A new approximation of the dispersion relations occurring in the sound-attenuation problem of turbofan aircraft engines

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Abstract: *The dispersion relations, appearing in the analysis of the stability of a gas flow in a straight acoustically-lined duct with respect to perturbations produced by a time harmonic source, beside the wave number and complex frequency contain the solution of a boundary value problem of the Pridmore-Brown equation depending on the wave number and frequency. For this reason, in practice the dispersion relations are rarely simple enough for carried out the zeros. The determination of zeros of these dispersion relations is crucial for the prediction of the perturbation attenuation or amplification. In this paper an approximation of the dispersion relations is given. Our approach preserves the general character of the mean flow, the general Pridmore-Brown equation and it's only in the shear flow that we replace the exact solution of the boundary value problem with its Taylor polynomial approximate. In this way new approximate dispersion relations are obtained which zero's can be found by computer.*

Key Words: *gas dynamics, dispersion relations, boundary value problem, Taylor polynomial approximate,* harmonic point source

1. INTRODUCTION

In the intakes and bypass ducts of turbofan aircraft engines acoustic linings are used to attenuate sound emissions. In the last 40 years a lot of experimental and theoretical results have been published concerning the subject (see for example Refs. [1]- [11] and the papers referred therein). The theoretical results concern acoustic perturbation in a gas flowing through an attenuating duct. In many cases concerning acoustic perturbation-liner interaction (mass-spring-damper impedance [1-2], or equivalently the three parameter impedance [3]; Helmholz-resonator or enhanced-Helmholz-resonator impedance [4]) it is assumed that the liner is locally reacting and the reaction is linear. The acoustic perturbation evolution has been analyzed using different methods [1]-[11]. The method which is currently used is that developed by Briggs-Bers in [12], [13] for infinitely extended homogeneous plasma and some variant of this, and uses Fourier (F) transform with respect to spatial variable *x* and Laplace (L) transform with respect to time variable *t*. In [12], [13], the perturbation is produced by a permanent harmonic point source. Frequently in the published papers concerning sound attenuation it is assumed that the acoustic perturbation is produced by a time harmonic point source of fixed frequency (see for example Ref. [5]). For the mathematical treatment of the acoustic perturbation evolution the nonlinear homogeneous Euler equations are linearized around the mean flow and the time harmonic point source is added as a right hand member to the linearized homogeneous Euler equations. The solution of the linear and non homogeneous Euler equations having zero initial value is that which describe the evolution of the acoustical perturbation.

In [14] a 2D straight lined duct and a non shipping mean flow, having symmetric velocity profile and being constant in the central part of the duct, was considered (see Fig. 1).

