Description of Mechanical Movements Using Fractional Order Derivatives is not Objective

Agneta M. BALINT[1], Stefan BALINT*[,2]

*Corresponding author
[1]Department of Physics, West University of Timisoara,
Blvd. V. Parvan 4, 300223 Timisoara, Romania,
agneta.balint@e-uvt.ro
*[,2]Department of Computer Science, West University of Timisoara,
Blvd. V. Parvan 4, 300223 Timisoara, Romania,
stefan.balint@e-uvt.ro

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Abstract: *The papers deals with the objectivity in science. It aims at objectively describing the material particle movement and the movement of a continuum body. This is followed by the Riemann-Liouville, Caputo and Cherbirt fractional order derivatives presentation. In the next part the loss of description objectivity due to the use of fractional order derivatives is presented. Finally, some recent papers proposing the use of the fractional order derivatives instead of the integer order derivatives in the description of some physical phenomena are cited.*

Key Words*: objectivity of a mathematical description; mechanical movement description; fractional order derivative*

1. OBJECTIVITY IN SCIENCE

The concept of objectivity in science means that qualitative and quantitative descriptions of phenomena remain unchanged when the phenomena are observed by different observers in other words, it is possible to reconcile observations of the process into a single coherent description. Galileo Galilee (1564-1642) said: "The mechanical event is independent from the observer. For frames moving uniformly with respect to each other, both states are mechanically equivalent".

Isaac Newton (1643- 1727) said: "The mechanical event is independent from the observer. This holds also for accelerated systems if the frames of references are fixed with respect to absolute space (with respect to the fixed stars)".

Albert Einstein (1879-1955) said: "The mechanical event is independent from the observer. There is no special reference point. The same holds for accelerated systems (general relativity)".

"Even more, if the theory is subjected to relativity, it should be generally covariant under all transformations, not just rigid body motions" [1].

2. OBJECTIVITY OF THE CLASSICAL DESCRIPTION OF THE A MATERIAL PARTICLE MOVEMENT

In the following, we define the objectivity of the classical description of a material particle movement [2].

2.1. Objectivity of the movement description

In classical mechanics an observer O represents a material particle from a point P called a material point in the three-dimensional affine Euclidian space E_3. To describe the movement of the material point, observer O chooses a fixed orthogonal reference frame $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$ in the space E_3, a moment of time M_O for fixing the origin of the time measurement and a unit [second] for the time measurement. A moment of time M which is earlier than M_O is represented by a negative real number $t_M < 0$, a moment of time M which is later than M_O is represented by a positive real number $t_M > 0$ and the moment of time M_O is represented by the real number $t_{M_O} = 0$.

At any moment of time M, represented by t_M, the observer considers the coordinate $(X_1(t_M), X_2(t_M), X_3(t_M))$ of the material point at the moment of time t_M with respect to the reference frame R_O, and describes the movement of the material point with the set of the real functions $X_1(t_M), X_2(t_M), X_3(t_M)$.

A second observer O^* uses a similar procedure and describes the same movement of the material point with the set of the real functions $X_1^*(t_M^*), X_2^*(t_M^*), X_3^*(t_M^*)$, representing the coordinates of the material point with respect to a second fixed reference frame $R_{O^*} = (O^*;\ \vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*)$. For the observer O^* the origin of the time measurement is M_{O^*} and the unit is [second]; a moment of time M which is earlier than M_{O^*} is represented by a negative number $t_M^* < 0$, the moment of time M_{O^*} is represented by the real number $t_{M_{O^*}}^* = 0$ and a moment of time M which is later than M_{O^*} is represented by a positive real number $t_M^* > 0$.

In case of the observer O, a moment of time M is described by the real number t_M and in case of the observer O^* by the real number t_M^*. For numbers t_M and t_M^* the following relations hold:

$$t_M = t_M^* + t_{M_{O^*}}, \tag{1}$$

$$t_M^* = t_M + t_{M_O}^*, \tag{2}$$

In the above relations $t_{M_{O^*}}$ is the real number which represents the moment M_{O^*} in time measuring system of observer O and $t_{M_O}^*$ is the real number which represents the moment of time M_O in time measuring system of observer O*.

For any moment of time M, the coordinates $(X_1(t_M), X_2(t_M), X_3(t_M))$ in R_O and $(X_1^*(t_M^*), X_2^*(t_M^*), X_3^*(t_M^*))$ represent points in the three dimensional affine Euclidian space E_3. These points have to coincide with the material point position at the moment of time M. Therefore, for the coordinates the following relations hold:

$$\begin{gathered} X_k(t_M) = X_{kO^*} + a_{1k}X_1^*(t_M^*) + a_{2k}X_2^*(t_M^*) + a_{3k}X_3^*(t_M^*) \quad k = 1,2,3 \\ t_M = t_M^* + t_{M_{O^*}} \end{gathered} \tag{3}$$

Or equivalently

$$\begin{gathered} X_k^*(t_M^*) = X_{kO}^* + a_{k1}X_1(t_M) + a_{k2}X_2(t_M) + a_{k3}X_3(t_M) \quad k = 1,2,3 \\ t_M^* = t_M + t_{M_O}^* \end{gathered} \tag{4}$$

The significance of the quantities appearing in the above relations are:
$a_{ij} = \langle \vec{e}_i^*, \vec{e}_j \rangle =$ constant = scalar product of the unit vectors $\vec{e}_i^*$ and $\vec{e}_j$ in E_3
$(X_{1O^*}, X_{2O^*}, X_{3O^*})$ are the coordinates of the point O^* in the reference frame R_O, $(X_{1O}^*, X_{2O}^*, X_{3O}^*)$ are the coordinates of the point O in the reference frame R_{O^*}.
Relations (3) or (4) reconcile the description made by the two observers and make possible the description of the movement by the set of functions $X_1(t_M), X_2(t_M), X_3(t_M)$ or by the set of functions $X_1^*(t_M^*), X_2^*(t_M^*), X_3^*(t_M^*)$. This means that this kind of the movement description is objective.

