# Numerical Methods for Transverse Vibration Analysis of Straight Euler and Timoshenko Beams

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***Abstract:*** *This paper presents a short review of numerical methods used for lateral vibration analysis in the case of straight Euler and Timoshenko beams. A particular integral method is described with more details. This approximate integral method is based on the use of flexibility influence functions (Green's functions). It leads to a matrix formulation and to an eigenvalue problem for vibration analysis. The presented approach is able to estimate separately the shear effects and the rotary inertia effects and also the combined effects. Simple numerical examples are also presented for comparisons with analytical and finite elements results. The results show good agreement.*

***Key Words****: Euler Beams, Timoshenko Beams, Green's Functions, Bending Vibration*

## 1. INTRODUCTION

The dynamic analysis of beam-like structures is a usual issue in many engineering areas. In particular, the calculation of the natural frequencies pays an important role in the design of different beam members. At present, a large number of numerical methods are available for this type of analysis. For many practical purposes the Euler-Bernoulli beam model, established in the 18th century, is used. This model is presented in more details in books like [1, 2]. In this case the depth of the beam is considered small and the mass is concentrated at its neutral axis. For short beams or deep cross section beams, the Timoshenko beam model, which includes the shear effects, is more adequate. Rotary inertia effects, introduced first by Lord Rayleigh in 1877, can also be taken into account in vibration analysis. For example, the Timoshenko beam model is described in works [3, 4]. All these books present different analytical or numerical methods used to solve the differential equations corresponding to transverse vibration of different configurations of such a beams. A lot of papers from literature also address this subject. An old classic method, the Myklestad method, which is a development of the Holzer method (used for torsion vibration), is described in [5] for the case of bending vibration analysis. The classic Rayleigh, Rayleigh-Ritz, Galerkin and collocation methods, all these considered as integral methods, are described in [6]. The Rayleigh method is used for the fundamental frequency estimation for both discrete and continuous systems (beams included). The Rayleigh-Ritz method (assumed modes method) is an extension of Rayleigh method. It is used for example in [7], while Galerkin's method is employed in [8]. The differential equations involved in vibration analysis were solved using a combination of Runge-Kutta and Regula-Falsi methods in the reference [9]. A new method, the so-called *cell discretization method* (CDM) is described in [10]. In paper [11], Wentzel, Kramers and Brillouin (WKB) expansion series is applied to find solutions for transverse vibration of beams. The equations involved for vibration of Euler beams can be also solved by Differential Transform Method (DTM), a transformation technique based on Taylor expansion series [12]. Other analytical approximate techniques, namely ADM (Adomian Decomposition Method), VIP (Variational Iteration Method) and HPM (Homotopy Perturbation Method) were described and employed in [13]. There are also some semi-analytical methods for vibration analysis of rotating beams as those presented in [14]. A lot of finite element formulations are available for vibration analysis of Timoshenko beams as for example [15-17] or [18] for Euler rotating beams vibration analysis. The use of Green's functions, for analysis of beam-like structures of interest for aeronautics, is presented in some books and papers as [19-21]. Concerning the Green functions approach for vibration analysis of beams, one can list here papers like [22-24].

In this work, the free vibration analysis of beams for both Euler and Timoshenko models is addressed based on the more general integral formulation described in [21] and using the Green's functions, as influence coefficients which have been determined in [20]. To illustrate the presented approach, rectangular uniform and non-uniform cross-section beams are analyzed in terms of natural frequencies calculations. Several examples are discussed in order to compare the obtained results with analytical results and other results from literature.

## 2. TIMOSHENKO VERSUS EULER BEAMS

According to the Timoshenko beam theory, when displaces laterally, the total rotation of the cross-section of a beam has two terms:

$$\frac{dv}{dx} = \theta + \gamma \tag{1}$$

with $\theta$ the bending rotation and $\gamma$ a rotation due to shear (Fig. 1).

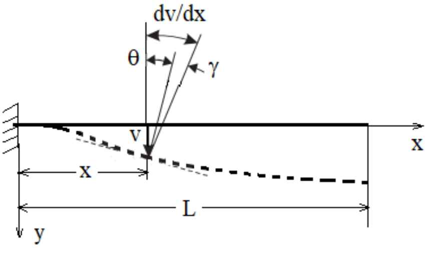


Fig. 1 – Elements of cross-section rotation for Timoshenko beams

The shear force in *y* direction and the bending moment (with respect to *z* axis) are for a uniform beam:

$$Q = kAG\left(\frac{dv}{dx} - \theta\right) \tag{2}$$

and respectively:

$$M = EI\frac{d\theta}{dx}. \tag{3}$$

In the case of harmonic vibration modes the distributed inertial forces and inertial moments (rotary inertia) are considered in the next two equations:

$$\frac{dQ}{dx} + \rho A\omega^2 v = 0 \tag{4}$$

and respectively,

$$\frac{dM}{dx} + \rho I\omega^2\theta = -Q. \tag{5}$$

In the equations above the notations are the followings: $E$ Young modulus of elasticity, $G$ shear modulus, $k$ shear coefficient of the cross-section, $I$ moment of inertia of the cross-section, ρ the mass density of the material, $A$ the cross-section area and ω is the natural circular frequency (or radians frequency) of vibration. Substituting (2) and (3) in (4) and (5), one can obtain the following two coupled equations:

$$kAG\left(\frac{d^2v}{dx^2} - \frac{d\theta}{dx}\right) = -\rho A\omega^2 v, \tag{6}$$

$$EI\frac{d^2\theta}{dx^2} + kAG\left(\frac{dv}{dx} - \theta\right) + \rho I\omega^2\theta = 0. \tag{7}$$

In the case of Euler beams $\gamma = 0$, $\theta = dv/dx$ and neglecting the rotary inertia, from the equations (3), (4) and (5) one obtains a single relation:

$$EI\frac{d^4v}{dx^4} = \rho A\omega^2 v. \tag{8}$$

It has analytical solutions allowing to determine the natural circular frequencies $\omega_i$ from the simple formula:

$$\omega_i = (\alpha_i L)^2\sqrt{\frac{EI}{\rho AL^4}}, \tag{9}$$

with $\beta_i = \alpha_i L$ depending on the boundary conditions.

For example, in the particular case of the cantilever (clamped-free beam) these values are given by [2]:

$$\alpha_1 L = 1.875;\ \alpha_2 L = 4.694;\ \alpha_3 L = 7.855;\ \alpha_4 L = 10.996;\ \alpha_i L = \frac{2i-1}{2}\pi. \tag{10}$$

In the finite element method, in order to obtain the stiffness matrix and mass matrix for a beam element having the length $l$, one can start from the expressions of the strain energy $U$ and kinetic energy $T$ respectively, as in [4]:

$$U = \frac{1}{2}\int_0^l EI\left(\frac{d\theta}{dx}\right)^2 dx + \frac{1}{2}\int_0^l GkA\left(\frac{dv}{dx} - \theta\right)^2 dx, \tag{11}$$

$$T=\frac{1}{2}\int_0^l \rho A\dot{v}^2dx+\frac{1}{2}\int_0^l \rho I\dot{\theta}^2dx\,. \qquad (12)$$

The second term in $U$ takes into account the shear effects and the second term in $T$ represents the rotary inertia effects. The Euler-Bernoulli beam elements consider only the first terms in these last two expressions.

## 3. INTEGRAL FORM OF THE BEAM EQUATIONS

Another idea in the Timoshenko beam approach is to consider the total lateral deflection of the beam $v$ as having two terms, $v_b$ due to bending and $v_s$, due to shear effects:

$$v=v_b+v_s \qquad (13)$$

According to (1) one have also: $\theta = dv_b/dx$ and $\gamma = dv_s/dx$. The shearing behavior of a straight non-uniform beam, having the length $L$ and loaded transversally by the distributed force $p(x)$, can be described by a differential equation:

$$\frac{d}{dx}\left(kAG\frac{dv_s}{dx}\right)=-p(x)\,, \qquad (14)$$

and based on the Green's functions approach takes the integral form:

$$v_s(x)=\int_0^L G_s(x,\xi)\,p(\xi)d\xi. \qquad (15)$$

In the case of the clamped-free beam, the Green's function values can be calculated as [20]:

$$G_s(x,\xi)=\int_0^{\min(x,\xi)}\frac{d\xi_1}{GkA(\xi_1)}\,. \qquad (16)$$

The differential equation governing the Saint Venant torsional behavior of a non-uniform beam, having the torsion constant $J$, the length $L$ and loaded by the distributed torsion moment $m_t(x)$, is:

$$\frac{d}{dx}\left(GJ\,\frac{d\phi}{dx}\right)+m_t(x)=0\,. \qquad (17)$$

It is similar with (14) and can be written in the integral form:

$$\phi(x)=\int_0^L G_t(x,\xi)m_t(\xi)d\xi\,. \qquad (18)$$

For the clamped-free beam case, the Green's function values are given by, [20]:

$$G_t(x,\xi)=\int_0^{\min(x,\xi)}\frac{d\xi_1}{GJ(\xi_1)}, \qquad (19)$$

representing the twist deflection angles $\phi(x,\xi)$ at distance $x$ due to a unit torsion moment applied at distance $\xi$ (Fig. 2-left). Replacing $GJ$ in (19) with $EI$, a Green's function $G_\theta$ representing the bending slope $\theta(x,\xi)$ is obtained (see also Fig. 2-middle). When no shear effects are taken into account, $v = v_b$ and the differential equation governing the bending behavior of a beam, having the moment of inertia $I$, the length $L$ and loaded by the distributed force $p(x)$, is:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = p(x). \tag{20}$$

which can take the integral form:

$$v(x) = \int_0^L G_v(x,\xi)\, p(\xi)\, d\xi. \tag{21}$$

The Green's function is in this case for the clamped-free beam, [20]:

$$G_v(x,\xi) = \int_0^{\min(x,\xi)} \frac{(x-\xi_1)(\xi-\xi_1)}{EI(\xi_1)} d\xi_1. \tag{22}$$