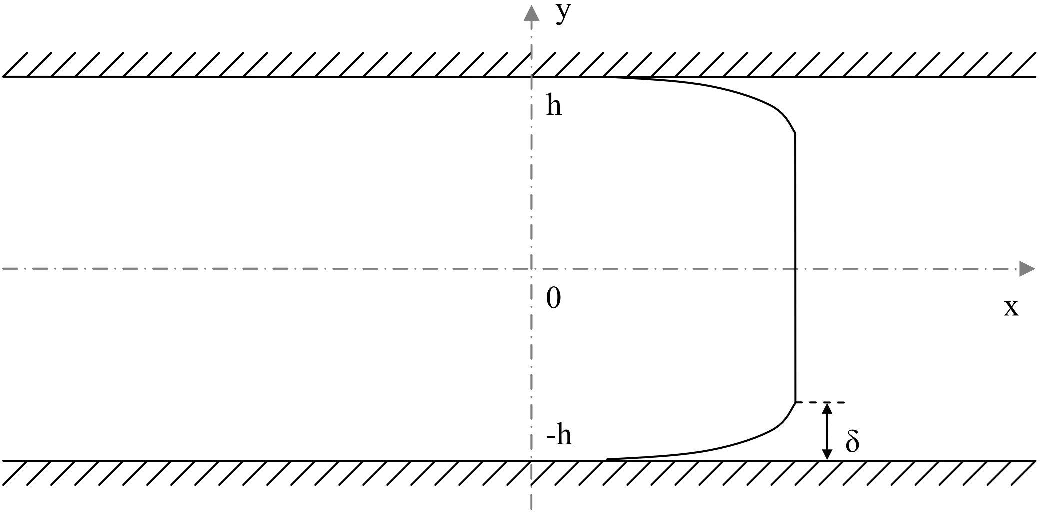


Fig. 1 - Flow in a rectangular acoustic lined duct

For a constant frequency ω_f time harmonic source, which is spatially distributed in a bounded neighborhood of a point in the duct, it was shown that the above considered initial value problem has a unique classical solution possessing F-transform with respect to x, and L-transform with respect to t `if and only if the imaginary parts of the complex frequencies ω ($\omega = iz$; $z - L$ transform variable) for which one of the dispersion relations vanishes for real wave numbers α, is bounded from above; i.e. $\text{Im}\{\omega(\alpha)\}$ is bounded from above for ω satisfying:

$$\mu \cdot \tan[\mu \cdot (-h+\delta)] + \frac{\frac{dq}{dy}(-h+\delta;\alpha,\omega)}{q(-h+\delta;\alpha,\omega)} = 0 \quad (1)$$

$$\mu \cdot \cos[\mu \cdot (-h+\delta)] - \frac{\frac{dq}{dy}(-h+\delta;\alpha,\omega)}{q(-h+\delta;\alpha,\omega)} = 0 \quad (2)$$

$$\mu^2 = \frac{\left(\omega - \alpha \overline{U_0}\right)^2}{c_o^2} - \alpha^2 \tag{3}$$

Here: h – duct half thickness; δ - shear flow thickness; $\overline{U_0}$ - constant velocity of the mean flow in the central part of the duct (Figure 1); q - solution of the problem:

$$\begin{cases} \dfrac{d^2 q}{dy^2} - \dfrac{2\alpha}{\alpha \cdot U_0(y) - \omega} \cdot \dfrac{dU_0}{dy} \cdot \dfrac{dq}{dy} + \left[\dfrac{(\alpha \cdot U_0(y) - \omega)^2}{c_0^2} - \alpha^2 \right] \cdot q = 0 \\ q(-h; \alpha_1 \cdot \omega) = 1 \quad ; \quad \dfrac{dq}{dy}(-h; \alpha, \omega) = -\dfrac{i \cdot \rho_0 \cdot \omega}{Z(\omega)} \end{cases} \tag{4}$$

c_0 - sound speed; $U_0(y)$- mean flow velocity profile (Figure 1); ρ_0 - mean flow density; the present paper we rederive the representation formula of the evolution of the acoustical perturbation established in [14]. Using this formula we discuss the mean flow stability with respect to the considered perturbation in the Locally Convex Linear Topological Space of the Rapidly Decreasing Functions (L.C.L.T.S.R.D.F.) i.e. the attenuation or amplification of the perturbation. It turns that this last problem is closely related to the location of the zeros of the dispersion eqs. (1) and (2). In practice eqs (1) and (2) are rarely simple enough for carried out the zeros. For this reason we derive new approximate dispersion equations.

2. REPRESENTATION FORMULA AND STABILITY ANALYSIS

Under the conditions presented in Introduction (see also Ref. [14]) the solution of the linearized non homogenous Euler equations:

$$\begin{cases} \dfrac{\partial u'}{\partial t} + U_0 \cdot \dfrac{\partial u'}{\partial x} + v' \cdot \dfrac{dU_0}{dy} + \dfrac{1}{\rho_0} \cdot \dfrac{\partial p'}{\partial x} = F(x,y) \cdot e^{i \cdot \omega_f \cdot t} \\ \dfrac{\partial v'}{\partial' t} + U_0 \cdot \dfrac{\partial v'}{\partial x} + \dfrac{1}{\rho_0} \cdot \dfrac{\partial p'}{\partial y} = G(x,y) \cdot e^{i \cdot \omega_f \cdot t} \\ \dfrac{\partial p'}{\partial t} + \gamma \cdot p_0 \cdot \dfrac{\partial u'}{\partial x} + \gamma \cdot p_0 \cdot \dfrac{\partial v'}{\partial y} + U_0 \cdot \dfrac{\partial p'}{\partial x} = H(x,y) \cdot e^{i \cdot \omega_f \cdot t} \end{cases} \tag{5}$$

