Integral Method for Structural Analysis of Beams with Bending-Torsion Coupling

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Abstract: *This paper deals with structural static or stability analysis of straight beams, when the bending-torsion coupling is considered. A practical case is the lateral buckling analysis of high cross-section beams which involves the coupled bending deflection and twist. Simple rectangular cross-section or thin walled open profile symmetric cross-section beams can buckle laterally when they are subjected to bending with respect to the stiffer plane. Bending-torsion coupling occurs also for some composite beams. The presented approximate integral method uses flexibility influence functions (Green functions) and a matrix formulation, leading to an eigenvalue problem in the case of stability analysis. The formulation can be also used for static or dynamic analysis. Some numerical examples are presented in comparison with the results of other methods.*

Key Words*: Straight Beams, Green Functions, Coupling, Lateral- Torsional Buckling*

1. INTRODUCTION

Beams are used as structural members in many engineering domains. They can be parts of complex mechanical structures. The tendency to reduce the structural weight led to the use of slender beams in civil metallic structures, machine structures and especially in aeronautics. The need for a careful structural analysis including static, dynamic and stability calculations also led to the use of various numerical methods, among which the FEM, implemented in different computer codes, has been imposed. Besides this, other methods have also been developed. They are used to solve more or less complex particular calculation situations. The flexibility influence functions (Green's functions) and their use in the structural and aeroelastic analysis of fixed aircraft wings were presented in several books as [1-3]. Some extensions for the vibration analysis of rotating beams and blades and for stability analysis of beams were reviewed in [4]. Concerning the particular aspect of the bending-torsion coupling, the vibration analysis of several blade configurations were presented in [5, 6]. The structural bending-torsion coupling, in the case of the static response of a composite beam was also analyzed by using Green's functions in [7]. The subject of using Green's functions in structural analysis still remains a topical one. Recent references as [8, 9] demonstrate this idea. In this work, several bending-torsion coupling cases for beam structural analysis are presented. Then the practical application of the lateral buckling analysis of high cross-section beams will be detailed. It involves an interesting coupling case between bending deflection and twist. For example simple rectangular cross-section beams can buckle laterally when they are subjected to bending with respect to the stiffer plane. The two coupled equations governing the bending and torsion behavior of the beam are put in integral form, finally obtaining an eigenvalue problem allowing calculating the critical bending moment. Two simple examples are also discussed in comparison with analytical results.

2. BENDING-TORSION COUPLING TERMS

A first example of the bending-torsion coupling is the case of the free harmonic vibration analysis of a rotating (angular velocity Ω) and untwisted blade (Fig. 1). The differential equations describing the flapwise bending (deflection w) and torsion behavior (angle ϕ) of such a blade are obtained in [5] taking only the linear terms of the more general equations from [10]:

$$\left[EI_y(x)w''\right]'' = \left[T(x)w'\right]' + \Omega^2 e\left[m(x)(x+e_1)\phi\right]' + \omega^2 m(x)(w+e\phi)\,, \tag{1}$$

$$\left[GJ(x)\phi'\right]' - \Omega^2 m(x)\left[(x+e_1)ew' + (k_{m2}^2 - k_{m1}^2)\phi\right] + \omega^2 m(x)(ew + k_m^2\phi) = 0\,, \tag{2}$$

where:

$$T(x) = \int_0^L m(x)\Omega^2(x+e_1)dx \tag{3}$$

is the tension force in a section due to the rotation. The main notations in above equations are: E, Young modulus of elasticity, G shear modulus, I_y moment of inertia of the cross-section, J torsional stiffness constant, e_1 root blade offset and e the distance between mass center and elastic center of the blade cross-section. The distributed mass is $m(x)$, ω is the natural circular frequency of the vibration, while k_{m1} and k_{m2} are mass radii of gyration of cross-sectional mass about major and minor neutral axis and k_m is the polar radius of gyration of cross-sectional mass about elastic axis:

$$k_m^2 = k_{m1}^2 + k_{m2}^2\,. \tag{4}$$

The previous equations (1) and (2) are coupled due to the offset e between the elastic center and mass center of the blade cross-section (Fig. 1).

