi\big|_{\varpi_1 \cup \lambda_1} = \frac{h}{2}, \quad \psi\big|_{\varpi_2 \cup \lambda_2} = -\frac{h}{2} \tag{1}$$

Outside the flow domain the fluid is at rest. From Bernoulli's law we deduce that

$$\left|\frac{df}{dz}\right|\Big|_{\varpi_2 \cup \lambda_2} = V_0 = const. \tag{2}$$

## 2. LEVI-CIVITA's FUNCTION

From (1) we deduce that the flow domain in the $f$ - plane a strip. The detachment points are

$$f_A = \varphi_1 + i\frac{h}{2}, \quad f_B = \varphi_2 - i\frac{h}{2}$$

The function

$$f = \frac{h}{\pi}\ln\left(\frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right) - \cos\sigma_0\right) + i\,\frac{h}{2}, \qquad \zeta = \xi + i\,\eta \tag{3}$$

with

$$\cos\sigma_0 = \frac{\Phi_2 - \Phi_1}{\Phi_2 + \Phi_1}, \qquad \Phi_1 = \exp\left(\frac{\pi\,\varphi_1}{h}\right), \qquad \Phi_2 = \exp\left(\frac{\pi\,\varphi_2}{h}\right)$$

is the conformal mapping of the unit half-disk from the $\zeta$ - plane onto the infinite horizontal strip of width $h$ in the $f$ $-$plane.

We introduce T. Levi -Civita's function [7] $\omega(z) = \theta(\xi,\eta) + i\,\tau(\xi,\eta)$

$$\omega(\zeta) = w(z(\zeta)) = \frac{d\,f}{d\,z} = V_0\,\exp(-i\,\omega(\zeta)) \tag{4}$$

From (4) it follows

$$\tau(\xi,\eta) = \ln\frac{|w(\zeta)|}{V_0}, \qquad \theta(\xi,\eta) = -\arg w(\zeta) \tag{5}$$

$\theta(\xi,\eta)$ is the angle of the velocity with the $Ox$ - axis. From relation (2) we deduce that

$$\tau(\xi,0) = 0, \quad \xi \in [-1,1] \tag{6}$$

On the unit half - circle the function $\theta(\cos s,\sin s), s \in [0,\pi]$ is discontinuous in $s = \sigma_0$ because

$$\lim_{s\uparrow\sigma_0}\theta(\cos s,\sin s) = -\mu\,\pi, \;\; \lim_{s\downarrow\sigma_0}\theta(\cos s,\sin s) = \mu\,\pi$$

We shall introduce therefore the continuous function $\Omega(\zeta) = \Theta(\xi,\eta) + i\,T(\xi,\eta)$, such that

$$\Theta(\cos s,\sin s) = -\mu\,\pi - \theta(\cos s,\sin s)\,, \; s \in [0,\sigma_0] \tag{7}$$

$$\Theta(\cos s,\sin s) = \mu\,\pi - \theta(\cos s,\sin s)\,, \; s \in [\sigma_0,\pi] \tag{8}$$

$$T(\xi,0) = 0\,, \qquad \xi \in [-1,1] \tag{9}$$

From (7) - (9) it follows

$$Re\left[\omega(\zeta) + \Omega(\zeta)\right] = -\,\mu\,\pi\,,\; \zeta = \exp(is)\,,\; s \in [0,\sigma_0] \tag{10}$$

$$Re\left[\omega(\zeta) + \Omega(\zeta)\right] = \mu\,\pi\,,\; \zeta = \exp(is)\,,\; s \in [\sigma_0,\pi] \tag{11}$$

$$Im\left[\omega(\zeta) + \Omega(\zeta)\right] = 0\,,\; \zeta = \xi \in [-1,1] \tag{12}$$

From (10) - (12) we deduce that

$$\omega(\zeta) = 2\mu i \ln\frac{\zeta - \exp(i\sigma_0)}{\zeta + \exp(-i\sigma_0)} + \mu\pi + 2\mu\sigma_0 - \Omega(\zeta) \tag{13}$$

From (4) and (13) it follows

$$\frac{dz}{d\zeta} = \frac{1}{V_0}\exp\left(-2\mu\ln\frac{\zeta - \exp(i\sigma_0)}{\zeta + \exp(-i\sigma_0)} + \mu i\pi + 2\mu i\ \sigma_0 - i\Omega(\zeta)\right)\frac{df}{d\zeta} \tag{14}$$