which satisfies the boundary conditions:

$$a \cdot \frac{\partial^2 v'}{\partial t^2} + b \cdot \frac{\partial v'}{\partial t} + c \cdot v' = \mp \frac{\partial p'}{\partial t} \qquad \text{for} \qquad y = -h \quad \text{and} \quad y = h \tag{6}$$

and the initial conditions:

$$u'(x,y,0) = v'(x,y,0) = p'(x,y,0) = 0 \tag{7}$$

can be represented by the formulas:

$$\begin{cases} u'(x,y,t)=\dfrac{1}{\sqrt{2\pi}}\cdot\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}\int_{+\infty+i\sigma}^{-\infty+i\sigma}\tilde{u}(\alpha,y,\omega)e^{i\cdot\alpha\cdot x}\cdot e^{-it\omega}d\alpha\, d\omega \\ v'(x,y,t)=\dfrac{1}{\sqrt{2\pi}}\cdot\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}\int_{+\infty+i\sigma}^{-\infty+i\sigma}\tilde{v}(\alpha,y,\omega)e^{i\cdot\alpha\cdot x}\cdot e^{-it\omega}d\alpha\, d\omega \\ p'(x,y,t)=\dfrac{1}{\sqrt{2\pi}}\cdot\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}\int_{+\infty+i\sigma}^{-\infty+i\sigma}\tilde{p}(\alpha,y,\omega)e^{i\cdot\alpha\cdot x}\cdot e^{-it\omega}d\alpha\, d\omega \end{cases} \tag{8}$$

In equations (5) the right-hand members represent the time harmonic source of fixed frequency $\omega_f \in \Re'$ which is spatially distributed in a bounded neighborhood of a point in the duct (i.e. F,G,H are smooth functions, with compact support).

In equation (6) a,b,c are positive real constants and represent the inertance, the resistance and the stiffness respectively of the liner (i.e. mass-spring-damper impedance).

In equations (8) σ is a real number which is an upper bound for the imaginary parts of the zero's ω of equations (1) and (2) and $\tilde{u}(\alpha,y,\omega)$, $\tilde{v}(\alpha,y,\omega)$, $\tilde{p}(\alpha,y,\omega)$ are given by the formulas:

$$\tilde{u}(\alpha,y,\omega)=\frac{1}{[\alpha\cdot U_0(y)-\omega]}\cdot\left[\frac{dU_0}{dy}\cdot\tilde{v}(\alpha,y,\omega)+\frac{i\cdot\alpha}{\rho_0}\cdot\tilde{p}(\alpha,y,\omega)+\frac{i\cdot\alpha}{\omega+\omega_f}\cdot\tilde{F}(\alpha,y)\right] \tag{9}$$

$$\tilde{v}(\alpha,y,\omega)=\frac{i}{\rho_0[\alpha\cdot U_0(y)-\omega]}\cdot\left[\frac{d\tilde{p}}{dy}-\frac{i\cdot\rho_0}{\omega+\omega_f}\cdot\tilde{G}(\alpha,y)\right] \tag{10}$$

$$\tilde{p}(\alpha,y,\omega)=\int_{-h}^{h}Q(y,y',\alpha,\omega)\cdot\frac{i\cdot\alpha}{c_0^2}\cdot R(\alpha,\omega,y')dy' \tag{11}$$