2.2 Objectivity of the material point velocity description

In the case of the above type of movement description, the velocity $\vec{V}_M$ at the moment *M* of the material particle description is objective.
In order to see what this means exactly, remember that in the description of *O* velocity $\vec{V}_M$ at the moment *M* of the material particle is the vector in $\boldsymbol{E_3}$ obtained by translating the vector $\vec{X}'(t_M) = X'_1(t_M)\vec{e}_1 + X'_2(t_M)\vec{e}_2 + X'_3(t_M)\vec{e}_3$ in the point of coordinates $(X_1(t_M), X_2(t_M), X_3(t_M))$. Here $X'_k(t_M)$ is the first order derivative of the function $X_k(t_M)$ at t_M for $k = 1,2,3$.
In the description of O^* velocity $\vec{V}_M^*$ at the moment *M* of the material particle is the vector in $\boldsymbol{E_3}$ obtained by translating the vector $\vec{X}^{*\prime}(t_M^*) = X_1^{*\prime}(t_M^*)\vec{e}_1^* + X_2^{*\prime}(t_M^*)\vec{e}_2^* + X_3^{*\prime}(t_M^*)\vec{e}_3^*$ in the point of coordinates $(X_1^*(t_M^*), X_2^*(t_M^*), X_3^*(t_M^*))$. Here $X_k^{*\prime}(t_M^*)$ is the first order derivative of the function $X_1^*(t_M^*)$at t_M^* for$k = 1,2,3$.
Objectivity of the velocity description means that the velocity of the material point is independent of the observer i.e. $\vec{V}_M = \vec{V}_M^*$. Equality $\vec{V}_M = \vec{V}_M^*$ can be obtained by differentiating in (3) and obtaining:

$$X'_k(t_M) = a_{1k}X_1^{*\prime}(t_M^*) + a_{2k}X_2^{*\prime}(t_M^*) + a_{3k}X_3^{*\prime}(t_M^*) \quad k = 1,2,3 \tag{5}$$

Equalities (5) shows that equality $\vec{V}_M = \vec{V}*_M$ holds.

2.3 Objectivity of the material point acceleration description

By using the observer *O* description, the acceleration $\vec{A}_M$ at the moment of time *M* of the material point is the vector in $\boldsymbol{E_3}$ obtained by translating the vector $\vec{X}''(t_M) = X''_1(t_M)\vec{e}_1 + X''_2(t_M)\bar{e}_2 + X''_3(t_M)\vec{e}_3$ in the point of coordinates$(X_1(t_M), X_2(t_M), X_3(t_M))$.
Here $X''_k(t_M)$ is the second order derivative of the function $X_k(t_M)$ at t_M for $k = 1,2,3$. Using observer O^* description, the acceleration $\vec{A}_M^*$ at the moment of time *M* of the material point is defined similarly. Objectivity of the acceleration description means that the acceleration is independent of the observer i.e. $\vec{A}_M = \vec{A}_M^*$. Equality $\vec{A}_M = \vec{A}_M^*$ can be obtained by differentiating twice in (3); thus we get:

$$X''_k(t_M) = a_{1k}X_1^{*\prime\prime}(t_M^*) + a_{2k}X_2^{*\prime\prime}(t_M^*) + a_{3k}X_3^{*\prime\prime}(t_M^*) \qquad k = 1,2,3 \tag{6}$$

Equalities (6) show that equality $\vec{A}_M = \vec{A}_M^*$ holds.

2.4 Objectivity of the material point dynamics description

The verbal expression of the second law of Newton is “the rate of change of momentum is proportional to the impressed force and takes place in the direction of the straight line in which the force acts” [2]. In terms of the observer *O* description the above verbal expression leads to

the conclusion that functions $X_1(t_M), X_2(t_M), X_3(t_M)$, describing the motion, verify the following system of differential equations:

$$mX''_k(t_M) = F_{kO}\big(t_M, X_1(t_M), X_2(t_M), X_3(t_M), X'_1(t_M), X'_2(t_M), X'_3(t_M)\big) \;\; k = 1,2,3, \quad (7)$$

Here: m represents the material point mass, $X_k(t_M)$, $k = 1,2,3$ represent the coordinates of position of the material point in R_O, $X'_k(t_M)$, $k = 1,2,3$ represent the velocity components of the material point in R_O, $X''_k(t_M)$, $k = 1,2,3$ represent the acceleration components of the material point and F_{kO}, $k = 1,2,3$ represents the components (in the reference frame R_O) of the force field **F** acting on the material point. In terms of the observer O^* description, the same verbal expression, leads to the conclusion that the functions $X_1^*(t_M^*), X_2^*(t_M^*), X_3^*(t_M^*)$ Xdescribing the motion under the action of the same field **F**, in the reference frame R_{O^*}, verifies the following system of differential equations:

$$\begin{gathered} m \cdot X_1^{*}{}''(t_M^*) = \\ F_{kO^*}^*(t_M^*, X_1^*(t_M^*), X_2^*(t_M^*), X_3^*(t_M^*), X_1^{*}{}'(t_M^*), X_2^{*}{}'(t_M^*), X_3^{*}{}'(t_M^*)) \quad k = 1,2,3 \end{gathered} \quad (8)$$

