Buckling Analysis of Straight Beams with different Boundary Conditions using an Integral Formulation of corresponding Differential Equations

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Abstract: *This work deals with structural stability analysis of straight beams, having different boundary conditions. The particular case of the beam on elastic foundation is also analyzed. The method of analysis is an approximate integral one, using structural flexibility functions (Green's functions). The differential equations governing the Euler buckling of such beams are put in integral form. This approach is a matrix one leading to an eigenvalues problem in the case of stability analysis. Different numerical examples concerning the calculation of the critical buckling loads are presented in comparison with available data from literature. The obtained results show good agreement from engineering point of view.*

Key Words*: Straight Euler Beam, Green's Functions, Elastic Foundation, Buckling*

1. INTRODUCTION

Buckling of beams subjected to compression represents an important topic in the fields of mechanical, structural and aeronautical engineering. The calculation of critical buckling loads pays a crucial role for such a compression members which are the object of many studies including static, dynamic and stability ones. Exact analytical solutions are available especially for particular uniform beams in the case of usual boundary conditions. Such solutions were resumed in several books as [1-3]. Apart from FEM, different numerical approaches have been developed for beams with variable flexural rigidity or with continuous restraints. For instant, one can enumerate the energy methods, finite difference method [4], DQM (differential quadrature method, [5]), VIM (variational iteration method, [6]) and HAM (homotopy perturbation method, [7]). The structural flexibility functions (Green's functions) and their use in the structural static, dynamic, stability and aeroelastic analysis of beam like structures were presented in several works as [8-12]. The equation describing the bending deflection $w(x)$ of a non-uniform beam having bending stiffness $EI(x)$ and resting on an elastic foundation (characterised by the constant k), subjected to constant axial compression force P and transverse distributed force $p(x)$ (see Fig. 1), takes the form [7]:

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)+P\cdot\frac{\partial^2 w}{\partial x^2}+kw=p(x)\ . \tag{1}$$

In order to obtain critical buckling loads, starting from the above differential equation, this work analyzes several beam configurations with different boundary conditions, including or not the presence of an elastic foundation and loaded in compression. The equation governing the bending behavior of the beam is solved using an integral form based on Green's functions. This approach leads to an eigenvalues problem allowing the computation of the critical buckling loads. Several examples are then discussed in comparison with available results from literature.

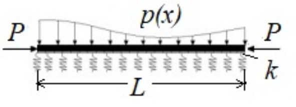

Fig. 1 – Beam resting on elastic foundation

2. INTEGRAL FORMULATION OF DIFFERENTIAL EQUATIONS

The simplest form of differential equation for static bending displacement *w(x)* of a straight beam loaded by the transverse load $p(x)$, is:

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)=p(x)\ . \tag{2}$$

It can take the following integral form, based on the use of structural flexibility functions (Green's functions) [8-10]:

$$w(x)=\int_0^L G_w(x,\xi)p(\xi)\,d\xi\,, \tag{3}$$

where $G_w(x,\xi)$ is the Green's function with the meaning of bending deflection w at location x on the beam due to a transverse unit force applied at location ξ on the same beam. Choosing n collocation points on the beam, the relation (3) takes the matrix form:

$$\{w\}=[G_w][W]\{p\}\,, \tag{4}$$

where:
$[G_w]$ is a (n,n) dimension matrix with the Green's functions values $G_w(x_i,\xi_j)$,
$[W]$ is a (n,n) diagonal weighting matrix containing the weighting numbers W_i, corresponding to Simpson method of integration,
$\{w\}$ and $\{p\}$ are column vectors containing bending deflections $w(\xi_j)$ and the distributed forces $p(\xi_j)$ respectively.
For the case when the external distributed load $p(x) = 0$, the equation (1) is re-written as:

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)=-P\frac{\partial^2 w}{\partial x^2}-kw\,. \tag{5}$$