It represents the lateral deflection $v(x,\xi)$ at distance $x$ due to a force of unity applied at distance $\xi$ (Fig. 2-right). For uniform clamped-free beam, relation (22) reduces to the expression for the Green's function obtained in [25].

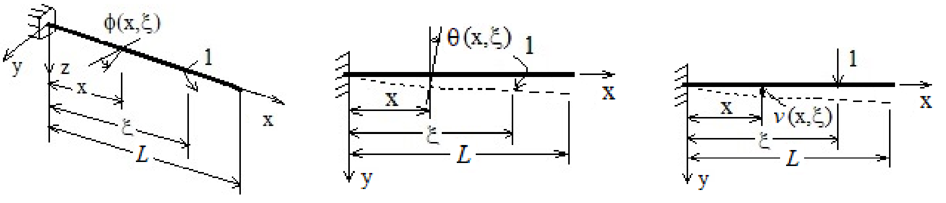


Fig. 2 – Physical significance of Green's functions

Using $n$ collocation points $\xi_i$ with $f_i = f(\xi_i)$ the value of an integral as those from relations (15), (18), (21) can be written as

$$\int_0^L f(\xi)d\xi = \sum_{i=1}^{n} f_i \cdot W_i, \tag{23}$$

with $W_i$ weighting numbers corresponding to the method of integration. Relation (15) gives the deflections $v_s$ for known distributed force $p(x)$, relation (18) gives the torsion deflection $\phi$ for applied distributed torsion moment $m_t(x)$ and relation (21) allows to obtain the bending deflection $v$ when the distributed force $p(x)$ is applied. These three relations take the followings matrix forms:

$$\{v_s\} = [G_s][W]\{p\}; \quad \{\phi\} = [G_t][W]\{m_t\}; \quad \{v\} = [G_v][W]\{p\}, \tag{24}$$

where one can notice the following $(n, n)$ matrices:

$[G_s]$ is a matrix containing the calculated influence coefficients $G_s(\xi_i,\xi_j)$

$[G_t]$ is a matrix containing the calculated influence coefficients $G_t(\xi_i,\xi_j)$

$[G_v]$ is a matrix containing the calculated influence coefficients $G_v(\xi_i,\xi_j)$

$[W]$ is a weighting matrix depending on the integration method (Simpson, here).

Other terms are column vectors containing the corresponding values (deflections and loads) in $n$ considered collocation points. The relation (6) has the form (14) with

$$p(x) = \rho A \omega^2 v\,, \tag{25}$$

and relation (8) is of the form (20) with $p(x)$ from above. The matrix forms for (6) and (8) are as follows:

$$\{v_s\} = \rho A \omega^2 [G_s][W]\{v\}, \tag{26}$$

$$\{v_b\} = \rho A \omega^2 [G_b][W]\{v\}. \tag{27}$$

In order to take into account the shear effects, the trick is to consider the distributed inertial forces in (25) as depending on total deflection $v$, and to obtain separately the components $v_s$ and $v_b$ by (26) and (27). Then, summing up these two relations, one obtain:

$$\{v\} = \rho A \omega^2 [[G_s] + [G_b]][W]\{v\} = \omega^2 [A_1]\{v\}\,. \tag{28}$$

The values of $\omega^2$ are eigenvalues of the matrix $[A_1]^{-1}$. Then the frequencies are $f = \omega/(2\pi)$. No rotary inertia effects are considered in such a formulation. Another idea is to eliminate $d\theta/dx$ between relations (6) and (7). In this manner one obtains:

$$EI\frac{d^4v}{dx^4} + \rho I \omega^2 \left(\frac{E}{kG} + 1\right)\frac{d^2v}{dx^2} - \rho A \omega^2 v + \frac{\rho^2 I}{kG}\omega^4 v = 0\,. \tag{29}$$

When the last term in the above equation is neglected, a truncated version( [26]) is obtained:

$$EI\frac{d^4v}{dx^4} = -\rho I \omega^2 \left(\frac{E}{kG} + 1\right)\frac{d^2v}{dx^2} + \rho A \omega^2 v\,. \tag{30}$$

It is similar with (8) and its integral form is:

$$\{v\} = -\rho I \omega^2 \left(\frac{E}{kG} + 1\right)[G_v][W][D_2]\{v\} + \rho A \omega^2 [G_v][W]\{v\}, \tag{31}$$

where the matrix $[D_2]$ is a differentiating matrix used to obtain values $d^2v/dx^2$. When the term $E/(kG)$ is neglected, $v = v_b$ and (31) can be used to estimate only the rotary inertia effects. Otherwise, the use of the matrix $[G_v]$ in (31) is another approximations as $v$ is in fact the total lateral deflection.