Note that in (8) $F_{kO^*}^*$ represents the components of the same force field **F** with respect to R_{O^*} and verifies:

$$F_{kO^*}^* = a_{k1}F_{1O} + a_{k2}F_{2O} + a_{k3}F_{3O} \quad k = 1,2,3 \quad (9)$$

Objectivity of the dynamics description means that the solutions of the systems of differential equations (7) and (8) describes the same movement of the material point under the action of the force **F** field. This can be proven showing that if $X_1(t_M), X_2(t_M), X_3(t_M)$ is a solution of (7), then the functions $X_1^*(t_M^*), X_2^*(t_M^*), X_3^*(t_M^*)$, given by (4), is a solution of (8) and vice versa. In other words, the dynamics of the material point can be described by system (7) or by system (8). This means that the dynamic description is independent of the observer.

3. OBJECTIVITY IN CLASSICAL CONTINUUM MECHANICS

3.1 Objectivity of the movement description in classical elasticity

To describe the movement of a continuous material body *B* in the classical theory of elasticity [3], observer *O* choose a fixed orthogonal reference frame $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$ in the affine euclidian space E_3, a moment of time M_O for fixing the origin of the time measurement and a unit [second] for time measuring. At a moment of time *M* a particle *P* of the material body *B* is represented by observer *O* as a material point P_M^O in E_3. So, at a moment of time *M* the material body is represented by a subset P_M^O of material points P_M^O of E_3. Observer *O* describes the movement of the material body *B* describing the movement in E_3 of each material point $P_{M_O}^O$ from $S_{M_O}^O$. This description is made by observer *O* with functions of the form:

$$Y_k = Y_k(t, X_1, X_2, X_3) \quad k = 1,2,3 \text{ and } (X_1, X_2, X_3) \in S_{M_O}^O, \quad (10)$$

where: (X_1, X_2, X_3) are the coordinates with respect to R_O of a material point $P_{M_O}^O$, representing the particle *P* of the material body *B* from $S_{M_O}^O$ at the moment M_O, i.e. $t = t_{M_O} = 0$; $(Y_1(t, X_1, X_2, X_3), Y_2(t, X_1, X_2, X_3), Y_3(t, X_1, X_2, X_3))$ are the coordinates with respect to R_O of the material point P_M^O, representing the same particle *P* of the material body *B*, at the moment of time *M* i.e. $t = t_M$. An obvious property of the functions appearing in the description (10) is that they verify equalities $Y_k(0, X_1, X_2, X_3) = X_k$ for $k = 1,2,3$ and any $(X_1, X_2, X_3) \in S_{M_O}^O$.

In a description of the form (10) it is assumed that for any fixed t the function $Y_t(X)$ defined as

$$(X_1, X_2, X_3) \to (Y_1(t, X_1, X_2, X_3), Y_2(t, X_1, X_2, X_3), Y_3(t, X_1, X_2, X_3)) \tag{11}$$

is bijective from $S_{M_O}^O$ to S_M^O also, that functions $Y_k(t, X_1, X_2, X_3)$ are continuously differentiable for $k = 1,2,3$ and Jacobi matrix of function $Y_t(X)$ is nonsingular.
Moreover, it is assumed that the inverse function $Y_t^{-1}(Y_t)$ defined by

$$(Y_1(t, X_1, X_2, X_3), Y_2(t, X_1, X_2, X_3), Y_3(t, X_1, X_2, X_3)) \to (X_1, X_2, X_3) \tag{12}$$

has the same properties.
Observer O^* describes the same movement of the same material body B with formulas

$$Y_k^* = Y_k^*(t^*, X_1^*, X_2^*, X_3^*) \text{ for } k = 1,2,3 \text{ and } (X_1^*, X_2^*, X_3^*) \in S_{M_{O^*}}^{O^*} \tag{13}$$

where: (X_1^*, X_2^*, X_3^*) are the coordinates with respect to R_{O^*} of a material point $S_{M_{O^*}}^{O^*}$ representing the same particle P of the material body B from $S_{M_{O^*}}^{O^*}$, at the moment of time M_{O*} i.e. $t^* = t_{M_{O^*}}^* = 0$ and $\left(Y_1^*(t^*, X_1^*, X_2^*, X_3^*),\ Y_2^*(t^*, X_1^*, X_2^*, X_3^*),\ Y_3^*(t^*, X_1^*, X_2^*, X_3^*)\right)$ are the coordinates with respect to R_{O^*} of the material point $P_M^{O^*}$, representing the same particle P of the material body B, at the moment of time M, i.e. $t^* = t_M^*$.
An obvious property of the functions appearing in description (13) is that they verify $Y_1^*(0, X_1^*, X_2^*, X_3^*) = X_k^*$ for $k = 1,2,3$ and any $(X_1^*, X_2^*, X_3^*) \in S_{M_{O^*}}^{O^*}$.
In description (13) it is assumed that for a fixed t^* the function $Y_{t*}(X^*)$ defined as

$$(X_1^*, X_2^*, X_3^*) \to \left(Y_1^*(t^*, X_1^*, X_2^*, X_3^*), Y_2^*(t^*, X_1^*, X_2^*, X_3^*), Y_3^*(t^*, X_1^*, X_2^*, X_3^*)\right) \tag{14}$$