## 3. SYMMETRICAL JETS. THE FUNCTIONAL EQUATION

If the walls of the jet are symmetric with respect to the $Ox$ - axis we have

$$\nu_1 = \nu_2 = \nu\,,\ \sigma_0 = \frac{\pi}{2}$$

Denoting by $l(s)$ the length of the arc from $\varpi_1$ having the endpoints $z(\exp(i\,s))$ and $z(0)$ we deduce from (3) and (14) that

$$\frac{dl}{ds} = \frac{h}{\pi V_0}\exp(T(s))\cot^{2\pi}\left(\frac{\pi}{2} - \frac{s}{2}\right)\tan s\,,\ s \in \left[0, \frac{\pi}{2}\right) \tag{15}$$

(In the sequel we shall use the notations $T(s) = T(\cos s, \sin s)$, $\Theta(s) = \Theta(\cos s, \sin s)$, $\theta(s) = \theta(\cos s, \sin s)$.)

The function $z(\zeta)$ maps the unit half - circle onto a curve consisting of half - lines and arcs of circle.

According to Schwarz's principle concerning the analytic continuation the function $z(\zeta)$ can be extended in a vicinity of the half - circle $\{\exp(i\,s)\ ;\ s \in 0, \pi]\}$.

Taking into account (14) one deduces that the function $\Omega(\zeta)$ can also be extended in a vicinity of the half – circle $\{\exp(i\,s)\ ;\ s \in 0, \pi]\}$.

The conjugate harmonic functions $T$ and $\Theta$ satisfy the relation

$$\frac{\partial T}{\partial n} = \frac{\partial \Theta}{\partial s} \tag{16}$$

where $\dfrac{\partial}{\partial n}$ is the inward normal derivative and $\dfrac{\partial}{\partial s}$ is the tangential derivative.

Using U. Dini's formula and seeking for $\Omega(\zeta)$ such that $\Omega(0) = 0$ we get

$$-i\Omega(\zeta) = \frac{1}{\pi}\int_0^{2\pi}\frac{\partial T(s)}{\partial n}\ln(\exp(i\,s) - \zeta)\,ds \tag{17}$$

From (9) it follows that the function $\Omega(\zeta)$ can be extended to the whole unit disk, according to Schwarz's continuation principle, by means of the relations

$$T(\xi, \eta) = -T(\xi, -\eta)\,,\ \Theta(\xi, \eta) = \Theta(\xi, -\eta) \tag{18}$$

On the other hand, because of the symmetry of the domain we have

$$T(\xi,\eta) = T(-\xi,\eta)\,,\ \Theta(\xi,\eta) = -\Theta(-\xi,\eta) \tag{19}$$

From (18) and (19) we deduce that

$$\frac{\partial T}{\partial n}(2\pi - s) = -\frac{\partial T}{\partial n}(s)\,,\ s \in [0,\pi]\,,\ \frac{\partial T}{\partial n}(\pi - s) = \frac{\partial T}{\partial n}(s)\,,\ s \in \left[0, \frac{\pi}{2}\right] \tag{20}$$

From (17) and (20) it follows:

$$-i\,\Omega(\zeta) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\partial T}{\partial n}(s)\ln\frac{(\exp(i\,s)-\zeta)(\exp(-i\,s)+\zeta)}{(\exp(-i\,s)-\zeta)(\exp(i\,s)+\zeta)}\,d\,s \tag{21}$$

From (21), putting $\zeta = \exp(i\,\sigma)$ and separating the real parts we obtain

$$T(\sigma) = -\frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \frac{\partial T}{\partial n}(s)\ln\left(\frac{\sin s + \sin\sigma}{\sin s - \sin\sigma}\right)^2 d\,s\,,\ \sigma \in\in \left[0, \frac{\pi}{2}\right) \tag{22}$$

From (16) and (22) it follows:

$$T(\sigma) = -\frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \frac{\partial \Theta}{\partial n}(s)\ln\left(\frac{\sin s + \sin\sigma}{\sin s - \sin\sigma}\right)^2 d\,s\,,\ \sigma \in\in \left[0, \frac{\pi}{2}\right) \tag{23}$$

Let $z(\exp(is_0)) \in \varpi_1$ represent the point where the rectilinear and the circular portions are matching. Obviously we have

$$\frac{d\,\theta}{d\,l}(s) = 0\,,\ s \in \left(s_0, \frac{\pi}{2}\right), \qquad \frac{d\,\theta}{d\,l}(s) = -\frac{1}{R}\,,\ s \in (0, s_0) \tag{24}$$