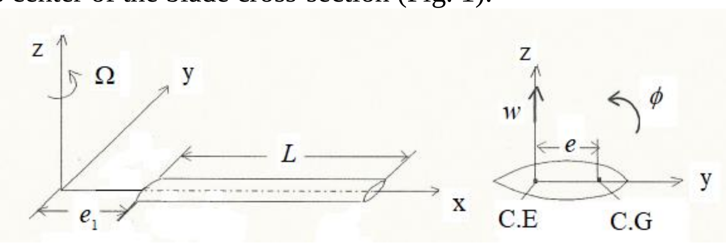

Fig. 1 – Rotating isotropic blade with bending-torsion coupling

Another case is the structural coupling which can be obtained in the case of a composite box beam with constant cross-section shape (Fig. 2). In [7] the vibration characteristics for rotating and nonrotating beams have been analyzed. The corresponding flap-torsion equations in the non-rotating case are those obtained also in [11]:

$$EI_y w^{IV} = \omega^2 m w - K\phi''', \tag{5}$$

$$GJ\phi'' + Kw''' + mk_m^2\omega^2\phi = 0, \tag{6}$$

where K is the bending-torsion or pitch-flap coupling stiffness depending on materials employed, lamination angles and stacking sequences of the walls of the box-beam. Another bending-torsion coupling case will be analyzed in section 3.

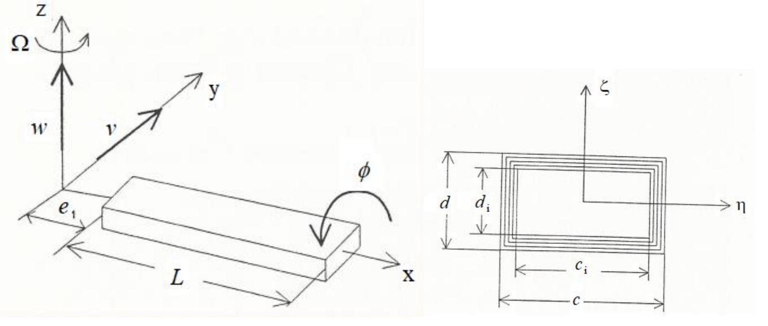

Fig. 2 – Rotating composite box beam

3. INTEGRAL FORM OF THE BENDING AND TORSION EQUATIONS

The bending behavior of a straight beam, having the length L and loaded transversally by the distributed force $p_y(x)$, can be described by a differential equation:

$$\left[EI_z(x)v''\right]'' = p_y(x). \tag{7}$$

It can take the integral form, [2]:

$$v(x) = \int_0^L G_v(x,\xi)\, p_y(\xi)\, d\xi. \tag{8}$$

The previous equation is based on the Green's function $G_v(x,\xi)$ representing the bending deflection $v(x,\xi)$ at distance x due to a unit force applied at ξ (Fig. 3). Similar results can be obtained in the case of the bending in the xz plane, where the equation is:

$$\left[EI_y(x)w''\right]'' = p_z(x) \tag{9}$$

and the corresponding integral form becomes:

$$w(x) = \int_0^L G_w(x,\xi)\, p_z(\xi)\, d\xi. \tag{10}$$

In this case, the Green's function $G_w(x,\xi)$ represents the bending deflections $w(x,\xi)$ at distance x due to a unit force applied at distance ξ.

The differential equation governing the Saint Venant torsional behavior of a beam, having the length L and loaded by the distributed torsion moment $m_t(x)$, is:

$$\left[GJ(x)\phi'\right]' + m_t(x) = 0. \tag{11}$$

It can be written in the integral form:

$$\phi(x) = \int_0^L G_t(x,\xi)\, m_t(\xi)\, d\xi \tag{12}$$

using the Green's function $G_t(x,\xi)$ representing the twist deflection angles $\phi\,(x,\xi)$ at distance x due to a unit torsion moment applied at distance ξ (Fig.3).