is bijective from $S_{M_{O^*}}^{O^*}$ to $S_M^{O^*}$, also that functions $Y_k^*(t^*, X_1^*, X_2^*, X_3^*)$ are continuously differentiable for $k = 1,2,3$ and Jacobi matrix of function $Y_{t^*}^*(X^*)$ is nonsingular. Moreover, it is assumed that the inverse function ${Y_{t^*}^*}^{-1}$ $(Y_{t^*}^*)$ defined by

$$\left(Y_1^*(t^*, X_1^*, X_2^*, X_3^*), Y_2^*(t^*, X_1^*, X_2^*, X_3^*), Y_3^*(t^*, X_1^*, X_2^*, X_3^*)\right) \to (X_1^*, X_2^*, X_3^*) \tag{15}$$

has the same properties.
The relations that reconcile description (10) and (13) and make possible the description of the movement by one of them, are the followings:

$$X_k^* = X_{k0}^* + a_{k1}X_1 + a_{k2}X_2 + a_{k3}X_3 \text{ for } k = 1,2,3 \tag{16}$$

$$Y_k^*(t^*, X_1^*, X_2^*, X_3^*) = X_{k0}^* + a_{k1} \cdot Y_1(t, X_1, X_2, X_3) + a_{k2} \cdot Y_2(t, X_1, X_2, X_3) + a_{k3} \cdot Y_3(t, X_1, X_2, X_3) \text{ for } k = 1,2,3 \tag{17}$$

$$t^* = t + t_{M_O}^*;\ t^* = t_M^*;\ t = t_M; \tag{18}$$

In other words, if: $(X_1^*, X_2^*, X_3^*) \in S_{M_{O^*}}^{O^*}$ and $(X_1, X_2, X_3) \in S_{M_O}^O$ verify (16); the coordinates $Y_k^*(t^*, X_1^*, X_2^*, X_3^*)$, $k = 1,2,3$ (appearing in (13)) and the coordinates $Y_k(t, X_1, X_2, X_3)$, $k = 1,2,3$ appearing in (10) verify (17), and $t^* = t + t_{M_O}^*$; $t^* = t_M^*$; $t = t_M$ verify (18), then the two observers describe the same movement of the body. This means that this kind of description of the movement is objective.

3.2 Objectivity of the displacement description and that of the deformation description

In the theory of elasticity observer O describes the displacement of the particle P of the material body B at the moment of time M by a vector valued function $\vec{D}_M(P)$ of variables M, P. For this purpose the material point P_M^O, representing the particle P of the material body B at the moment of time M is considered and the vector

$$\vec{U} = \sum_{j=1}^{3} (Y_j(t, X_1, X_2, X_3) - X_j) \cdot \vec{e}_j \tag{19}$$

is translated in E_3 into the material point P_M^O. The vector obtained in this way is denoted by $\vec{D}_M(P)$. In (19) $t = t_M$ and (X_1, X_2, X_3) are the coordinates of the material point P_M^O with respect to $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$. Observer O^* describes the displacement of the same particle P of the material body B at the moment of time M by a vector valued function $\vec{D}_M^*(P)$ of variables M, P. For this purpose the material point $P_M^{O^*}$, representing the same particle P of the material body B at the moment of time M is considered and the vector

$$\vec{U}^* = \sum_{j=1}^{3} \left(Y_j^*(t^*, X_1^*, X_2^*, X_3^*) - X_j^*\right) \cdot \vec{e}_j^* \tag{20}$$

is translated in E_3 into the material point $P_M^{O^*}$. The vector obtained in this way is denoted by $\vec{D}_M^*(P)$. In (20) $t^* = t_M^*$ and (X_1^*, X_2^*, X_3^*) are the coordinates of the material point $P_M^{O^*}$, with respect to $R_{O^*} = (O^*; \vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*)$. The description of the displacement is objective if and only if the equality$\vec{D}_M(P) = \vec{D}_M^*(P)$ holds. This equality can be proven using formulas (16) - (20).
In the theory of elasticity observer O describes the deformation of the material body B at the particle P at the moment of time M by a tensor valued function $\Gamma_M(P)$ of variables M, P (the Cauchy deformation tensor). For this purpose the material point P_M^O, representing the particle P of the material body B at the moment of time M is considered, and the tensor which components with respect to $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$ translated into P_M^O are given by

$$\gamma_{jk} = \frac{1}{2} \cdot \left(\frac{\partial U_j}{\partial X_k} + \frac{\partial U_k}{\partial X_j} + \sum_{i=1}^{3} \frac{\partial U_i}{\partial X_k} \cdot \frac{\partial U_i}{\partial X_j} \right) \tag{21}$$

is considered.
In (21) $t = t_M$ and (X_1, X_2, X_3) are the coordinates of the material point P_M^O with respect to $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$. The tensor obtained in this way is denoted by $\Gamma_M(P)$ and is called the Cauchy deformation tensor in P at the moment of time M.
Observer O^* describes the deformation of the material body B at the particle P at the moment of time M by a tensor valued function $\Gamma_M^*(P)$ of variables M, P (the Cauchy deformation tensor). For this purpose the material point $P_M^{O^*}$, representing the particle P of the material body B at the moment of time M is considered, and the tensor which components with respect to $R_{O^*} = (O^*; \vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*)$ translated into $P_M^{O^*}$, given by

$$\gamma_{jk}^* = \frac{1}{2} \cdot \left(\frac{\partial U_j^*}{\partial X_k^*} + \frac{\partial U_k^*}{\partial X_j^*} + \sum_{i=1}^{3} \frac{\partial U_i^*}{\partial X_k^*} \cdot \frac{\partial U_i^*}{\partial X_j^*} \right) \tag{22}$$

is considered.