From (7), (15) and (24) it follows

$$\frac{\partial\,\Theta}{\partial\,s}(s) = 0\,,\ s \in \left(s_0, \frac{\pi}{2}\right) \tag{25}$$

$$\frac{\partial\,\Theta}{\partial\,s}(s) = \frac{h}{\pi R V_0}\exp(T(s))\cot^{2\mu}\left(\frac{\pi}{4} - \frac{s}{2}\right)\tan s\,,\ s \in (0, s_0) \tag{26}$$

From (23), (25) and (26) we deduce:

$$T(\sigma) = -\int_0^{s_0} \frac{h\exp(T(s))}{2\pi^2 R V_0}\cot^{2\mu}\left(\frac{\pi}{4} - \frac{s}{2}\right)\tan s \ln\left(\frac{\sin s + \sin\sigma}{\sin s - \sin\sigma}\right)^2 d\,s \tag{27}$$

## 4. THE EXISTENCE OF THE SOLUTION

We shall consider the operator $\mathsf{F}(T(s), s_0, h, k) : \mathsf{D} \times [0,1] \to C\left[0, \frac{\pi}{2}\right] \times \mathbf{R}^2$ given by the right hand side of the system of equations

$$T(\sigma) = -\int_0^{s_0} \frac{h\exp(T(s))}{2\pi^2 RV_0}\cot^{2\mu}\left(\frac{\pi}{4}-\frac{s}{2}\right)\tan s\ln\left(\frac{\sin s+\sin\sigma}{\sin s-\sin\sigma}\right)^2 d\,s \tag{28}$$

$$s_0 = s_0 + k\nu\pi R - \frac{h}{\pi RV_0}\int_0^{s_0}\exp(T(s))\cot^{2\mu}\left(\frac{\pi}{4}-\frac{s}{2}\right)\tan s\,d\,s \tag{29}$$

$$h = \frac{V_0 L + 2RV_0\left[\cos(\mu-\nu)\pi - \cos(\mu-k\nu)\pi\right]}{1+\frac{2}{\pi}\int_0^1 \sin\theta(\xi,0)\frac{\xi-\xi^{-1}}{\xi+\xi^{-1}}\frac{d\,\xi}{\xi}} \tag{30}$$

From (28) - (30) we easily check that $\mathsf{F}(T(s),s_0,h,k)$ is continuous with respect to $T(s)$, $s_0$, $h$ and uniformly continuous with respect to $k$.

In [2] one demonstrates that there is a constant $s_0^* \in \left(0,\frac{\pi}{2}\right)$ such that for every solution $\left(T_k(s),s_{0k},h_k\right)$ of the system (28) - (30) we have $0 \le s_{0k} \le s_0^* 0$ and $h_k < V_0 L_0$, $L_0 = L + 2R\left[\cos(\mu-\nu)\pi - \cos\mu\pi\right]$. We denote by

$$M = \max_{\sigma\in\left[0,\frac{\pi}{2}\right]} \frac{L_0}{2\pi^2 R}\int_0^{s_0^*}\cot^{2\mu}\left(\frac{\pi}{4}-\frac{s}{2}\right)\tan s\ln\left(\frac{\sin s+\sin\sigma}{\sin s-\sin\sigma}\right)^2 d\,s$$

We consider the domain $\mathsf{D} = \left\{T(s)\in C\left[0,\frac{\pi}{2}\right];\left|T(s)\right| < M+1\right\}\times\left[0,V_0 L_0\right]\times\left[0,s_0^*\right]$. For any solution $\left(T_k(s),s_{0k},h_k\right)$of (28) - (30) we have

$$0 < \max_{s\in\left[0,\frac{\pi}{2}\right]}\left|T(s)\right| + \left|s_{0k}\right| + \left|h_k\right| < M + V_0 L_0 + s_0^* + 1$$

whence it follows that $\mathsf{F}(T(s),s_0,h,k)$ has no fixed point on $\partial\mathsf{D}$.

Taking into account the expression of the kernel of the integral equation (28) it follows that $\mathsf{F}$ maps the Cartesian product of an arbitrary bounded set of continuous functions with an arbitrary bounded set from $\mathbf{R}^2$ onto the product of a bounded set of equi-continuous functions with a bounded set from $\mathbf{R}^2$. Taking into account Arzela's theorem we deduce that $\mathsf{F}(T(s),s_0,h,k)$ is a compact operator.

