# Linear Stability with Respect to the Initial Value Perturbations in the Presence of Solutions of the Linearized Equation Having Strictly Positive Exponential Growth Rate

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***Abstract:*** *The objective is to emphasize the importance of the functional framework specification when the linear stability of a gas flow is analyzed. For this the linear stability with respect to the initial value perturbations of the constant 1-D gas flow is presented in three different functional frameworks. For the linearized Euler equations in each framework the initial value problem is well-posed in sense of Hadamard and the zero solution is stable, i.e. the constant flow is linearly stable. However, in the first framework the set of the exponential growth rate of the solutions of the linearized equations is the whole real axis and the Briggs-Bers stability analysis can not be applied. In the second framework even if the exponential growth rate of the solutions of the linearized equation is equal to zero, the Briggs-Bers stability analysis fails, because the Fourier transform can not be applied. In the third framework the exponential growth rate of the solutions of the linearized equation is equal to zero, the Briggs-Bers stability analysis works, but there are solutions which satisfy the linearized equations only in a generalized sense (almost everywhere). These considerations can be useful in a better understanding of some apparently strange results obtained in different mathematical models of the sound propagation in a gas flowing in a lined duct.*

***Key Words:*** *Liapunov stability; instantaneous perturbation; exponential growth rate; 1D analysis aeroacoustics.*

## 1. MOTIVATION OF THE MATHEMATICAL CONSIDERATIONS

Due to the practical importance of the sound attenuation, in the last 60 years more than five hundred papers were published, reporting experimental and theoretical results on the subject. The papers referred here concern mainly acoustic perturbation attenuation in a gas flowing through a lined duct and represent just a small part of the literature concerning the subject.

For describing the instantaneous acoustic perturbation propagation in a gas flowing through a lined duct, the authors of [1-18] consider the nonlinear Euler equations governing the gas flow and the solution of these equations describing the gas flow. After that, the nonlinear Euler equations are linearized at the specified solution and the so-called homogeneous linearized Euler equations are derived.

It is assumed that the homogeneous linearized Euler equations describe the evolution of an instantaneous acoustic perturbation. More precisely, it is assumed that if at the moment, let say, $t=0$, an instantaneous acoustic perturbation occurs, then its evolution is described by that solution of the homogeneous linearized Euler equations which satisfies on the duct wall the boundary condition and at $t=0$ is equal to the perturbation.

In [1-18] for the above initial-boundary value problem, mode type solutions are researched. The conditions for the existence of the mode type solutions are found. The space-time behavior of a mode is investigated by using the zeros of the dispersion relations.

The mode type solutions correspond to a particular set of initial data. The extension of the results obtained in this particular set to a larger set of initial data requires a precise definition of the new set of data and the investigation of the fact that the initial-boundary value problem is well-posed in the new framework. The necessity of such type of investigations was emphasized in [10] only in 2009.

In [10] a "concept" of well-posed problem was introduced, showing that in a lot of papers published before, the initial-boundary value problem is ill-posed, because the set of the exponential growth rates of the solutions of the homogeneous linearized equations is unbounded from above. Later in [11-14] considerable efforts were undertaken for modifying the boundary conditions on the duct wall in order to make the initial-boundary value problem well-posed in sense of [10].

The objective of this paper is to underline that the precise description of the functional framework is crucial even in the case of 1-D gas flow, where the wall and its effect do not appear. For this purpose, the linear stability analysis with respect to the initial value perturbation of the constant 1-D gas flow is presented in three different functional frameworks.

The concept of linear stability with respect to the initial value perturbations of the constant 1-D gas flow is defined in ***Section 2*** and it has to be mentioned that this is not necessarily hydrodynamic stability in the sense defined in [19]. In each framework the initial value problem is well-posed in sense of Hadamard [20, 21] and the constant flow is linearly stable with respect to the initial value perturbations.

However, in the first framework the set of exponential growth rates of the solutions of the homogeneous linearized Euler equations is the whole real axis. In this framework the Briggs-Bers stability analysis [22, 23] can not be applied.

In the second framework, even if the exponential growth rate of the solutions of the homogeneous linearized Euler equations is equal to zero, the Briggs-Bers stability analysis fails. In the third framework the exponential growth rate of the solutions of the homogeneous linearized Euler equations is equal to zero, the Briggs-Bers stability analysis works, but there are solutions which satisfy the linearized Euler equations only almost everywhere (i.e. there are generalized solutions).