In (22) $t^* = t_M^*$ and (X_1^*, X_2^*, X_3^*) are the coordinates of the material point $P_M^{O^*}$, with respect to $R_{O^*} = (O^*; \vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*)$. The tensor obtained in this way is denoted by $\Gamma_M^*(P)$ and is called the Cauchy deformation tensor in *P* at moment of time *M*. The description of the deformation is objective if and only if the equality $\Gamma_M(P) = \Gamma_M^*(P)$ holds. This equality can be proven using formulas (16) - (20) and relations

$$\gamma_{jk}^* = \sum_{r=1}^{3}\sum_{q=1}^{3} \gamma_{rq} a_{jr} a_{kq} \tag{23}$$

3.3 Objectivity of the small deformation tensor

In case of small displacements, the deformation tensor components $\Gamma_M(P)$ in the description of *O*, can be approximated by:

$$\varepsilon_{jk} = \frac{1}{2} \cdot \left(\frac{\partial U_j}{\partial X_k} + \frac{\partial U_k}{\partial X_j} \right) \tag{24}$$

and the deformation tensor components $\Gamma_M^*(P)$ in description of *O** can be approximated by

$$\varepsilon_{jk}^* = \frac{1}{2} \cdot \left(\frac{\partial U_j^*}{\partial X_k^*} + \frac{\partial U_k^*}{\partial X_j^*} \right) \tag{25}$$

The components ε_{jk} (obtained using observer *O* description) are related to components ε_{jk}^* (obtained using observer O^* description) by relations

$$\varepsilon_{jk}^* = \sum_{r=1}^{3}\sum_{q=1}^{3} \varepsilon_{rq} a_{jr} a_{kq} \tag{26}$$

Relations (26) show that in case of small displacements the deformation description by the tensor $E_M(P)$,called the Cauchy small deformation tensor of the body in *P* at the moment of time *M*, which components in $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$ translated into P_M^O are given by (24) is objective.

3.4 Objectivity of the stress description

In the description of observer *O* in a material body *B* the surface force acting per unit area of a surface passing through the particle *P* at a moment of time *M*, is a vector valued function, of the form [3]:

$$\vec{T} = \vec{T}(t, X_1, X_2, X_3; \vec{n}) \tag{27}$$

Here: $\vec{T}$ represents the force at the moment of time *M* acting on the unit surface which passes through the material point P_M^O representing the particle *P* of the material body *B* at the moment of time *M*; (X_1, X_2, X_3) are the coordinates of the material point $P_{M_O}^O$ representing the particle *P* of the material body *B* at the moment of time M_O with respect to $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$; $t = t_M$; $\vec{n}$ is the unit normal to the surface at the material point P_M^O. The dependence of $\vec{T}(t, X_1, X_2, X_3; \vec{n})$ on $\vec{n}$ is given by [3]:

$$\vec{T}(t, X_1, X_2, X_3; \vec{n}) = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \sigma_{ij}(t, X_1, X_2, X_3) \cdot n_j \right) \vec{e}_i \tag{28}$$

The coefficients $\sigma_{ij}(t, X_1, X_2, X_3)$ $i, j = 1,2,3$ are the components of the Cauchy stress tensor with respect to $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$ translated at P_M^O and n_j are the components of the unit normal $\vec{n}$ with respect to R_O. In the description of observer O^* in a material body *B* the surface force acting per unit area of a surface passing through the particle *P* at a moment of time *M*, is a vector valued function, of the form [3]

$$\vec{T}^* = \vec{T}^*(t^*, X_1^*, X_2^*, X_3^*; \vec{n}^*) \tag{29}$$

Here: $\vec{T}^*$ represents the force at the moment of time *M* acting on the unit surface which passes through the material point $P_M^{O^*}$ representing the particle *P* of the material body *B* at the moment of time *M*; (X_1^*, X_2^*, X_3^*) are the coordinates of the material point $P_{M_{O^*}}^{O^*}$ representing the particle *P* of the material body *B* at the moment of time M_{o*} with respect to $R_{O^*} = (O^*; \vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*)$; $t^* = t_M^*$; $\vec{n}^*$ is the unit normal to the surface at the material point $P_M^{O^*}$. The dependence of $\vec{T}^*(t^*, X_1^*, X_2^*, X_3^*; \vec{n}^*)$ on $\vec{n}^*$ is given by:

$$\vec{T}^*(t^*, X_1^*, X_2^*, X_3^*; \vec{n}^*) = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} \sigma_{ij}^*(t^*, X_1^*, X_2^*, X_3^*) \cdot n_j^*\right)\vec{e}_i^* \tag{30}$$

The coefficients $\sigma_{ij}^*(t^*, X_1^*, X_2^*, X_3^*)$ $i, j = 1,2,3$ are the coefficients of the Cauchy stress tensor with respect to $R_{O^*} = (O^*; \vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*)$; translated at $P_M^{O^*}$ and n_j^* are the components of the unit normal $\vec{n}^*$ with respect to R_{O^*}. The coefficients σ_{jk} of the Cauchy stress tensor, obtained using the observer *O* description, are related to the coefficients σ_{ij}^* of Cauchy stress tensor, obtained using the observer O^* description, by relations

$$\sigma_{ij}^* = \sum_{r=1}^{3}\sum_{q=1}^{3} \sigma_{rq} a_{jr} a_{kq} \tag{31}$$

Relations (31) reconcile the meaning of concepts of Cauchy stress components in the description made by the two observers and show that this kind of the stress description is objective.