For $k=0$, the operator $\mathsf{F}(T(s),s_0,h,k)$ is constant and it is given by the right hand side of the following relations

$$T(s)=0,\ s_0=0,\ h = \frac{V_0 L + 2RV_0\left[\cos(\mu-\nu)\pi - \cos\mu\pi\right]}{1+\frac{2}{\pi}\int_0^1 \sin\left(2\mu\arcsin\frac{2\xi}{\xi^2-1}\right)\frac{\xi-\xi^{-1}}{\xi+\xi^{-1}}\frac{d\,\xi}{\xi}} \tag{31}$$

Hence the topological degree of the operator $I-\mathsf{F}(T(s),s_0,h,0)$ in $\mathbf{0}\in C\left[0,\frac{\pi}{2}\right]\times\mathbf{R}^2$ is 1. From Leray - Schauder [6] fixed point theorem it follows that $\mathsf{F}(T(s),s_0,h,k)$ has at least one fixed point, $\forall k\in[0,1]$.

## 5. THE SEMI - INVERSE METHOD. NUMERICAL RESULTS

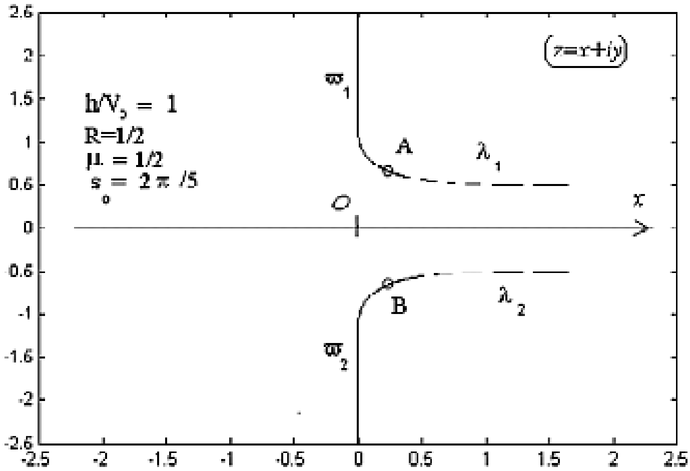


Fig. 2 Example

Let us consider the operator $\mathsf{L}:C\left[0,\frac{\pi}{2}\right]\to C\left[0,\frac{\pi}{2}\right]$ where

$$\mathsf{L}(T)(\sigma)=-\int_0^{s_0}\frac{h\exp(T(s))}{2\pi^2 RV_0}\cot^{2\mu}\left(\frac{\pi}{4}-\frac{s}{2}\right)\tan s\ln\left(\frac{\sin s+\sin\sigma}{\sin s-\sin\sigma}\right)^2 ds \tag{32}$$

Considering $T_0(\sigma)=0,\ \sigma\in\left[0,\frac{\pi}{2}\right]$ we can establish the relations

$$\begin{aligned}&\mathsf{L}(T_0)(\sigma)\le\mathsf{L}^3(T_0)(\sigma)\le\ldots\le\mathsf{L}^{2n+1}(T_0)(\sigma)\le\ldots\le T(\sigma)\le\\&\le\mathsf{L}^{2n}(T_0)(\sigma)\le\ldots\le\mathsf{L}^2(T_0)(\sigma)\le 0,\ \forall\sigma\in\left[0,\frac{\pi}{2}\right]\end{aligned} \tag{33}$$

whence we deduce the lower and upper bounds for $T$:

$$\lim_{n\to\infty}\mathsf{L}^{2n+1}(T_0)(\sigma)\le T(\sigma)\le\lim_{n\to\infty}\le\mathsf{L}^{2n}(T_0)(\sigma)\le,\ \forall\sigma\in\left[0,\frac{\pi}{2}\right] \tag{34}$$

For $h$ and $s_0$ small enough the two bounds coincide because $\mathsf{L}$ is a contraction. In the sequel for various values of $h$ and $s_0$ we shall verify numerically that the lower and upper bounds coincide. For calculating $T=\lim_{n\to\infty}\mathsf{L}^{2n+1}(T_0)(\sigma)$ we consider an equidistant grid on $\left[0,\frac{\pi}{2}\right]$ consisting of the nodes $0=\sigma_0,\sigma_1,\ldots,\sigma_m=\frac{\pi}{2}$ and another grid on $[0,s_0]$ consisting of the nodes $0=\tilde{s}_0,\tilde{s}_1,\ldots,\tilde{s}_p=s_0$. Using Simpson's quadrature formula one calculates $\mathsf{L}(T_0)(\sigma_i)$ and then, iteratively $\mathsf{L}^2(T_0)(\sigma_i)$, $\mathsf{L}^3(T_0)(\sigma_i)\ldots$.