It can be regarded as an equation of the form (2) allowing the use of integral form (3) and the matrix form (4). In above equation E is the Young modulus of elasticity, $I(x)$ the moment of inertia of the cross-section and k the elastic coefficient for the Winkler foundation model. For constant value k the equation (5) takes the matrix form:

$$\{w\} = -P[G_w][W][D_2]\{w\} - k[G_w][W]\{w\}, \tag{6}$$

where $[D_2]$ is a (n,n) differentiation matrix used to obtain the second derivative of the bending deflection w. Using the notations:

$$[G_1] = [[G_w][W][D_2]]^{-1}; \quad [G_2] = [G_1][G_w][W], \tag{7}$$

after a left multiplication of (6) with the matrix $[G_1]$ one obtains the relation:

$$[G_1]\{w\} = -P[I_n]\{w\} - k[G_2]\{w\}, \tag{8}$$

where $[I_n]$ is the (n,n) unity matrix. Relation (8) represents an n dimensional eigenvalues and eigenvectors problem of the form:

$$[[G_1] + k[G_2]]\{w\} = [A_n]\{w\} = -P\{w\}. \tag{9}$$

The critical buckling loads $\lambda = -P_{cr}$ are the eigenvalues λ of the matrix $[A_n]$.

As was showed in [10], the dimension of the eigenvalues problem (9) can be reduced using collocation functions, by writing the bending displacement w in the form:

$$w(x) = \sum_{k=1}^{p} C_k \cdot f_k(x) \tag{10}$$

with $f_k(x)$ a number of p known functions and p constant coefficients C_k. Then, for the n chosen collocation points, one can obtain the following relations:

$$\{w\} = [F]\{C\}; \quad \{w'\} = [F_1]\{C\}; \quad \{w''\} = [F_2]\{C\} \tag{11}$$

where the (n,p) dimension matrices $[F]$, $[F_1]$, $[F_2]$ contain the values f_k and the derivatives values f_k', f_k'' in the collocation points. Thus, the differentiating matrices are no more necessary and (6) can be written as:

$$[F]\{C\} = -P[G_w][W][F_2]\{C\} - k[G_w][W][F]\{C\}, \tag{12}$$

Multiplying the above relation at left with the transpose of the matrix $[F]$ and using the following notations:

$$[A_1] = [F]^t[F]; \quad [B_1] = [F]^t[G_w][W][F_2]; \quad [B_2] = [F]^t[G_w][W][F], \tag{13}$$

the relation (12) becomes:

$$[A_1]\{C\} = -P[B_1]\{C\} - k[B_2]\{C\}. \tag{14}$$

After a left multiplication of (14) with the inverse of the matrix $[B_1]$ one obtains:

$$[B_1]^{-1}[A_1]\{C\} = -P[I_p]\{C\} - k[B_1]^{-1}[B_2]\{C\}, \tag{15}$$

where $[I_p]$ is the (p,p) unity matrix. Relation (15) represents a p dimensional eigenvalues and eigenvectors problem of the form:

$$[B_1]^{-1}[[A_1] + k[B_2]]\{C\} = [A_p]\{C\} = -P\{C\}. \tag{16}$$

The critical buckling loads $\lambda = -P_{cr}$ are now the eigenvalues λ of the matrix $[A_p]$.

3. BUCKLING OF UNIFORM CROSS-SECTION BEAMS

The first examples for buckling load calculation concerning the uniform cross-section beams (EI = const.) are taken from [7]. The results are presented using two non-dimensional coefficients namely β (of the elastic foundation) and α a critical buckling load parameter:

$$\beta = \frac{kL^4}{EI}, \quad \alpha = \frac{P_{cr}L^2}{EI}. \tag{17}$$

The comparison of results is presented for five different usual boundary conditions (B.C.) shown in figure 2, namely P-P (pinned-pinned), C-F (clamped-free), C-P (clamped-pinned), C-C (clamped-clamped) and C-S (clamped-sliding restraint). Table 1 shows the results concerning the first two buckling modes when using n = 100 equally spaced collocation points on the beam.