3.5 Objectivity of the Hooke constitutive law description

In case of a homogeneous and isotropic material body *B*, the law oh Hooke concerning small deformations, regarding observer *O*, according to [3], is described by the following relations:

$$\sigma_{ij}(t, X_1, X_2, X_3) = \lambda \cdot \theta \cdot \delta_{ij} + 2\mu \cdot \varepsilon_{ij} \tag{32}$$

where λ and μ are the Lame constants δ_{ij} are the Kronecker coefficients and $\theta = \sum_{i=1}^{3} \varepsilon_{ii}$.
In case of the same material body *B*, the law oh Hooke concerning small deformations, in terms of the observer O^*, according to [3], is described by the following relations:

$$\sigma_{ij}^*(t^*, X_1^*, X_2^*, X_3^*) = \lambda \cdot \theta^* \cdot \delta_{ij} + 2\mu \cdot \varepsilon_{ij}^* \tag{33}$$

Relations (16) - (20), (25) and (31) reconcile descriptions (32) and (33) showing that the Hooke constitutive law description is objective. Therefore, equation (32) is independent of the observer (it is objective) and is called the constitutive law of Hooke in case of small deformations.

4. CAPUTO, RIEMANN-LIOUVILLE AND THE CHERBIRT FRACTIONAL ORDER DERIVATIVES

The Caputo fractional order derivative was introduced by M. Caputo in 1967 [4]. According to [5] for a continuously differentiable function $f: [0, \infty) \to R$ the Caputo fractional derivative of orderα, $(0 < \alpha < 1)$ is defined by:

$$D_C{}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau \tag{34}$$

where Γ is the Euler gamma function. For a continuous function $f: [0, \infty) \to R$ the Riemann-Liouville fractional derivative of order α,$(0 < \alpha < 1)$, according to [5], is defined by:

$$D_{R-L}{}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau \tag{35}$$

The Cherbit fractional order derivative was introduced by G. Cherbit in [6].
According to [7] a function $f: (-\infty, \infty) \to R$ is Cherbit differentiable of orderα, $(0 < \alpha < 1)$ at t if at least one of the limits

$$lim \frac{f(t \pm \varepsilon) - f(t)}{\varepsilon^{\alpha}} \tag{36}$$

exist finitely for ε tends to 0. The finite limit(s) is called the Cherbit fractional derivative of orderα.

5. OBJECTIVITY LOST

According to [7] Cherbit in [6] and later Ben Ada and Creson in [8] introduced the Cherbit fractional order derivative (36) calling, fractional velocity „for the study of fractal phenomena and physical processes for which the instantaneous velocity was not well defined”.
In the context of objectivity, discussed in the present paper, **we want to show that the objectivity is lost when fractional order velocity is defined by using Riemann-Liouville or Caputo fractional order derivatives.**

5.1 Velocity description with Riemann-Liouville fractional order derivatives is not objective.

For that we consider a material point *P* which moves on a line *L* and the movement is observed by two observers *O* and O^* placed in two different geometrical points *O* and O^* on the line. The reference frame of *O* is $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$, and that of O^* is $R_{O^*} = (O^*; \vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*)$. Assume that the support of the vectors $\vec{e}_1$ and $\vec{e}_1^*$; is the line *L* and $a_{11} =< \vec{e}_1^*, \vec{e}_1 >= 1$. Each observer has his own chronometer which is started at the moment of time M_O, and M_{O^*} when the material point *P* passes for the first time in front of observer *O* and O^* respectively. In case of observer *O* a moment of time *M* is described by the real number t_M and in case of observer O^* by the real number t_M^*. Remember that for the numbers t_M and t_M^* the following relations hold:

$$t_M = t_M^* + t_{M_{O^*}} \qquad\qquad t_M^* = t_M + t_{M_O}^*$$

In the above relations $t_{M_{O^*}}$ is the real number which represent the moment M_{O^*} in the system of time measuring of observer *O*, and $t_{M_O}^*$ is the real number which represent the moment M_O

in the system of time measuring of observer O^*. Assume that $t^*_{M_O} < 0 < t_{M_{O*}} < t_M$. For the coordinates of the material point P in the representation of observers O and O^* the following equalities hold:

$X_1(t_M) = X_{1O^*} + X_1^*(t_M^*)$; $X_2(t_M) = 0$; $X_3(t_M) = 0$; and $X_1^*(t_M^*) = X_{1O}^* + X_1(t_M)$; $X_2^*(t_M^*) = 0$; $X_3^*(t_M^*) = 0$.

Using the observer O description, the first component of the velocity computed with the Riemann-Liouville fractional order derivativeα, $(0 < \alpha < 1)$ is

$$D_{R-L}{}^{\alpha} X_1(t_M) = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{d}{dt_M} \int_0^{t_M} \frac{X_1(\tau)}{(t_M - \tau)^{\alpha}} d\tau$$

while the second and the third components are equal to zero.
Using the observer O^* description, the first component of velocity computed with the Riemann-Liouville fractional order derivativeα, $(0 < \alpha < 1)$ is

$$D_{R-L}{}^{\alpha} X_1^*(t_M^*) = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{d}{dt_M^*} \int_0^{t_M^*} \frac{X_1^*(\xi)}{(t_M^* - \xi)^{\alpha}} d\xi$$

while the second and the third components are equal to zero.
The velocity description with Riemann-Liouville fractional order derivativeα, $(0 < \alpha < 1)$ is objective if and only if the following equality holds:

$$D_{R-L}{}^{\alpha} X_1(t_M) = D_{R-L}{}^{\alpha} X_1^*(t_M^*).$$

Or using equalities $t_M = t_M^* + t_{M_{O^*}}$ and $X_1(t_M) = X_{1O*} + X_1^*(t_M^*)$ it is easy to see that the following equality holds

$$D_{R-L}{}^{\alpha} X_1(t_M) = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{d}{dt_M} \int_0^{t_{M_{O*}}} \frac{X_1(\tau)}{(t_M - \tau)^{\alpha}} d\tau + D_{R-L}{}^{\alpha} X_1^*(t_M^*)$$

Hence, the velocity description with Riemann-Liouville fractional order derivative α is not objective.