We stop the calculations when $\left|\mathsf{L}^n(T_0)(\sigma_i)-\mathsf{L}^{n+1}(T_0)(\sigma_i)\right|<\varepsilon$ where $\varepsilon$ is an a priori given small number. Using Cauchy's formula we calculate then $\Theta(\cos\sigma_i,\sin\sigma_i)$. We calculate also $\Theta(\xi_i,0),\xi_i\in[0,1]$ by means of Schwarz - Villat formula. From (29) and (30) we then calculate $\nu$ and $L$.

Then by means of (14) one may calculate numerically the positions of the points belonging to the free lines. In figure 2, we give an example of flow domain and calculated free lines.

## 6. NON-SYMMETRICAL JETS WITH RECTILINEAR WALLS

If the walls of the jet are rectilinear we have $\nu=0$ and $\Omega=0$. The detachment points are

$$z_A=\frac{\Lambda}{2},\quad z_B=-\frac{\Lambda}{2}$$

With

$$\Lambda=\frac{h\exp(\mu i\pi+2\mu i\sigma_0)}{\pi V_0}\int_{-1}^{1}\left(\frac{\exp(-i\sigma_0)-\zeta}{\exp(i\sigma_0)-\zeta}\right)^{2\mu}\frac{\zeta^2-1}{\zeta^2-2\zeta\cos\sigma_0+1}\frac{d\zeta}{\zeta}$$

The parametric equations of the free lines are

$$\lambda_1: z=z_A+$$

$$\frac{h\exp(\mu i\pi+2\mu i\sigma_0)}{\pi V_0}\int_{1}^{\xi}\left(\frac{\exp(-i\sigma_0)-\zeta}{\exp(i\sigma_0)-\zeta}\right)^{2\mu}\frac{\zeta^2-1}{\zeta^2-2\zeta\cos\sigma_0+1}\frac{d\zeta}{\zeta},\ 0<\xi<1$$

$$\lambda_2: z=z_B+$$

$$\frac{h\exp(\mu i\pi+2\mu i\sigma_0)}{\pi V_0}\int_{-1}^{\xi}\left(\frac{\exp(-i\sigma_0)-\zeta}{\exp(i\sigma_0)-\zeta}\right)^{2\mu}\frac{\zeta^2-1}{\zeta^2-2\zeta\cos\sigma_0+1}\frac{d\zeta}{\zeta},\ 0>\xi>-1$$

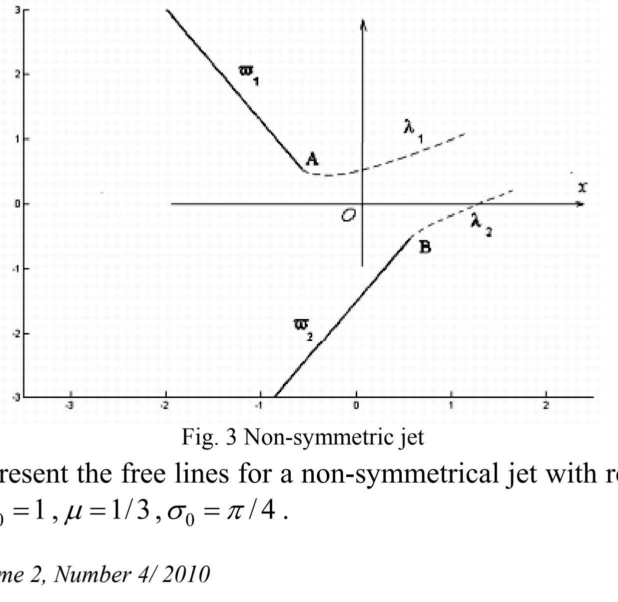


Fig. 3 Non-symmetric jet

In figure 3 we present the free lines for a non-symmetrical jet with rectilinear walls. We have considered $h/V_0=1$, $\mu=1/3$, $\sigma_0=\pi/4$.

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