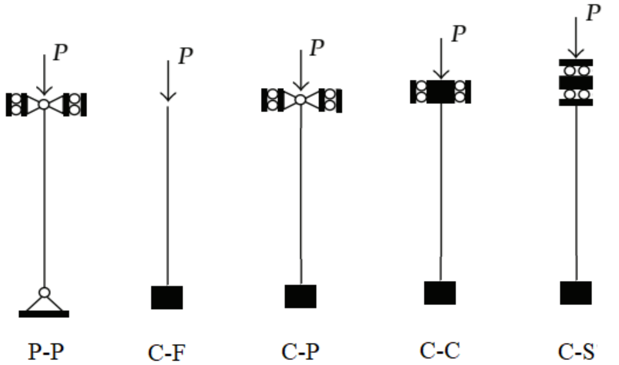

Fig. 2 – The five types of boundary conditions for an uniform beam

Table 1 – Results for α parameter obtained with n=100 collocation points and relation (9)

B.C.	β = 0		β = 50		β = 100	
	[7]	Present	[7]	Present	[7]	Present
P-P-1	9.8696	9.9619	14.9357	15.0757	20.0017	20.1888
P-P-2	39.4779	39.8683	40.7449	41.1487	42.0114	42.4294
C-F-1	2.4674	2.4676	8.8614	8.8627	11.9964	11.9993
C-F-2	22.2066	22.2230	33.0879	33.1107	45.2659	45.3001
C-P-1	20.1907	20.2045	24.2852	24.3018	28.3066	28.3261
C-P-2	59.6795	59.7993	61.0966	61.2192	62.5613	62.6865
C-C-1	39.4784	39.5288	43.2606	43.3159	47.0660	47.0670
C-C-2	80.7629	80.9741	81.7943	82.0082	82.8246	83.0414
C-S-1	9.8696	10.2801	23.5717	24.6760	32.6690	34.4636
C-S-2	39.4784	39.5288	44.4104	43.7216	52.8965	51.2938

In this work, the necessary Green's functions values (the terms of the matrix [G_w]) are obtained using a specific approach from Strength of Materials, namely Mohr-Maxwell method. These values are bending displacements of the beam in static determinate system cases for S-S and C-F boundary conditions. For the other three boundary conditions cases, the bending displacements are numerically computed in static indeterminate systems. This calculation process consists in numerical integration which leads to a first source of errors, which are added to those of the subsequent calculations. Especially the numerical differentiation using the matrix [D_2] is a source of numerical errors which can be avoided using the approach based on collocation points and collocation functions [10]. Some corrections were applied in the general matrix formulation (6) for the C-F and C-S beams, as in these two cases the application point (the beam tip) of the compression force P has also a transverse displacement. The results obtained are still in good agreement from engineering point of view.

As a special treatment has to be done in the case of a clamped-free beam, an example from [6] is analyzed. It concerns the study of buckling in the case of a clamped-free beam resting on a transverse spring of stiffness K located at the free end (Fig. 3).

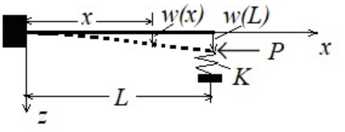

Fig. 3 – Clamped-free beam having a spring at the end

The bending behavior of such a beam is in this case described by:

$$EI(x)\frac{\partial^2 w}{\partial x^2} = EI(x)w'' = -P\left[w(x) - w(L)\right] \tag{18}$$

which has in the right hand a supplementary term, a bending moment $Pw(L)$. By differentiation with respect to x, the above equation becomes:

$$\left[EI(x)w''\right]' = -P\left[w'(x) - w'(L)\right] \tag{19}$$

The term $Pw'(L)$ represents a transverse concentrated force located at the end. A second differentiation with respect to x leads to:

$$\left[EI(x)w''\right]'' = -P\left[w'' - w''(L)\right] \tag{20}$$

It has the form (2) and neglecting the term containing $w''(L)$ its integral form is:

$$w(x) = -P\int_0^L G_w(x,\xi)w''(\xi)d\xi + P\cdot G_w(x,L)\cdot w'(L) - K\cdot G_w(x,L)\cdot w(L) \tag{21}$$

The second term of (21) takes into account the concentrated tip transverse force $Pw'(L)$. In the same manner, the effect of the tip spring was added in the third term in (21). When using n collocation points, the matrix form of (21) is:

$$\{w\} = -P[G_w][W][D_2]\{w\} + P[G_w^*]\{w\} - K[G_w^{**}]\{w\}. \tag{22}$$

$[G_w^*]$ is a (n, n) sparse correction matrix used to obtain the first derivative of the bending deflection at beam end ($x = L$), in the form:

$$w'(L) = \frac{w(L) - w(L-\Delta x)}{\Delta x} = \frac{w_n - w_{n-1}}{x_n - x_{n-1}}. \tag{23}$$