These facts can be helpful in a better understanding of some apparently strange results obtained in different mathematical models of the sound propagation in a gas flowing in a lined duct.

## 2. THE 1-D GAS FLOW MODEL

Consider an inviscid, non-heat conducting, compressible, isentropic, perfect gas. In the 1-D gas flow model the nonlinear Euler equations governing the gas flow are [24]:

$$\begin{cases} \frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial x} = 0 \\ \frac{\partial \rho}{\partial t} + u \cdot \frac{\partial \rho}{\partial x} + \rho \cdot \frac{\partial u}{\partial x} = 0 \end{cases} \tag{1}$$

Here: $t$-time; $u$- velocity along the $Ox$ axis; $p$- pressure; $\rho$-density. Eqs. (1) are considered for $x \in R'$ and $t \geq 0$.

It is assumed that, $p$, $\rho$ and the absolute temperature $T'$ satisfy the equation of state of perfect gas:

$$p = \rho \cdot R \cdot T' \tag{2}$$

with: $R = c_p - c_v$; $c_p$, $c_v$ being the specific heat capacities at constant pressure and constant volume, respectively.

Let: $u \equiv U_0 = const > 0$, $\rho \equiv \rho_0 = const > 0$, $p \equiv p_0 = const > 0$ be a constant solution of the system of partial differential equations (1). According to (2): $p_0 = \rho_0 \cdot R \cdot T_0$ and the associated sound speed $c_0$ verifies $c_0^{\ 2} = \gamma \cdot \frac{p_0}{\rho_0} = \gamma \cdot R \cdot T_0$, where $\gamma = \frac{C_p}{C_v}$.

Linearizing Eq.(1) at $u = U_0$, $\rho = \rho_0$, $p = p_0$ and assuming that the perturbations $p'$, $\rho'$ of $p_0, \rho_0$ satisfy:

$$\left( \frac{\partial}{\partial t} + U_0 \cdot \frac{\partial}{\partial x} \right) \left( p' - c_0^{\ 2} \rho' \right) = 0 \tag{3}$$

for the perturbations $u'$, $p'$ of $U_0, p_0$ the following system of homogeneous linear partial differential equations is obtained:

$$\begin{cases} \frac{\partial u'}{\partial t} + U_0 \cdot \frac{\partial u'}{\partial x} + \frac{1}{\rho_0} \cdot \frac{\partial p'}{\partial x} = 0 \\ \frac{\partial p'}{\partial t} + U_0 \cdot \frac{\partial p'}{\partial x} + \gamma \cdot p_0 \cdot \frac{\partial u'}{\partial x} = 0 \end{cases} \tag{4}$$

It is assumed that this system governs the evolution of an instantaneous acoustic perturbation, i.e. it at $t = 0$ an acoustic perturbation occurs, then the evolution is given by that solution of the system (4) which at $t = 0$ is equal to the instantaneous perturbation.

The solution $u = U_0$, $p = p_0$ is linearly stable with respect to the initial value perturbation if the solution $u'(x,t)$, $p'(x,t)$ of (4) is small all time $t \geq 0$ provided it is small at the beginning $t = 0$ (i.e. $u'(x,0)$, $p'(x,0)$ are small). This stability is not necessarily equivalent to that defined in [21].

The precise meaning of the concepts "solution" and "small" has to be designed by definition and there is some freedom here.

By using the above freedom in the next section three different functional frameworks are considered, having the property that in each of them the concepts "solution" and "small" have specific meanings. Due to that, the mathematical tools and some of the results concerning linear stability are specific.

## 3. LINEAR STABILITY WITH RESPECT TO THE INITIAL VALUE PERTURBATION OF THE CONSTANT 1-D GAS FLOW

First, let $X$ be the set of initial data defined as:

$X = \{H = (F,G) \vdots F,G : R' \rightarrow R' \textit{ continously differentiable}\}$.