5.2 Velocity description with Caputo fractional order derivatives is not objective

Using the observer O description, the first component of the velocity described with Caputo fractional order derivativeα, $(0 < \alpha < 1)$ is

$$D_C{}^{\alpha} X_1(t_M) = \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t_M} \frac{X'_1(\tau)}{(t_M - \tau)^{\alpha}} d\tau$$

while the second and the third components are equal to zero. Using the observer O* description, the first component of the velocity described with Caputo fractional derivative α, $(0 < \alpha < 1)$ is

$$D_C{}^{\alpha} X_1^*(t_M^*) = \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t_M^*} \frac{X_1^{*\prime}(\xi)}{(t_M^* - \xi)^{\alpha}} d\xi$$

while the second and the third components are equal to zero. The velocity described by the Caputo fractional order derivativeα, $(0 < \alpha < 1)$ is objective if and only if the following equality holds:

$$D_C{}^{\alpha}X_1(t_M) = D_C{}^{\alpha}X_1^*(t_M^*)$$

Using equalities $t_M = t_M^* + t_{M_{O*}}$ and $X_1(t_M) = X_{1O*} + X_1^*(t_M^*)$ it is easy to see that the following equality holds:

$$D_C{}^{\alpha}X_1(t_M) = \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t_{M_{O*}}} \frac{X'_1(\tau)}{(t_M-\tau)^{\alpha}} d\tau + D_C{}^{\alpha}X_1^*(t_M^*)$$

Hence, the velocity description with Caputo fractional order derivativeα, $(0 < \alpha < 1)$ is not objective.

5.3 Acceleration description with Riemann-Liouville fractional order derivative is not objective.

That is because the following equality holds:

$$D_{R-L}{}^{\alpha}X_1(t_M) = \frac{1}{\Gamma(2-\alpha)} \cdot \frac{d^2}{dt^2{}_M} \int_0^{t_{M_{O*}}} \frac{X_1(\tau)}{(t_M-\tau)^{\alpha-1}} d\tau + D_{R-L}{}^{\alpha}X_1^*(t_M^*) \tag{37}$$

5.4 Acceleration description with Caputo fractional order derivative is not objective

That is because the following equality holds:

$$D_C{}^{\alpha}X_1(t_M) = \frac{1}{\Gamma(2-\alpha)} \cdot \int_0^{t_{M_{O*}}} \frac{X''_1(\tau)}{(t_M-\tau)^{\alpha-1}} d\tau + D_C{}^{\alpha}X_1^*(t_M^*) \tag{38}$$

5.5 Dynamic description with Riemann-Liouville fractional order derivative is not objective

In terms of the description of observer *O*, the second law of Newton, leads to the conclusion that functions $X_1(t_M), X_2(t_M), X_3(t_M)$, describing the motion under the action of a force field which acts along the line *L* and depends only on the position, satisfy the following system of differential equations:

$$mD_{R-L}{}^{\alpha}X_1(t_M) = F_{1O}(X_1(t_M)), mD_{R-L}{}^{\alpha}X_2(t_M) = 0, mD_{R-L}{}^{\alpha}X_3(t_M) = 0 \tag{39}$$

In terms of the description of observer *O**, the second law of Newton leads to the conclusion that functions $X_1^*(t_M^*)$, $X_2^*(t_M^*)$, $X_3^*(t_M^*)$, describing the motion under the action of the same force field, satisfy the following system of differential equations:

$$m \cdot D_{R-L}{}^{\alpha}X_1^*(t_M^*) = F_{1O}^*(X_1^*(t_M^*)), m \cdot D_{R-L}{}^{\alpha}X_2^*(t_M^*) = 0, \\ m \cdot D_{R-L}{}^{\alpha}X_3^*(t_M^*) \tag{40}$$

Equalities $m \cdot D_{R-L}{}^{\alpha}X_1(t_M) = F_{1O}(X_1(t_M))$, $m \cdot D_{R-L}{}^{\alpha}X_1^*(t_M^*) = F_{1O}^*(X_1^*(t_M^*))$ and (37) implies:

$$F_{1O}(X_1(t_M)) = F_{1O}^*(X_1^*(t_M^*)) + \frac{m}{\Gamma(2-\alpha)} \cdot \frac{d^2}{dt^2{}_M} \int_0^{t_{M_{O*}}} \frac{X_1(\tau)}{(t_M-\tau)^{\alpha-1}} d\tau \tag{41}$$

from where:

$$\frac{m}{\Gamma(2-\alpha)} \cdot \frac{d^2}{dt^2{}_M} \int_0^{t_{M_{O*}}} \frac{X_1(\tau)}{(t_M-\tau)^{\alpha-1}} d\tau = 0 \tag{42}$$

In general, equality (42) is not valid. For this reason, the dynamic description with the system of differential equations (39) is not objective.