In a more general case, for the non-uniform beams the equations (6) and (7) become:

$$\frac{d}{dx}\left[kAG\left(\frac{dv}{dx} - \theta\right)\right] = -\rho A \omega^2 v\,, \tag{32}$$

$$\frac{d}{dx}\left(EI\frac{d\theta}{dx}\right) + kAG\left(\frac{dv}{dx} - \theta\right) + \rho I \omega^2 \theta = 0\,. \tag{33}$$

where,

$$v = v_b + v_s; \quad \theta = \frac{dv_b}{dx}; \quad \frac{dv_s}{dx} = \frac{dv}{dx} - \theta. \tag{34}$$

The matrix form for (32) is similar with (26) but the mass distribution is considered in the diagonal matrix $[\rho A]$ containing the values $m(x) = \rho A(x)$ in the collocation points:

$$\{v_s\} = \omega^2 [G_s][W][\rho A](\{v_s\} + \{v_b\}). \tag{35}$$

For (33) another idea is to consider this equation of the form (17) and to obtain the bending slope θ using $[G_\theta]$, a Green's function similar with (19) but with *EI* instead *GJ*. For a given number *n* of collocation points, the obtained integral form is:

$$\{\theta\} = [G_\theta][W][kAG][D_1]\{v_s\} + \omega^2 [G_\theta][W][\rho I]\{\theta\}, \tag{36}$$

where the matrix $[D_1]$ is a differentiating matrix used to obtain values $dv_s/dx$. The same differentiating matrix, based on central differences is used to replace in (36):

$$\{\theta\} = [D_1]\{v_b\}. \tag{37}$$

Then multiplying (36) with the inverse of $[D_1]$ one obtains:

$$\{v_b\} = [D_1]^{-1}[G_\theta][W][kAG][D_1]\{v_s\} + \omega^2 [D_1]^{-1}[G_\theta][W][\rho I][D_1]\{v_b\}. \tag{38}$$

In the above matrix relation, for the non-uniform beam, the diagonal matrices $[kAG]$ and $[\rho I]$ contain the corresponding values in the collocation points. Then, making the following notations:

$$[G_1] = [G_s][W][\rho A];\ [G_2] = [D_1]^{-1}[G_\theta][W][kAG][D_1];\ [G_3] = [D_1]^{-1}[G_\theta][W][\rho I][D_1], \tag{39}$$

the coupled relations (35) and (38) are written as:

$$\begin{Bmatrix} \{v_s\} \\ \{v_b\} \end{Bmatrix} = \begin{bmatrix} \omega^2[G_1] & \omega^2[G_1] \\ [G_2] & \omega^2[G_3] \end{bmatrix} \begin{Bmatrix} \{v_s\} \\ \{v_b\} \end{Bmatrix}. \tag{40}$$

It has the form:

$$\{z\} = [\omega^2 [A_2] + [B_2]]\{z\}, \tag{41}$$

where $\{z\}=[\{v_s\}\ \{v_b\}]^T$ and $[A_2]$, $[B_2]$ are $(2n, 2n)$ dimensional matrices. Relation (41) can be written in the form:

$$[[A_3] - \omega^2 [I_{2n}]]\{z\} = \{0\}. \tag{42}$$

with $[I_{2n}]$ the $2n$ dimensional unity matrix and

$$[A_3] = [A_2]^{-1}[[I_{2n}] - [B_2]]. \tag{43}$$

In this manner one can obtain the values of $\omega^2$ as eigenvalues of matrix $[A_3]$. The corresponding natural frequencies are then obtained with $f = \omega/(2\pi)$.

## 4. NUMERICAL EXAMPLES

As numerical application one consider two examples of cantilever beams having rectangular cross section (*b*x*h*) and the length *L* = 1m. The material characteristics are the following: *E* = 2.1e11 Pa, *G* = 8.1e10 Pa, ρ = 7860 kg/m$^3$. The first case is for a slender beam with *b* = 0.02 m, *h* = 1.5*b* = 0.03 m and the second case is for a stocky beam with *b* = 0.02 m, *h* = 4*b* = 0.08 m (Fig. 3). These examples was analyzed also in [27].