$[G_w^{**}]$ is a (n, n) sparse correction matrix used to add the last term from (21) namely the contribution of the free end transverse spring (located at $x = L$). The collocation points x_i are chosen equally spaced with the step Δx. The first point x_1 is placed near the $x = 0$ and the last point is located exactly at the beam free end ($x_n = L$). Using the notations:

$$[G_3] = \left[[G_w][W][D_2] - [G_w^*]\right]^{-1}; \quad [G_4] = [G_3][G_w^{**}], \tag{24}$$

after a left multiplication of (22) with the matrix [G_3] one obtains the relation:

$$[G_3]\{w\} = -P[I_n]\{w\} - K[G_4]\{w\}, \qquad (25)$$

where [I_n] is the (n, n) unity matrix. Relation (25) represents a n dimensional eigenvalues and eigenvectors problem of the form:

$$[[G_3] + K[G_4]]\{w\} = [B_n]\{w\} = -P\{w\}. \qquad (26)$$

The critical buckling loads $\lambda = -P_{cr}$ are the eigenvalues λ of the matrix [B_n].

Using also the collocation functions approach, the matrix form for (21) becomes:

$$[F]\{C\} = -P[G_w][W][F_2]\{C\} + P[G_w^*][F_1]\{C\} - K[G_w^{**}][F]\{C\}, \qquad (27)$$

where $[G_w^*]$ and $[G_w^{**}]$ are sparse corrections matrices used to add the two last terms from (21). The above relation is then multiplying with the transpose of the matrix [F] and using the following notations:

$$[A_1] = [F]^t[F]; \quad [B_1] = [F]^t[G_w][W][F_2]; \quad [B_3] = [F]^t[G_w^*][F_1]; \quad [B_4] = [F]^t[G_w^{**}][F], \qquad (28)$$

the relation (27) becomes:

$$[A_1]\{C\} = -P[B_1]\{C\} + P[B_3]\{C\} - K[B_4]\{C\}. \qquad (29)$$

Putting as supplementary notations:

$$[B_5] = [[B_1] - [B_3]]^{-1}; \quad A_p = [B_5][[A_1] + K[B_4]], \qquad (30)$$

after a left multiplication of (29) with the matrix [B_5] one obtains:

$$[A_p]\{C\} = -P\{C\}, \qquad (31)$$

where [A_p] is a (p,p) matrix whose eigenvalues λ are the critical buckling loads $\lambda = -P_{cr}$.
The results are presented using two non-dimensional coefficients:

$$\overline{K} = \frac{KL^3}{EI}, \quad \alpha = \frac{P_{cr}L^2}{EI}. \qquad (32)$$

For the beam configuration presented in Fig. 3, the results obtained with (26) and (31) for p = 10 collocation functions, are given in Tables 2 and 3.

Table 2 – Results for α parameter, n =100 collocation points (C-F beam with tip spring)

$\overline{K}$	0.1	1	2.5	5	7.5	10
Results [6]	2.5484	3.2734	4.4644	6.3920	8.2309	9.9563
Present,(26)	2.5486	3.2737	4.4647	6.3926	8.2317	9.9574
Present,(31)	2.5485	3.2736	4.4646	6.3923	8.2313	9.9570

Table 3 – Results for α parameter, n=100 collocation points (C-F beam with tip spring)

$\overline{K}$	25	50	75	100	1000
Results [6]	16.6435	18.9922	19.4958	19.7035	20.1496
Present,(26)	16.6495	19.0031	19.5079	19.7160	20.1631
Present,(31)	16.6482	19.0003	19.5047	19.7126	20.1594

All these results are in good agreement. The further increasing of K value leads to a bad conditioning of the eigenvalues problems (26) and (31).