5.6 Dynamics description with Caputo fractional order derivatives is not objective

In terms of the description of observer *O*, the second law of Newton, leads to the conclusion that functions $X_1(t_M), X_2(t_M), X_3(t_M)$, describing the motion under the action of a force field which acts along the line *L* and depends only on the position, satisfy the following system of differential equations:

$$mD_C{}^{\alpha}X_1(t_M) = F_{1O}(X_1(t_M)), mD_C{}^{\alpha}X_2(t_M) = 0, mD_C{}^{\alpha}X_3(t_M) = 0 \tag{43}$$

In terms of the description of observer *O**, the second law of Newton, leads to the conclusion that functions $X_1^*(t_M^*)$, $X_2^*(t_M^*)$, $X_3^*(t_M^*)$, describing the motion under the action of the same force field, satisfy the following system of differential equations:

$$m \cdot D_C{}^{\alpha}X_1^*(t_M^*) = F_{1O}^*(X_1^*(t_M^*)), \quad m \cdot D_C{}^{\alpha}X_2^*(t_M^*) = 0, \quad m \cdot D_C{}^{\alpha}X_3^*(t_M^*) = 0 \tag{44}$$

Equalities $m \cdot D_C{}^{\alpha}X_1(t_M) = F_{1O}(X_1(t_M)), m \cdot D_C{}^{\alpha}X_1^*(t_M^*) = F_{1O}^*(X_1^*(t_M^*))$ and (38) imply

$$F_{1O}(X_1(t_M)) = F_{1O}^*(X_1^*(t_M^*)) + \frac{m}{\Gamma(2-\alpha)} \cdot \int_0^{t_{M_{O*}}} \frac{X''_1(\tau)}{(t_M-\tau)^{\alpha-1}} d\tau \tag{45}$$

from where:

$$\frac{m}{\Gamma(2-\alpha)} \cdot \int_0^{t_{M_{O*}}} \frac{X''_1(\tau)}{(t_M-\tau)^{\alpha-1}} d\tau = 0 \tag{46}$$

This last equality in general is not valid. For this reason the dynamic description with the system of differential equations (43) is not objective.

5.7 The Hooke constitutive law description with Riemann-Liouville fractional order derivativesα, $(0 < \alpha < 1)$ is not objective

In [9] Bagley and Torvik instead of the description of the constitutive law of Hooke given by (31) in terms of the observer *O*, describe the constitutive law of Hooke in terms of the same observer *O* by:

$$\sigma_{ij}(t, X_1, X_2, X_3) = \lambda \cdot \theta(t) \cdot \delta_{ij} + 2\mu \cdot D_{R-L}{}^{\alpha}\varepsilon_{ij}(t) \tag{47}$$

In terms of observer $O*$ this description become

$$\sigma_{ij}^*(t^*, X_1^*, X_2^*, X_3^*) = \lambda \cdot \theta^*(t) \cdot \delta_{ij} + 2\mu \cdot D_{R-L}{}^{\alpha}\varepsilon_{ij}^*(t^*) \tag{48}$$

In order to see that description (47) is not objective start with (31) use (47) consider i different from *j*, use the equality

$$D_{R-L}{}^{\alpha}\varepsilon_{ij}(t_M) = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{d}{dt_M} \int_0^{t_{M_{O*}}} \frac{\varepsilon_{ij}(\tau)}{(t_M-\tau)^{\alpha}} d\tau + D_{R-L}{}^{\alpha}\varepsilon_{ij}^*(t_M^*) \tag{49}$$

and obtain the following equality:

$$\sigma_{ij}^*(t^*, X_1^*, X_2^*, X_3^*) = 2\mu \cdot D_{R-L}{}^{\alpha}\varepsilon_{ij}^*(t^*) + \frac{2 \cdot \mu}{\Gamma(1-\alpha)} \sum_{k=1}^{3}\sum_{l=1}^{3} a_{ki}a_{lj} \cdot \frac{d}{dt_M} \int_0^{t_{M_{O*}}} \frac{\varepsilon_{ij}(\tau)}{(t_M-\tau)^{\alpha}} d\tau$$

This equality shows that equality (48) in general is not verified.

For this reason the Hooke constitutive law description using Riemann-Liouville fractional order derivativesα, $(0 < \alpha < 1)$ is not objective.

6. FINAL REMARK

In the scientific literature there exists a lot of papers using fractional order derivatives for describing real world phenomena. For example, in [10] the authors use a fractional conservation of mass equation to model fluid flow when the control volume is not large enough compared to the scale of heterogeneity and when the flux within the control volume is non-linear; in [11], [12] Atangana et al. described some groundwater flow problems using the concept of derivative with fractional order; the authors of [13]-[15] use fractional order differential equations for modeling contaminant flow in heterogeneous porous media; for describing anomalous diffusion processes in complex media fractional-order diffusion equation models are considered in [16] and [17]; in [18] the author uses fractional order derivatives in linear viscoelasticity; in [19]-[21] the authors use fractional order derivatives for acoustical wave propagation in complex media; in [22], [23] the fractional order Schrodinger equation and the variable –order fractional Schrodinger equation are considered. It would be interesting to know if these equations are objective (frame invariants).

7. CONCLUSIONS

1. Velocity description with Riemann-Liouville or Caputo fractional order derivatives is not objective.
2. Acceleration description with Riemann-Liouville or Caputo fractional order derivatives is not objective.
3. Dynamic description with Riemann-Liouville or Caputo fractional order derivatives is not objective.
4. Hooke constitutive law description with Riemann-Liouville fractional order derivatives is not objective. Hence an important question appears: what is the interpretation of the results reported in [9] and how this result has to be used in case of an aircraft wing?

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