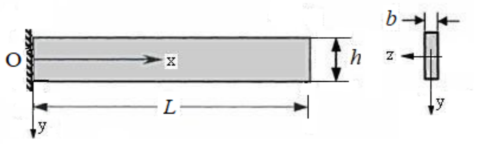


Fig. 3 – Clamped-free beam with rectangular cross-section

Table 1 shows the results comparison for the slender clamped-free beam obtained with relations (27, (28), (31) or (42).

The shear coefficient of the cross-section was taken $k$ = 5/6 and the number of collocation points was considered $n$ = 100.

Table 1 – Results for the slender beam in terms of ω

| $\omega$ | Analytic (9) Euler beam | $n$ = 100 (27) Euler beam | $n$ = 100 (31) with rotary inertia | $n$ = 100 (28) with shear effects | $n$ = 100 (31) reduced model | $n$ = 100 (42) complete model |
|---|---|---|---|---|---|---|
| $\omega_1$ | 157.37 | 157.39 | 157.39 | 157.30 | 157.31 | 157.26 |
| $\omega_2$ | 986.31 | 986.35 | 985.86 | 982.64 | 982.16 | 981.07 |
| $\omega_3$ | 2761.9 | 2761.8 | 2757.1 | 2737.2 | 2732.6 | 2727.5 |
| $\omega_4$ | 5412.5 | 5412.1 | 5392.2 | 5323.9 | 5305.2 | 5290.6 |
| $\omega_5$ | 8946.3 | 8946.5 | 8889.9 | 8717.1 | 8663.9 | 8634.3 |

Table 2 plots the same results in the case of the stocky clamped-free beam, obtained with the same number of collocation points $n$ = 100.

It can be seen that the influence of shear effects are bigger than those given by rotary inertia.

The impact is also larger for the stocky (deep) beam. The natural frequencies are smaller in the case of Timoshenko beam with respect to Euler beam, the differences increasing for higher natural modes.

Table 2 – Results for the deep beam in terms of ω

| $\omega$ | Analytic (9) Euler beam | $n$ = 100 (27) Euler beam | $n$ = 100 (31) with rotary inertia | $n$ = 100 (28) with shear effects | $n$ = 100 (31) reduced model | $n$ = 100 (42) complete model |
|---|---|---|---|---|---|---|
| $\omega_1$ | 419.66 | 419.7 | 419.8 | 418.1 | 420.1 | 417.55 |
| $\omega_2$ | 2630.17 | 2630.27 | 2620.99 | 2562.15 | 2592.65 | 2540.8 |
| $\omega_3$ | 7365.29 | 7364.84 | 7276.52 | 6933.56 | 7020.6 | 6813.8 |
| $\omega_4$ | 14433.3 | 14432.1 | 14067.5 | 12975.9 | 13087.4 | 12626.7 |
| $\omega_5$ | 23856.8 | 23857.3 | 22842.5 | 20324.1 | 20356.6 | 19605.6 |

In order to check our results obtained for Timoshenko beam (with the complete model), two finite element models were built in ANSYS APDL for the clamped-free deep beam case.

The modeling and calculations are performed using the experience from [28] and the FE code ANSYS [29].

One of them is a beam model and the other is a solid model (Fig. 4). Table 3 shows good agreement of results concerning the natural frequencies for the first 5 vibration natural modes.

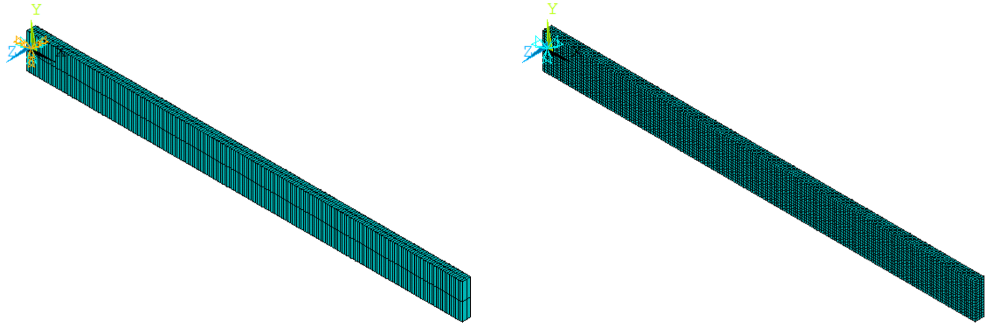

Fig. 4 – Beam FE model and solid FE model for the clamped-free deep beam

Table 3 – Results comparison with FEM for the deep beam in terms of ω

| Method | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|---|---|---|---|---|---|
| $n$ = 100, relation (42) | 417.55 | 2540.8 | 6813.8 | 12626.7 | 19605.6 |
| ANSYS, Beam 188, 100 elements | 417.6 | 2542.4 | 6822.3 | 12652.4 | 19661.3 |
| ANSYS, Solid 186, 12800 elements | 418.3 | 2547.6 | 6840.5 | 12697.6 | 19751.2 |