The functions $Q(y,y',\alpha,\omega)$ and $R(\alpha,\omega,y')$ appearing in (11) are given by:

$$Q(y,y',\alpha,\omega)=\begin{cases}\dfrac{1}{E_0}\cdot q_1(y',\alpha,\omega)\cdot q_2(y,\alpha,\omega) & for \quad -h\le y'\le y\le h \\ \dfrac{1}{E_0}\cdot q_1(y,\alpha,\omega)\cdot q_2(y',\alpha,\omega) & for \quad -h\le y\le y'\le h\end{cases} \tag{12}$$

$$\begin{aligned} R(\alpha,\omega,y')=&-\frac{\rho_0\cdot\alpha}{\omega+\omega_f}\cdot\tilde{F}(\alpha,y')-\frac{2\cdot\alpha\cdot i\cdot\rho_0}{[\alpha\cdot U_0(y)-\omega]\cdot(\omega+\omega_f)}\cdot\frac{dU_0}{dy}\cdot\tilde{G}(\alpha,y')+ \\ &+\frac{i\cdot\rho_0}{\omega+\omega_f}\cdot\frac{d\tilde{G}}{dy}(\alpha,y')+\frac{\alpha\cdot U_0(y)-\omega}{c_0^2(\omega-\omega_f)}\cdot\tilde{H}(\alpha,y')\end{aligned} \tag{13}$$

with $E_0(\alpha,\omega)$ given by:

$$E_0(\alpha,\omega)=\frac{1}{[\omega-\alpha\cdot U_0(0)]^2}\cdot\left[q_1(0,\alpha,\omega)\cdot\frac{dq_2}{dy}(0,\alpha,\omega)-q_2(0,\alpha,\omega)\cdot\frac{dq_1}{dy}(0,\alpha,\omega)\right] \tag{14}$$

and $q_1(y,\alpha,\omega)$, $q_2(y,\alpha,\omega)$ solutions of the boundary value problems:

$$\begin{cases}\dfrac{d^2q_1}{dy^2}-\dfrac{2\alpha}{\alpha\cdot U_0(y)-\omega}\cdot\dfrac{dU_0}{dy}\cdot\dfrac{dq_1}{dy}+\left[\dfrac{(\alpha\cdot U_0(y)-\omega)^2}{c_0^2}-\alpha^2\right]\cdot q_1=0\\ q_1(-h,\alpha,\omega)=1 \quad ; \quad \dfrac{dq_1}{dy}(-h,\alpha,\omega)=-\dfrac{i\cdot\rho_0\cdot\omega}{Z(\omega)}\end{cases} \tag{15}$$

$$\begin{cases}\dfrac{d^2q_2}{dy^2}-\dfrac{2\alpha}{\alpha\cdot U_0(y)-\omega}\cdot\dfrac{dU_0}{dy}\cdot\dfrac{dq_2}{dy}+\left[\dfrac{(\alpha\cdot U_0(y)-\omega)^2}{c_0^2}-\alpha^2\right]\cdot q_2=0\\ q_2(h,\alpha,\omega)=1 \quad ; \quad \dfrac{dq_2}{dy}(h,\alpha,\omega)=\dfrac{i\cdot\rho_0\cdot\omega}{Z(\omega)}\end{cases} \tag{16}$$

$\tilde{F}(\alpha,y),\tilde{G}(\alpha,y),\ \tilde{H}(\alpha,y)$ are the F - transforms of $F(x,y),G(x,y),\ H(x,y)$ with respect to x.

The representation formulas (8) can be used for the stability analysis of the mean flow with respect to the considered perturbation in the L.C.L.T.S.R.D.F. That is because the F - transform is continuous in this space (see [15] chap VI, pg. 146).

For the stability analysis it is sufficient to examine the singularities of the integrals occurring in (8). This singularities are the zeros of equations (1) or (2) (see (11) and (12)) and $\omega=-\omega_f$.

If the zeros of equations (1) and (2) have negative imaginary parts for every α (i.e. $Sup\ \mathrm{Im}\{\omega(\alpha)\}<0$) then the L - contour, $L=[+\infty+i\sigma,-\infty+i\sigma]$ can be lowered under the real axis of the ω plane and the perturbation is not amplified i.e. the mean flow is stable with respect to this perturbation.

If there exist a zero of the eq. (1) or (2) with strictly positive imaginary part then the perturbation is amplified (see [15] chap. IX, pg. 231-272).

3. APPROXIMATE DISPERSION RELATIONS

In practice the dispersion relations are rarely simple enough for carried out the zeros. That is mainly because beside the unknown frequency ω they contains the solution $q(y;\alpha,\omega)$ of (4) which depends also on ω and in general can not be find explicitly (i.e. there is no explicit formula for that).