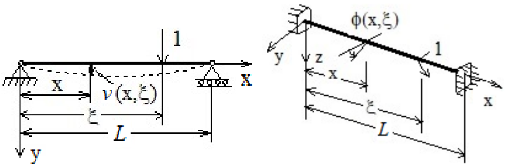

Fig. 3 – Physical significance of Green's functions

The integrals involved in such type of approach can be approximated by a summation using n collocation points ξ_i with $f_i = f(\xi_i)$:

$$\int_0^L f(\xi)d\xi = \sum_{i=1}^{n} f_i \cdot W_i\,, \tag{13}$$

where W_i represents the weighting numbers corresponding to Simpson's method of integration adopted here.

For example, the equations (8),(10) and (12) give the possibility to obtain the static bending and torsion deflections for known distributed forces $p_y(x)$ or $p_z(x)$ and distributed torsion moment $m_t(x)$.

These three relations can be written in matrix form:

$$\{v\} = [G_v][W]\{p_y\};\qquad \{w\} = [G_w][W]\{p_z\};\qquad \{\phi\} = [G_t][W]\{m_t\}. \tag{14}$$

In the previous equations one can remark the following (n,n) matrices:

$[G_v]$ is a matrix containing the measured or calculated influence coefficients $G_v(\xi_i,\xi_j)$,

$[G_w]$ is a matrix containing the measured or calculated influence coefficients $G_w(\xi_i,\xi_j)$,

$[G_t]$ is a matrix containing the measured or calculated influence coefficients $G_t(\xi_i,\xi_j)$,

$[W]$ is a weighting matrix depending on the integration method (Simpson, here).

Other terms are column vectors containing the corresponding values of the beam deflections in the n considered collocation points.

4. LATERAL STABILITY ANALYSIS OF A BEAM

A simply supported beam having a rectangular cross section (bxh) with $b << h$, loaded with the constant bending moment $M_y = M$ (axial force $N = 0$ and bending moment $M_z = 0$) can be seen as an example . In this case the beam is much stiffer in xz bending plane and weaker in the lateral xy bending plane. The notations are those from Fig. 4. For a small torsion deflection ϕ of the beam cross-section, one can obtain the component $M_z = -M\sin\phi = -M\phi$ of the bending moment acting in the weaker bending plane, where the minimum moment of inertia is I_z and a torsion moment component $M_x = M\alpha$, with $\alpha = v'$. The corresponding bending behavior equations is:

$$EI_z\,\frac{d^2v}{dx^2} = M_z = -M_y\phi = -M\phi\,. \tag{15}$$

The torsion equation is in this case:

$$GJ\phi'' + m_t(x) = GJ\phi'' - Mv'' = 0\,. \tag{16}$$

These two equations are coupled as ϕ is involved in the first equation (15) and the lateral bending deflection v is involved in the torsion equation (16).

Elimining $v(x)$ between the two equations, one can obtain the well known second order differential equation Prandtl-Michell:

$$GJ\frac{d^2\phi}{dx^2}+\frac{[M_y(x)]^2}{EI_z}\phi(x)=0. \tag{17}$$

This equation can be solved for different particular cases. In the case of a constant bending moment $M_y(x) = M$, it becomes:

$$\frac{d^2\phi}{dx^2}+\frac{M^2}{GJ\cdot EI_z}\phi=0. \tag{18}$$

For this equation, one can look for a solution of the form:

$$\phi(x)=a_0\cdot\sin\left(\frac{\pi x}{L}\right), \tag{19}$$

which respects the boundary conditions for the torsion angle ϕ at the two simply supported ends ($\phi = 0$). Replacing (19) in (18) one obtains:

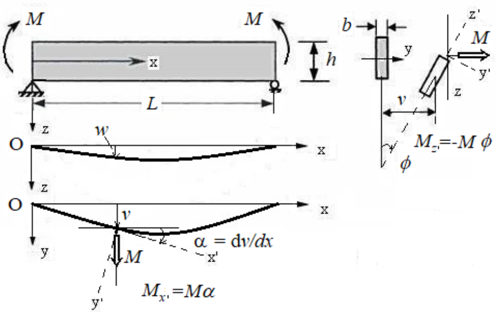

Fig. 4 – Lateral buckling for a simply-supported rectangular cross-section beam

$$\left[-\left(\frac{\pi}{L}\right)^2+\frac{M^2}{GJ\cdot EI_z}\right]\cdot a_0\cdot\sin\left(\frac{\pi x}{L}\right)=0. \tag{20}$$

From the above relation, the critical(buckling) moment obtained also in [12] is:

$$M_{cr}=\frac{\pi}{L}\sqrt{GJ\cdot EI_z}. \tag{21}$$

In order to use the formulation described in the previous section, the equation governing the bending displacements (15) can be written as:

$$[EI_z(x)v'']''=-M\phi''. \tag{22}$$

The matrix form of this equation is:

$$\{v\}=-M[G_v][W][D_2]\{\phi\}=-M[G_1]\{\phi\}. \tag{23}$$

where $[D_2]$ is a differentiating matrix based on central difference operator.

The more general form of the torsion equation (16) is:

$$\left[GJ(x)\phi'\right]' - Mv'' = 0 \tag{24}$$

and in matrix form:

$$\{\phi\} = -M[G_t][W][D_2]\{v\} = -M[G_2]\{v\}. \tag{25}$$

Elimining $\{v\}$ in the equations (23) and (25) one can obtain:

$$\{\phi\} = -M[G_2]\{v\} = M^2[G_2][G_1]\{\phi\}. \tag{26}$$

It represents an eigenvalue problem:

$$\left[[A_1] - M^2[I]\right]\{\phi\} = \{0\}, \tag{27}$$

where $[A_1]$ = inv $([G_2]\ [G_1])$ and $[\ I\ ]$ is a unity matrix having also the dimension $n\times n$. The eigenvalues λ_i of the matrix $[A_1]$ give the square of the critical moments. The value of interest is the minimum buckling moment, obtained from the first (minimum) eigenvalue:

$$M_{cr} = \sqrt{\lambda_1}\ . \tag{28}$$

One can also eliminate $\{\phi\}$ in equations (23) and (25) and the result is:

$$\{v\} = -M[G_1]\{\phi\} = M^2[G_1][G_2]\{v\}. \tag{29}$$

This is also an eigenvalue problem of the form:

$$\left[[A_2] - \frac{1}{M^2}[I]\right]\{v\} = \{0\}, \tag{30}$$

where $[A_2] = [G_1][G_2]$ and $[\ I\ ]$ a unity $n\times n$ dimension matrix.The eigenvalues λ_i of the matrix $[A_2]$ give the inverse of square of the critical moments. The value of interest is the minimum buckling moment, which in this case is obtained starting from the last (maximum) eigenvalue:

$$M_{cr} = 1/\sqrt{\lambda_n}\ . \tag{31}$$

The first numerical example is the case of a uniform simply-supported beam for which the critical buckling moment is given by relation (21). In order to test the presented matrix formulation one can take unitary values for the beam characteristics: length L =1, stiffness EI_z = 1 and GJ = 1. Table 1 shows the convergence of the results obtained with relations (28) or (31), when one increases the number of collocation points. The convergence is relative slow due to the differentiating process using the matrix $[D_2]$. These results can be improved using collocation functions in order to better represent the vectors containing the values $\{v\}$ and $\{\phi\}$ as shown in [4]. The bending displacement v and the torsion angle ϕ are written as:

$$v(x) = \sum_{k=1}^{p} C_{vk}\cdot f_k(x); \qquad \phi(x) = \sum_{k=1}^{p} C_{\phi k}\cdot f_k(x), \tag{32}$$

where $f_k(x)$ are p known functions and C_{vk}, $C_{\phi k}$ are constant coefficients. For the n collocation points, one obtains relations of the form:

$$\begin{aligned} \{v\} &= [F]\{C_v\}; \quad \{v'\} = [F_1]\{C_v\}; \quad \{v''\} = [F_2]\{C_v\}, \\ \{\phi\} &= [F]\{C_\phi\}; \quad \{\phi'\} = [F_1]\{C_\phi\}; \quad \{\phi''\} = [F_2]\{C_\phi\}, \end{aligned} \tag{33}$$

where [F], [F_1], [F_2] are matrices of dimension (n, p) containing the values f_k, $f_k^{'}$, $f_k^{''}$ in the collocation points.