These results can be also obtained using collocation functions in order to represent better the vectors containing the values $\{\theta\}=\{dv_b/dx\}$ as presented in [21]. The bending and shear displacements $\{v_b\}$ and $\{v_s\}$ are written as:

$$v_b(x)=\sum_{k=1}^{p} C_{bk}\cdot f_{bk}(x); \quad v_s(x)=\sum_{k=1}^{p} C_{sk}\cdot f_{sk}(x), \tag{44}$$

where $f_{bk}(x)$, $f_{sk}(x)$ are $p$ known functions and $C_{bk}$, $C_{sk}$ are constant coefficients. For the $n$ collocation points, one obtains relations of the form:

$$\begin{aligned} &\{v_b\}=[F_b]\{C_b\}; \quad \{v_b^{'}\}=\{\theta\}=[F_{b1}]\{C_b\}; \quad \{v_b^{''}\}=[F_{b2}]\{C_b\}; \\ &\{v_s\}=[F_s]\{C_s\}; \quad \{v_s^{'}\}=[F_{s1}]\{C_s\}, \end{aligned} \tag{45}$$

where $[F_b]$, $[F_{b1}]$, $[F_{b2}]$, $[F_s]$, $[F_{s1}]$ are matrices of dimension $(n, p)$ containing the values $f_{bk}$, $f_{bk}^{'}$, $f_{bk}^{''}$, $f_{sk}$, $f_{sk}^{'}$, in the collocation points.

The equation (35) becomes:

$$[F_s]\{C_s\}=\omega^2[G_s][W][\rho A]([F_s]\{C_s\}+[F_b]\{C_b\}). \tag{46}$$

Multiplying at left with the transpose of the matrix $[F_s]$, one obtains:

$$[A_s]\{C_s\}=\omega^2[F_s]^t[G_s][W][\rho A]([F_s]\{C_s\}+[F_b]\{C_b\}), \tag{47}$$

where:

$$[A_s]=[F_s]^t[F_s], \tag{48}$$

is a $(p, p)$ dimension matrix. Then, multiplying the equation (47) at left with the inverse of the matrix $[A_s]$, we have:

$$\{C_s\}=\omega^2[A_s]^{-1}[F_s]^t[G_s][W][\rho A]([F_s]\{C_s\}+[F_b]\{C_b\}). \tag{49}$$

The equation (36) becomes:

$$[F_{b1}]\{C_b\}=[G_\theta][W][kAG][F_{s1}]\{C_s\}+\omega^2[G_\theta][W][\rho I][F_{b1}]\{C_b\}. \tag{50}$$

Multiplying at left with the transpose of the matrix [$F_{b1}$], one obtains:

$$[A_{b1}]\{C_b\}=[F_{b1}]^t[G_\theta][W][kAG][F_{s1}]\{C_s\}+\omega^2[F_{b1}]^t[G_\theta][W][\rho I][F_{b1}]\{C_b\}, \tag{51}$$

where:

$$[A_{b1}]=[F_{b1}]^t[F_{b1}], \tag{52}$$

is a ($p$, $p$) dimension matrix. Then, multiplying the equation (51) at left with the inverse of the matrix [$A_b$], the result is:

$$\{C_b\}=[A_{b1}]^{-1}[F_{b1}]^t[G_\theta][W][kAG][F_{s1}]\{C_s\}+\omega^2[A_{b1}]^{-1}[F_{b1}]^t[G_\theta][W][\rho I][F_{b1}]\{C_b\}. \tag{53}$$

Working now with the coupled equations (49) and (53), these can take the following form:

$$\begin{Bmatrix}\{C_s\}\\ \{C_b\}\end{Bmatrix}=\begin{bmatrix}\omega^2[G_4] & \omega^2[G_5]\\ [G_6] & \omega^2[G_7]\end{bmatrix}\begin{Bmatrix}\{C_s\}\\ \{C_b\}\end{Bmatrix}, \tag{54}$$

where the matrices [$G_4$] to [$G_7$] are products of several matrices obtained by terms identifications.
The last equation has the form:

$$\{C\}=[\omega^2[A_4]+[B_4]]\{C\}, \tag{55}$$

where $\{C\}=[\{C_s\}\ \{C_b\}]^T$ and [$A_4$], [$B_4$] are (2$p$, 2$p$) dimensional matrices. Relation (55) can be written as:

$$[[A_5]-\omega^2[I_{2p}]]\{C\}=\{0\}, \tag{56}$$

with [$I_{2p}$] the 2$p$ dimensional unity matrix and

$$[A_5]=[A_4]^{-1}[[I_{2p}]-[B_4]]. \tag{57}$$

In this manner one can obtain the values of $\omega^2$ as eigenvalues of matrix [$A_5$]. The dimension of the eigenvalue problem is reduced at 2$p$ instead 2$n$. For $v_b$ one can use the family of the mode shapes for transverse vibrations of uniform beams taken from [2], based on Krylov-Duncan functions:

$$S(x)=\frac{ch\,x+\cos x}{2},\quad T(x)=\frac{sh\,x+\sin x}{2}, \tag{58}$$

and:

$$U(x)=\frac{ch\,x-\cos x}{2},\quad V(x)=\frac{sh\,x-\sin x}{2}. \tag{59}$$

In the case of clamped-free beam the collocation functions are of the form:

$$f_{bk}(x) = T(\beta_k)\cdot U\left(\beta_k \frac{x}{L}\right) - S(\beta_k)\cdot V\left(\beta_k \frac{x}{L}\right), \tag{60}$$

where $\beta_k$ are those given by (10).

For $v_s$ the family of mode shapes for longitudinal vibrations of uniform fixed-free beam have been used, [2]:

$$f_{sk}(x) = \sin\left(\frac{2k-1}{2}\pi \frac{x}{L}\right). \tag{61}$$

The collocation functions can be used also for equation (27) which, for the case of non-uniform beams becomes:

$$[F_b]\{C_b\} = \omega^2 [G_b][W][\rho A][F_b]\{C_b\}. \tag{62}$$

Multiplying at left with the transpose of the matrix [$F_b$], the result is:

$$[A_b]\{C_b\} = \omega^2 [F_b]^t [G_b][W][\rho A][F_b]\{C_b\}, \tag{63}$$

where:

$$[A_b] = [F_b]^t [F_b] \tag{64}$$

is now a ($p$, $p$) dimension matrix.

Equation (63) is then multiplied at left with the inverse of the matrix [$A_b$], obtaining:

$$\{C_b\} = \omega^2 [A_b]^{-1} [F_b]^t [G_b][W][\rho A][F_b]\{C_b\} = \omega^2 [G]\{C_b\}. \tag{65}$$

The values of $\omega^2$ are eigenvalues of matrix $[G]^{-1}$. It is used for the Euler beam case. For an estimation of the shear effects one can replace matrix [$G_b$] in the above relation with the sum [$G_b$]+[$G_s$].

Other approximate results are obtained starting from (31) which can be written as follows, if using collocation functions for the uniform beams:

$$[F_b]\{C_b\} = -\omega^2 \rho I\left(\frac{E}{kG}+1\right)[G_b][W][F_{b2}]\{C_b\} + \omega^2 \rho A [G_b][W][F_b]\{C_b\}. \tag{66}$$

Multiplying at left with the transpose of the matrix [$F_b$], and then with the inverse of the matrix [$A_b$], it becomes:

$$\{C_b\} = \omega^2 [A_b]^{-1} [F_b]^t \left[-\rho I\left(\frac{E}{kG}+1\right)[G_b][W][F_{b2}] + \rho A [G_b][W][F_b]\right]\{C_b\}. \tag{67}$$

When $v = v_b$ and the term $E/(kG)$ is neglected, the above relation can be used to include only the rotary inertia effects.

Taking into account all terms, (67) provides an approximate solution for $\omega^2$ which are also the eigenvalues of the resulting matrix form (67).

Table 4 presents the results for the deep beam using the approach with $p$ = 10 collocation functions and $n$ = 100 collocation points.

Table 4 – Results for the deep beam in terms of ω using collocation functions

| $\omega$ | Analytic (9) Euler beam | $p$ = 10 (65) Euler beam | $p$ = 10 (67) with rotary inertia | $p$ = 10 (65) with shear effects | $p$ = 10 (67) reduced model | $p$ = 10 (56) complete model |
|---|---|---|---|---|---|---|
| $\omega_1$ | 419.66 | 419.71 | 419.8 | 418.1 | 420.1 | 417.58 |
| $\omega_2$ | 2630.17 | 2630.27 | 2620.98 | 2562.17 | 2592.61 | 2541.9 |
| $\omega_3$ | 7365.29 | 7364.84 | 7276.29 | 6933.81 | 7019.76 | 6819.1 |
| $\omega_4$ | 14433.3 | 14432.1 | 14065.8 | 12979.1 | 13081.8 | 12645.5 |
| $\omega_5$ | 23856.8 | 23857.3 | 22835.4 | 20334.5 | 20337.1 | 19639.9 |

The results are very close to those presented in Table 2 but now the dimensions of the eigenvalue problems were reduced from $n$ to $p$ or from $2n$ to $2p$, respectively.

Using collocation functions, the differentiation by using matrices $[D_1]$, $[D_2]$ is no more necessary.