In some of the published papers the authors overcome this difficulty by considering linear mean flow velocity profile in the shear flow and considering for the Pridmore-Brown equation the incompressible limit in the shear flow. In this way a boundary value problem is obtained for which the solution can be obtained explicitly (see for example [16]). For this situation the determination of zeros of the dispersion relations can be made.

In this section of the paper we will present other approach. Our approach preserves the general character of the mean flow, the general Pridmore-Brown equation and the requirements at the boundaries but in the shear flow we've replaced the exact solution of the boundary value problem by its Taylor polynomial approximate. In this way new approximate dispersion relations are obtained which can be handled by computer.

In the shear flow (i.e. for $y \in [-h, -h+\delta]$) the second order Taylor polynomial approximation of $q(y,\alpha,\omega)$ is given by:

$$T_2(y,\alpha,\omega) = 1 - \frac{i \cdot \rho_0 \cdot \omega}{Z(\omega)} \cdot (y+h) + \frac{1}{2}\left[\frac{2i \cdot \alpha \cdot \rho_0}{Z(\omega)} \cdot \frac{dU_0}{dy} \cdot (-h) + \alpha^2 - \frac{\omega^2}{c_0^2}\right] \cdot (y+h)^2 \tag{17}$$

The values of $T_2(y,\alpha,\omega)$ and $\frac{dT_2}{dy}(y,\alpha,\omega)$ at $y = -h+\delta$ are given by:

$$\left.\begin{aligned} T_2(-h+\delta,\alpha,\omega) &= 1 - \frac{i \cdot \rho_0 \cdot \omega}{Z(\omega)} \cdot \delta + \frac{1}{2}\left[\frac{2 \cdot i \cdot \alpha \cdot \rho_0}{Z(\omega)} \cdot \frac{dU_0}{dy} \cdot (-h) + \alpha^2 - \frac{\omega^2}{c_0^2}\right] \cdot \delta^2 \\ \frac{dT_2}{dy}(-h+\delta,\alpha,\omega) &= -\frac{i \cdot \rho_0 \cdot \omega}{Z(\omega)} + \left[\frac{2 \cdot i \cdot \alpha \cdot \rho_0}{Z(\omega)} \cdot \frac{dU_0}{dy} \cdot (-h) + \alpha^2 - \frac{\omega^2}{c_0^2}\right] \cdot \delta \end{aligned}\right\} \tag{18}$$

Using the above formulas the following approximate dispersion relations are obtained:

$$\mu \cdot \tan[\mu \cdot (-h+\delta)] + \frac{\frac{dT_2}{dy}(-h+\delta,\alpha,\omega)}{T_2(-h+\delta,\alpha,\omega)} = 0 \tag{19}$$

$$\mu \cdot \cot[\mu \cdot (-h+\delta)] + \frac{\frac{dT_2}{dy}(-h+\delta,\alpha,\omega)}{T_2(-h+\delta,\alpha,\omega)} = 0 \tag{20}$$

As concern the accuracy of this approximation we remark that we have:

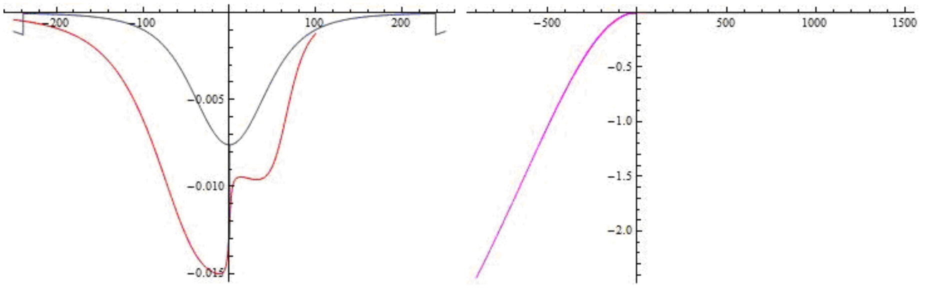
$$|q(y,\alpha,\omega) - T_2(y,\alpha,\omega)| \le \frac{\delta^3}{6} \cdot M \tag{21}$$

where M is the upper bound of the third order derivative of $q(y,\alpha,\omega)$.