For an initial data $H = (F,G) \in X$ the couple of functions $H'(x,t) = (u'(x,t), p'(x,t))$ given by:

$$u'(x,t) = \frac{F[x-(U_0-c_0)t] + F[x-(U_0+c_0)t]}{2} + \frac{G[x-(U_0+c_0)t] - G[x-(U_0-c_0)t]}{2c_0\rho_0}$$
$$p'(x,t) = c_0\rho_0 \cdot \frac{F[x-(U_0+c_0)t] + F[x-(U_0-c_0)t]}{2} + \frac{G[x-(U_0+c_0)t] + G[x-(U_0-c_0)t]}{2} \tag{5}$$

is continuously differentiable, verifies Eqs. (4) and the initial condition:

$$H'(x,0) = H(x) \tag{6}$$

The couple of functions $H'(x,t)$, having the above properties is unique and is called classical solution of (4), (6). In other words, for the set $X$ of the initial data, the initial value problem (4), (6) has a unique, continuously differentiable, classical solution [24].

If a sequence of initial data $H_n = (F_n, G_n) \in X$ converges uniformly (on $R'$) to the initial data $H = (F,G) \in X$, then for every $t \geq 0$ the sequence of the corresponding solutions $H_n'(\cdot,t)$ converges uniformly to the solution $H'(\cdot,t)$ corresponding to $H$.

This means that for the set of the initial data $X$ the initial value problem (4), (6) is well-posed in sense of Hadamard on $[0,T]$ for every $T > 0$ [20], [21].

With respect to the usual algebraic operations and topology generated by the uniform convergence [25] the set $X$ is a topological vector space [26].

A neighborhood of $H_0 = (F_0, G_0) \in X$ is a set $V_{H_0} \subset X$ having the property that there exists a strictly positive real number $\varepsilon > 0$ such that if $|F(x) - F_0(x)| < \varepsilon$ and $|G(x) - G_0(x)| < \varepsilon$ for any $x \in R'$, then $H = (F,G) \in V_{H_0}$. For $\varepsilon > 0$ and $H_0 \in X$, the set $V_{H_0}^{\varepsilon}$ defined by:

$$V_{H_0}^{\varepsilon} = \{H \in X \vdots |F(x) - F_0(x)| < \varepsilon \textit{ and } |G(x) - G_0(x)| < \varepsilon \; \forall x \in R'\}$$

is a neighborhood of $H_0$.

The fact that $H$ is close to $H_0$ can be expressed by saying that $H \in V_{H_0}^{\varepsilon}$ for $\varepsilon$ small.

The fact that $H$ is small can be expressed by saying that $H \in V_0^{\varepsilon}$ for $\varepsilon$ small. It can be seen that for every strictly positive real number $\varepsilon > 0$ if $H \in V_0^{\delta(\varepsilon)}$ with $\delta(\varepsilon) = \varepsilon / \max\left(1 + c_0\rho_0, 1 + \frac{1}{c_0\rho_0}\right)$, then $H'(\cdot,t) \in V_0^{\varepsilon}$ for $t \geq 0$. This means that the solution $u = U_0$, $p = p_0$, $\rho = \rho_0$ of (1) is linearly stable with respect to the initial value perturbation with initial data from $X$.

On the other hand, the exponential growth rate of the solution $H'(x,t)$ corresponding to the initial perturbation $H(x) = (\exp(-x), \exp(-2x))$ is equal to $2(U_0 + c_0) > 0$. Here the exponential growth rate of $H'(x.t) = (u'(x,t), p'(x,t))$ is defined as:

$$\max\left\{\lim_{t\to\infty}\frac{\ln|u'(x,t)|}{t}, \lim_{t\to\infty}\frac{\ln|p'(x,t)|}{t}\right\}$$

Moreover, the set of exponential growth rates of the solutions of the initial value problem (4), (6) is the whole real axis and there exist solutions whose exponential growth rate is equal to $+\infty$ (for instance that of the solution corresponding to the initial data $F(x) = G(x) = \exp(x^2)$).

The linear stability in the presence of solutions having strictly positive or $+\infty$ growth rate can be surprising. That is because usually the stability of the linear evolutionary equations is analyzed in functional framework in which the Hille-Yoshida theory can be applied [27], [28], [20] (i.e. locally convex and sequentially complete topological vector spaces). The change of this framework is the source of surprise. In the new framework in general the Fourier transform with respect to $x$ and Laplace transform with respect to $t$ of a solution do not exist. So, the Briggs-Bers linear stability analysis [22], [23] cannot be applied.

Finally, it has to be noted that the real and the imaginary parts of the initial data for which mode type solutions (i.e. $u' = A\exp i(\omega t - kx)$, $p' = B\exp i(\omega t - kx)$) exist, belong to $X$ for arbitrary real or complex wave numbers $k$.