Equation (23) for bending becomes:

$$\{v\} = [F]\{C_v\} = -M[G_v][W][F_2]\{C_\phi\}, \tag{34}$$

and equation (25) for torsion becomes:

$$\{\phi\} = [F]\{C_\phi\} = -M[G_t][W][F_2]\{C_v\}. \tag{35}$$

Multiplying the last two equations at left with the transpose of the matrix [F], one obtains:

$$[A]\{C_v\} = -M[F]^t[G_v][W][F_2]\{C_\phi\} = -M[G_3]\{C_\phi\}, \tag{36}$$

$$[A]\{C_\phi\} = -M[F]^t[G_t][W][F_2]\{C_v\} = -M[G_4]\{C_v\}, \tag{37}$$

where:

$$[A] = [F]^t[F], \tag{38}$$

is a (p, p) dimension matrix. Then, multiplying equations (36), (37) at left with the inverse of the matrix [A], one obtains:

$$\{C_v\} = -M[A]^{-1}[G_3]\{C_\phi\}, \tag{39}$$

$$\{C_\phi\} = -M[A]^{-1}[G_4]\{C_v\}. \tag{40}$$

Elimining $\{C_v\}$ in equations (39) and (40) one can also obain:

$$\{C_\phi\} = -M[A]^{-1}[G_4]\{C_v\} = M^2[A]^{-1}[G_4][A]^{-1}[G_3]\{C_\phi\} = M^2[G_5]\{C_\phi\}. \tag{41}$$

By elimining $\{C_\phi\}$ in equations (39) and (40), the result is:

$$\{C_v\} = -M[A]^{-1}[G_3]\{C_\phi\} = M^2[A]^{-1}[G_3][A]^{-1}[G_4]\{C_v\} = M^2[G_6]\{C_v\}. \tag{42}$$

These are eigenvalues problems which can be written for example as:

$$[[I] - M^2[G_5]]\{C_\phi\} = \{0\}; \quad \left[[I] - \frac{1}{M^2}[G_6]^{-1}\right]\{C_v\} = \{0\}, \tag{43}$$

relations similar with (27), respectively (30).

Table 2 presents the results when using a number of k = 1..p from the following collocation functions, compatible with the boundary conditions:

$$v_k(x) = v_0 \sin\left(\frac{k\pi x}{L}\right); \quad \phi_k(x) = \phi_0 \sin\left(\frac{k\pi x}{L}\right). \tag{44}$$

The approach based on collocation functions utilizes for the bending deflection v and for the twisting angle ϕ the same from above functions. In this case for a given number of collocation points (n = 100) the results does not depend on p, as for the first buckling mode the first functions for k = 1 from (44) are the exact ones. The precision is also increased because the numerical differentiation is no longer necessary. When the cross-section of the beam is non-uniform, the collocation functions approach is useful as one can see from the next example.

Table 1 – Results for the critical moment (collocation points)

	$n = 20$	$n = 60$	$n = 100$	$n = 150$	$n = 200$	Exact (21)
M_{cr}	3.2599	3.1764	3.1623	3.1507	3.1493	$\pi = 3.1415$

Table 2 – Results for the critical moment (n =100 and p collocation functions)

	$p = 1$	$p=2$	$p = 4$	$p = 6$	$p = 10$	Exact (21)
M_{cr}	3.1412	3.1412	3.1412	3.1412	3.1412	$\pi = 3.1415$

Another example is the case of a linearly tapered beam having rectangular cross-section with $b(x) = b_o$ and:

$$h(x) = h_o\left(1 + \frac{x}{L}\delta\right) \tag{45}$$

For these notations see also Fig. 5. In this case, for the simply-supported (pin-ended) beam, the critical bending moment obtained in [13] is:

$$M_{cr} = \frac{\pi\delta}{L\ln(1+\delta)}\sqrt{GJ_o \cdot EI_o}\,, \tag{46}$$

where $I_o = I_z$ at $x = 0$ and $J_o = J$ at $x = 0$.