4. NUMERICAL ILLUSTRATION

Computations were performed for the following numerical values:
$a = 0.1215\,[kg/m^3]$, $b = 100\,[kg/(m^2 \cdot s^2)]$, $c = 8166\,[kg/(m^2 \cdot s^2)]$, $\rho_0 = 1.225\,[kg/m^3]$,
$c_0 = 340\,[m/s]$, $\overline{U}_0 = 82\,[m/s]$, $h = 1\,[m]$, $\delta = 0.001\,[m]$, $U_0'(-h) = \frac{\overline{U}_0}{\delta}$.

The imaginary parts of the obtained zeros of (19) and (20) are represented in Fig. 2a and 2b, respectively.



a) b)

Fig. 2 - Imaginary parts of the frequencies ω, obtained using (19) *a)* and (3.4) *b)*, respectively

The computed results suggest stability. This prediction can be tested against observation and if verified, lends authenticity to the approximation.

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REFERENCES

[1] B. J. Tester, Some aspects of "sound" attenuation in lined ducts containing inviscid mean flow with boundary layers, *J. Sound and Vibration* vol. **28** pp. 217-245, 1973.

[2] S. W. Rienstra, Impedance models in time domain, including the extended Helmholtz resonator model, *AIAA Paper 2006-2686*.

[3] C. K. W. Tam, L. Auriault, Time-domain impedance boundary conditions for computational aeroacustics, *AIAA Journal* vol**. 34** no. 5 pp. 917-923, 1996.

[4] M. Brandes, D. Ronneberger, Sound amplification in flow ducts lined with a periodic sequence of resonators, *AIAA Paper 95-126*, 1995.

[5] E. J. Brambley, Fundamental problems with the model of uniform flow over acoustic lines, *J. Sound and Vibration* vol. **322** pp. 1026-1031, 2009.

[6] D. C. Pridmore-Brown, Sound propagation in a fluid flowing through an attenuating duct, *J. Fluid Mechanics,* vol. **4** pp. 393-406, 1958.

[7] W. Eversman, R. J. Beckemeyer, Transition of sound in ducts with the shear layers-convergence to the uniform flow case*, J. Acoustical Soc. of America* vol. **52** pp. 216-220, 1972.

[8] M. M. Myers, S. L. Chunang, Uniform asymptotic approximations for duct acoustic modes in a thin boundary-layer flow, *AIAA Journal* vol. **22** no. 9 pp. 1234-1241, 1984.

[9] E.J. Rice, Propagation of waves in an acoustically lined duct with a mean flow, *Technical Report SP-207, NASA 1962.*

[10] A. H. Nayfeh, J. E. Kaiser, D.P. Telionis, Acoustics of aircraft engine-duct system, *AIAA Journal* vol. **13** no. 2 pp. 130-153, 1975.

[11] S. W. Rienstra, A classification of duct modes based on surface waves, *Wave Motion* vol. **37** pp. 119-135, 2003.

[12] A. Bers, *Space-Time evolution of plasma-instabilities-absolute and convective*, Handbook of Plasma Physics, Ed. A.A. Galeev and R.N. Sudan North-Holland Publ. CO, Amsterdam 1983 pg. 452-484

[13] R.I. Briggs, *Electron-Stream Interaction with Plasmas*, M.I.T. Press, 1964.

[14] St. Balint, A. M. Balint, M. Darau, Partial Fourier and Laplace transforms in stability analysis with respect to perturbation produced by a spatially distributed time harmonic source, *Mathematics in Engineering Science and Aerospace,* to appear.

[15] K. Yoshida, *Functional Analysis*, Springer, Berlin-Gottingen-Heidelberg, 1965.

[16] S. W. Rienstra, M. Darau, Boundary layer thickness effect of the hydrodynamic instability along an impedance wall, *J. Fluid Mechanics*, to appear.

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