Now let $Y$ be the set of initial data defined as:

$$Y = \{H = (F,G) : F,G : R^1 \to R^1 \text{ continuously differentiable and bounded}\}$$

$Y$ is a subset of $X$ which does not contain initial data for which modes having complex wave number exist. That is because such an initial data is unbounded.

For $H = (F,G) \in Y$ the couple of functions $H'(x,t) = (u'(x,t), p'(x,t))$ given by (5) is the *unique bounded classical solution* of (4), (6). For every $t \geq 0$ $H'(\cdot,t) \in Y$.

The set of initial data $Y$ is a normed space [27], [28] with respect to the usual algebraic operations and the norm defined as:

$$\|H\|_Y = \max\left\{\sup_{x\in R'}|F(x)|, \sup_{x\in R'}|G(x)|\right\}.$$

A sequence $H_n = (F_n, G_n) \in Y$ converges to $H = (F,G) \in Y$ if $\|H_n - H\|_Y \underset{n\to\infty}{\to} 0$.

It can be seen that if the sequence $H_n$ converges to $H$, then for every $t \geq 0$ the sequence of the corresponding solutions $H_n'(,t) = (u_n'(\cdot,t), p_n'(\cdot,t))$ converges to the solution $H'(\cdot,t) = (u'(\cdot,t), p'(\cdot,t))$ corresponding to $H$. This means that for the set of the initial data $Y$ the initial value problem (4), (6) is well posed in sense of Hadamard on $[0,T]$ for every $T > 0$ [20], [21].

The fact that $H$ is small, can be expressed by saying that $\|H\|_Y < \varepsilon$ for $\varepsilon$ small.

It can be seen that for every prior given $\varepsilon > 0$ if $\|H\|_Y < \varepsilon / \max(1 + c_0\rho_0, 1 + 1/c_0\rho_0)$, then $\|H'(\cdot, t)\|_Y < \varepsilon$ for $t \geq 0$. This means that the constant solution $u = U_0$, $p = p_0$, $\rho = \rho_0$ of (1) is linearly stable with respect to the initial value perturbation with initial data from $Y$.

A significant difference between the results obtained in the frameworks $X$ and $Y$ is that in $Y$ the exponential growth rate of the solutions of the Eqs.(4) is equal to zero. Though in $Y$ for $\operatorname{Re} z > 0$ the Laplace transform with respect to $t$ is defined for every solution the Briggs-Bers linear stability analysis fails. That is because there exist solutions for which the Fourier transform with respect to $x$ cannot be applied. For instance, in the case of the solution $u'(x,t) \equiv 1$ and $p'(x,t) \equiv 1$.

Finally, let $Z$ be the set of initial data defined as:

$$Z = \left\{ H = (F, G) \vdots F, G : R' \to R'; \int_{R'} |F(x)|^2 dx < +\infty \ and \int_{R'} |G(x)|^2 dx < +\infty \right\}.$$

When the linear stability analysis is made by using the Briggs-Bers method [22], [23], the assumptions $\int_{R'} |F(x)|^2 dx < +\infty$ and $\int_{R'} |G(x)|^2 dx < +\infty$ are necessary. That is because the method uses Fourier transform with respect to the spatial variable $x$ of the solution and of the initial condition too.

For $H = (F,G) \in Z$ formulas (5) provide a couple of functions $H'(x,t) = (u'(x,t), p'(x,t))$. In general $H'(x,t)$ is not differentiable. See for instance, $H'(x,t)$ corresponding to $G = F$, where $F : R' \to R'$ is defined as:

$$\begin{cases} F(x) = f_n(x) & for \quad x \in \left[ -2^n - \frac{1}{2^{3n+1}}, -2^n + \frac{1}{2^{3n+1}} \right] \\ F(x) = f_0(x) & for \quad x \in [-1/2, 1/2] \\ F(x) = 0 & for \quad x \in [-1/2, 1/2] \cup \bigcup_{n=1}^{\infty} \left[ -2^n - \frac{1}{2^{3n+1}}, -2^n + \frac{1}{2^{3n+1}} \right] \end{cases} \tag{7}$$

where:

$$f_n(x) = \begin{cases} 2^{4n+1}\left( x + 2^n + \frac{1}{2^{3n+1}} \right) & for \ x \in \left[ -2^n - \frac{1}{2^{3n+1}}, -2^n \right] \\ -2^{4n+1}\left( x + 2^n - \frac{1}{2^{3n+1}} \right) & for \ x \in \left[ -2^n, -2^n + \frac{1}{2^{3n+1}} \right] \end{cases}$$

for $n = 1, 2, 3, \ldots$

$$f_0(x) = \begin{cases} 2(x + 1/2) & for \ x \in [-1/2, 0] \\ -2(x - 1/2) & for \ x \in [0, 1/2] \end{cases}$$

But, if $H = (F,G) \in Z$ is continuously differentiable, then the couple $H'(x,t) = (u'(x,t), p'(x,t))$, given by (5), is continuously differentiable and verifies Eqs.(4).