The next example concerns a non-uniform cross-section beam. One considers a linear tapered cantilever beam having the dimensions according to [30]: length $L$ = 0.5 m, $h$ = 5 mm, $b_1$ = 75 mm, $b_n$ = 20 mm (see Fig. 5).

The material is steel with $E$ = 2.05e11Pa, ν = 0.3, ρ = 7850 kg/m$^3$. Experimental and numerical results concerning the first 5 natural frequencies for transverse (bending) vibration in the $xy$ plane, are available from [30].

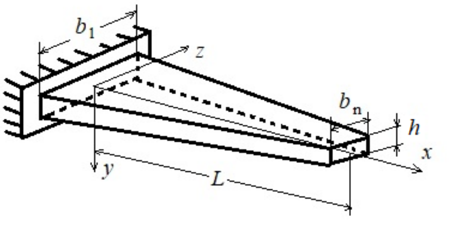


Fig. 5 – Tapered Timoshenko beam according to [30]

Table 5 shows a results comparison for this case including results from [30] and results obtained in ANSYS using SHELL 281 elements and SOLID 186 elements. The results obtained with the proposed approach use $n$ = 100 collocation points and $p$ = 10 collocation functions with $k$ = 5/6.

They are in good agreement with those obtained with FEM using ANSYS. The experimental values are slightly inferior as a rigid clamped end is difficult to obtain under laboratory conditions.

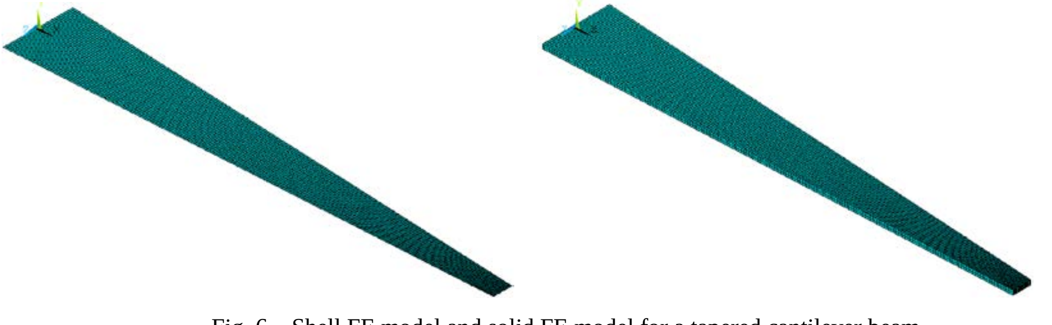

Fig. 6 – Shell FE model and solid FE model for a tapered cantilever beam

Table 5 – Results for the tapered beam in terms of $f = \omega/(2\pi)$ [Hz]

| Method | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ |
|---|---|---|---|---|---|
| Experimental, [30] | 21 | 114 | 294 | 565 | 923 |
| Numerical, [30] | 23.4 | 114.4 | 296.4 | 566.7 | 925.9 |
| ANSYS, Elm. Solid 186 | 23.9 | 117.9 | 306.3 | 586.7 | 959.9 |
| ANSYS, Elm. Shell 281 | 23.9 | 117.9 | 306.2 | 586.4 | 959.4 |
| Present, relation (42) $n$ = 100 | 23.9 | 117.7 | 305.5 | 584.2 | 953.9 |
| Present, rel.(56) $n$ = 100, $p$ = 10 | 23.7 | 116.5 | 302.3 | 578.8 | 945.5 |

## 5. CONCLUSIONS

This work addresses the natural frequencies calculation for transverse (bending) vibration of Euler and Timoshenko beams. A short review of a multitude of numerical methods used on this topic is presented. Then, an *integral method* based on the use of *flexibility influence functions* (Green's functions) is described and adapted for this type of analyses. These functions are computed here for the clamped-free beam case with relations from [20], using their physical significance of influence coefficients (displacements or rotation angles). The method described is a simple matrix one, using for numerical integration, *integration (weighting) matrices*. *Differentiating matrices* are also necessary. However, the differentiating matrices can be avoided by using *collocation functions* which meet the boundary conditions. The simple examples presented here show the viability of the proposed method, as good agreement of the obtained results with analytical or other numerical results can be seen. The presented approach is a simple integral one, easy to be used in matrix form. The good accuracy of the method from an engineering point of view depends on the number of collocation points and functions.

In this type of approach, the boundary conditions are included in the formulation by using appropriate Green's functions. The use of Timoshenko beam model is important in order to obtain accurate results, especially for high vibration modes, for short beams and for stocky (deep) beams. The beam equations are written usually in terms of total lateral deflection and bending slope. In the presented approach, the same equations are written in terms of components $v_b$ (due to bending) and $v_s$ (due to shearing) of the total lateral deflection $v$. It is an alternative way used also in other works dedicated to Timoshenko beam studies.

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