4. BUCKLING OF NON-UNIFORM CROSS-SECTION BEAMS

As example, one consider a non-uniform cross-section beam with the bending stiffness EI_0 at $x = 0$ and an exponential variation of the stiffness along the length:

$$EI(x) = EI_0 \cdot e^{\frac{ax}{L}} \tag{33}$$

or a power-low type variation of the form:

$$EI(x) = EI_0 \cdot \left(1 - b\frac{x}{L}\right)^a \tag{34}$$

with a and b positive constants. The results obtained using the present formulation with n = 120 collocation points are compared with exact solutions given in [2] used for comparisons also in [4] in terms of the stability parameter:

$$\alpha = \frac{P_{cr}L^2}{EI_0}. \tag{35}$$

Tables 4 to 7 present result comparisons for several cases given by different values of the a and b parameters. The used approach is based on collocation points only (n = 120). Good agreement was obtained also for these non-uniform beam cases concerning the stability analysis.

Table 4 – Results for α parameter, bending stiffness given by (33)

a	P-P beam		C-F beam		C-C beam		C-P beam	
	[2]	Present	[2]	Present	[2]	Present	[2]	Present
0.0	9.870	9.946	2.467	2.476	39.480	39.510	20.190	20.200
-0.1	9.380	9.456	2.394	2.393	37.550	37.563	19.200	19.202
-0.5	7.634	7.686	2.110	2.107	30.600	30.563	15.640	15.614
-1.0	5.827	5.861	1.782	1.774	23.490	23.416	11.990	11.944
-1.5	4.389	4.412	1.480	1.471	17.860	17.773	9.098	9.046
-2.0	3.264	3.279	1.209	1.199	13.460	13.367	6.839	6.786

Table 5 – Results for α parameter, bending stiffness given by (34) with a = 1

b	P-P beam		C-F beam		C-C beam		C-P beam	
	[2]	Present	[2]	Present	[2]	Present	[2]	Present
0.1	9.372	9.442	2.393	2.391	37.48	37.493	19.17	19.169
0.3	8.343	8.402	2.235	2.232	33.27	33.254	17.03	17.018
0.5	7.256	7.303	2.062	2.056	28.70	28.64	14.74	14.703

Table 6 – Results for α parameter, bending stiffness given by (34) with a = 2

b	P-P beam		C-F beam		C-C beam		C-P beam	
	[2]	Present	[2]	Present	[2]	Present	[2]	Present
0.1	8.893	8.959	2.319	2.317	35.56	35.561	18.19	18.181

0.3	7.005	7.050	2.012	2.006	27.91	27.850	14.29	14.255
0.5	5.198	5.227	1.683	1.674	20.48	20.381	10.53	10.469

Table 7 – Results for α parameter, bending stiffness given by (34) with *a* = 3

b	P-P beam		C-F beam		C-C beam		C-P beam	
	[2]	Present	[2]	Present	[2]	Present	[2]	Present
0.1	8.436	8.495	2.246	2.243	33.73	33.715	17.25	17.237
0.3	5.840	5.875	1.798	1.790	23.29	23.210	11.92	11.875
0.5	3.628	3.644	1.336	1.325	14.35	14.236	7.362	7.299

The tables 8 to 11 show the same result comparisons when the used approach is based on collocation points (n = 120) and collocation functions (p = 5). The obtained results show a slight improvement.

Table 8 – Results for α parameter, bending stiffness given by (33)

a	P-P beam		C-F beam		C-C beam		C-P beam	
	[2]	Present	[2]	Present	[2]	Present	[2]	Present
0.0	9.870	9.868	2.467	2.468	39.480	39.493	20.190	20.192
-0.1	9.380	9.384	2.394	2.394	37.550	37.544	19.200	19.195
-0.5	7.634	7.633	2.110	2.109	30.600	30.549	15.640	15.608
-1.0	5.827	5.825	1.782	1.776	23.490	23.410	11.990	11.939
-1.5	4.389	4.388	1.480	1.473	17.860	17.773	9.098	9.042
-2.0	3.264	3.263	1.209	1.202	13.460	13.373	6.839	6.783

Table 9 – Results for α parameter, bending stiffness given by (34) with *a* = 1

b	P-P beam		C-F beam		C-C beam		C-P beam	
	[2]	Present	[2]	Present	[2]	Present	[2]	Present
0.1	9.372	9.370	2.393	2.393	37.48	37.47	19.17	19.16
0.3	8.343	8.342	2.235	2.233	33.27	33.24	17.03	17.01
0.5	7.256	7.254	2.062	2.058	28.70	28.63	14.74	14.70