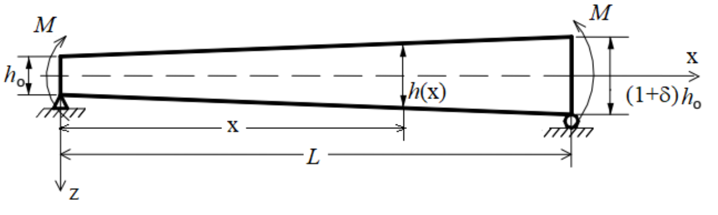

Fig. 5 – Lateral buckling for a simply-supported rectangular non-uniform cross-section beam

As example, one can also take the unit values for the beam characteristics: length $L = 1$ and stiffness $EI_o = 1$ and $GJ_o = 1$. In this case, the bending and torsion stiffness have linear variations:

$$EI_z(x) = EI_o\left(1 + \frac{x}{L}\delta\right); \quad GJ(x) = GJ_o\left(1 + \frac{x}{L}\delta\right). \tag{47}$$

Table 3 shows the convergence of the results obtained with relations (28) or (31), when the number of the collocation points is increased.

Table 4 presents the results when using the p collocation functions and n collocation points. One can see that the increased number of the considered collocation functions can also improve the precision.

Table 3 – Results for the critical moment (non-uniform beam, collocation points)

M_{cr}	$n = 20$	$n = 60$	$n = 100$	$n = 150$	$n = 200$	Exact (46)
$\delta = 0.3$	3.7587	3.6426	3.6219	3.6119	3.6069	3.5922
$\delta = 0.5$	4.0741	3.9352	3.9093	3.8959	3.8643	3.8740
$\delta=1$	4.8218	4.6222	4.5855	4.5676	4.5587	4.5324

Table 4 – Results for the critical moment (non-uniform beam, collocation functions and points: *p*, *n*)

M_{cr}	$p = 1$, $n = 100$	$p = 2$, $n = 100$	$p = 10$, $n = 100$	$p = 10$, $n = 150$	$p = 10$, $n = 200$	Exact (46)
$\delta = 0.3$	3.6043	3.5943	3.5941	3.5936	3.5933	3.5922
$\delta = 0.5$	3.9036	3.8780	3.8775	3.8764	3.8759	3.8740
$\delta = 1$	4.6289	4.5444	4.5401	4.5375	4.5363	4.5324

5. CONCLUSIONS

This work presents several examples of structural analysis of beams having bending-torsion coupling. Generally, the coupling terms can be structural, inertial or aerodynamic ones. The case of the lateral buckling of beams having a stiff bending plane and a weak bending plane is then analyzed using an integral formulation (I.F.) based on the use of *flexibility influence functions* (Green's functions).

These functions are computed here for the simply supported beam in bending and for a fixed-fixed beam subjected to Saint-Venant torsion, using their physical significance of displacements(or rotation angles) in a collocation point of the beam due to a unit force(or unit moment) applied in another collocation point.

For the numerical integration, *integration (weighting) matrices* based of Simpson's method of integration are here employed. *Differentiating matrices* are also necessary in order to obtain the second derivatives of the beam deflections. The numerical differentiation is a source of errors and therefore, the use of *collocation functions* can lead to a better accuracy of calculations. It is of particular importance in the case of non-uniform cross-section beams.

The simple examples presented here show good agreement from an engineering point of view, when compared with the analytical results concerning the critical bending moment calculation for lateral buckling. The presented approach is a simple matrix one, its accuracy depending on the number of the used collocation points and functions.

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