Table 10 – Results for α parameter, bending stiffness given by (34) with *a* = 2

b	P-P beam		C-F beam		C-C beam		C-P beam	
	[2]	Present	[2]	Present	[2]	Present	[2]	Present
0.1	8.893	8.892	2.319	2.318	35.56	35.54	18.19	18.17
0.3	7.005	7.004	2.012	2.007	27.91	27.84	14.29	14.25
0.5	5.198	5.197	1.683	1.676	20.48	20.38	10.53	10.46

Table 11 – Results for α parameter, bending stiffness given by (34) with *a* = 3

b	P-P beam		C-F beam		C-C beam		C-P beam	
	[2]	Present	[2]	Present	[2]	Present	[2]	Present
0.1	8.436	8.433	2.246	2.244	33.73	33.70	17.25	17.23
0.3	5.840	5.839	1.798	1.792	23.29	23.21	11.92	11.87
0.5	3.628	3.627	1.336	1.325	14.35	14.24	7.362	7.296

For the P-P beam case, the used collocation functions, corresponding to the boundary conditions, are the family of functions:

$$w_i(x) = \sin\left(\frac{i\pi x}{L}\right), i = 1...p. \quad (36)$$

For other boundary conditions one uses the family of the mode shapes for transverse vibrations of uniform beams from [13], based on Krylov-Duncan functions:

$$S(x) = \frac{ch\,x + \cos x}{2}, \quad T(x) = \frac{sh\,x + \sin x}{2}, \quad (37)$$

and:

$$U(x) = \frac{ch\,x - \cos x}{2}, \quad V(x) = \frac{sh\,x - \sin x}{2}. \quad (38)$$

For the C-F beam the collocation functions are of the form:

$$w_i(x) = T(\beta_i) \cdot U\left(\beta_i \frac{x}{L}\right) - S(\beta_i) \cdot V\left(\beta_i \frac{x}{L}\right) \quad (39)$$

where:

$$\beta_1 = 1.8751, \beta_2 = 4.6941, \beta_i = \frac{2i-1}{2}\pi, \quad i = 3...p. \quad (40)$$

For the C-C beam the collocation functions are the next ones:

$$w_i(x) = V(\beta_i) \cdot U\left(\beta_i \frac{x}{L}\right) - U(\beta_i) \cdot V\left(\beta_i \frac{x}{L}\right) \quad (41)$$

where:

$$\beta_1 = 4.73, \beta_2 = 7.8532, \beta_i = \frac{2i+1}{2}\pi, \quad i = 3...p. \quad (42)$$

For the C-P beam the collocation functions are also of the form, (41) but with:

$$\beta_1 = 3.9266, \beta_2 = 7.0685, \beta_i = \frac{4i+1}{4}\pi, \quad i = 3...p. \quad (43)$$

All these collocation functions in such beam analyzes are somewhat similar with the shape functions used in FEM.

5. CONCLUSIONS

This paper presents critical buckling load calculations of several beam cases with different boundary conditions. The differential equations governing the bending behavior of such beams are put in integral form based on the use of *structural flexibility functions* (Green's functions). The effects of an elastic foundation (according to the Winkler model) can be also considered in the formulation.

The necessary Green's functions are numerically computed for five different and usual boundary conditions using a specific integral method of Strength of Materials, for static determinate or static indeterminate cases. In fact the values of such a function are bending displacements in a collocation point of the beam due to a unit force applied in another collocation point(influence coefficients).

The integral formulation uses *integration matrices* based on Simpson's method. *Differentiating matrices* are also necessary in order to obtain the second derivatives of the bending deflections. The examples presented here show good agreement from engineering point of view when compare with the analytical results concerning the critical buckling loads.

The presented matrix approach is an interesting alternative in such stability analysis, their accuracy depending on the number of the used collocation points. In order to decrease the dimension of the eigenvalues problem and to reduce the errors that come from the differentiation process, the use of *collocation functions* can be also taken into account with good results. This approach is also easy to implement and to use for parametric studies in the case of buckling analysis of non-uniform beams with different end boundary conditions.

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