In general for $H=(F,G)\in Z$ the couple $H'(x,t)$, defined by (5), verifies the initial condition (6).

For $H=(F,G)\in Z$ the couple $H'(x,t)=(u'(x,t),p'(x,t))$, obtained with formula (5), is called *generalized solution* of the initial value problem (4), (6).
For $t$ fixed $H'(\cdot,t)\in Z$ and satisfies the inequality:

$$\|H'(\cdot,t)\|_Z\le \max\left\{1+c_0\rho_0,\,1+\frac{1}{c_0\rho_0}\right\}\cdot\|H'(\cdot,0)\|_Z \tag{8}$$

where the norm in $Z$ is defined as:

$$\|H\|_Z=\max\left\{\int_{R'}|F(x)|^2dx,\int_{R'}|G(x)|^2dx\right\}.$$

Inequality (8) implies that if the sequence $H_n'(\cdot,0)\in Z$ converges in $Z$ to $H'(\cdot,0)\in Z$, then for every $t\ge 0$ the sequence $H_n'(\cdot,t)$ converges in $Z$ to $H'(\cdot,t)$. This means that the initial value problem (4), (6) is well-posed for the set of initial data $Z$.

According to the same inequality, for any $\varepsilon>0$ and $H\in Z$ with $\|H\|_Z<\varepsilon/\max\left\{1+c_0\rho_0,\,1+\frac{1}{c_0\rho_0}\right\}$, the solution $H'(x,t)$ which corresponds to $H$ (i.e. $H'(x,0)=H(x)$) satisfies $\|H'(\cdot,t)\|_Z<\varepsilon$.

This means that the considered constant solution $u=U_0$, $p=p_0$, $\rho=\rho_0$ of (1) is linearly stable with respect to the initial value perturbation with initial data from $Z$. The exponential growth rate of any generalized solution $H'(x,t)=(u'(x,t),p'(x,t))$ defined as:

$$\max\left\{\lim_{t\to+\infty}\frac{\ln\int_{R'}|u'(x,t)|^2dx}{t},\ \lim_{t\to+\infty}\frac{\ln\int_{R'}|p'(x,t)|^2dx}{t}\right\}$$

is equal to zero.
For every generalized solution the Laplace transform is defined for $\operatorname{Re} z>0$ and the Fourier transform with respect to $x$ exists. So, the Briggs-Bers stability analysis can be applied.

What can be strange (mainly for engineers) in this framework is the presence of solutions which satisfy the linearized Euler equations only almost everywhere and the fact that initial data for which modes type solution exists are not in $Z$.

Some conclusions based on the above considerations:

1. In the functional frameworks $X, Y, Z$ the initial value problem is well-posed in sense of Hadamard and the constant 1-D gas flow is linearly stable with respect to the initial value perturbation.

2. Though there exist solutions having strictly positive exponential growth rate, the null solution of the linearized Euler equations is stable with respect to the initial value perturbation from $X$.

3. Even if in the framework $Y$ the exponential growth rate of the solutions of the linearized Euler equations is equal to zero, the Briggs-Bers stability analysis cannot be applied.

4. In the framework $Z$ there exist solutions which are not differentiable. These solutions satisfy Eqs.(4) only in a generalized sense. Only in the frameworks $X$ and $Y$ the solutions satisfy Eqs.(4) in a classical sense.

## 4. COMMENTS

1. The condition that the set of the exponential growth rates of the solutions of the linearized Euler equation has to be bounded from above by zero is not necessary in every functional framework for the initial value problem to be well-posed and the null solution to be linearly stable.
2. Linear stability cannot be denied just because the set of the exponential growth rates of the solutions of the linearized equation is not bounded from above.
3. The condition that the set of the exponential growth rates of the solutions of the linearized Euler equation is bounded from above is not sufficient in every functional framework for the Briggs-Bers stability analysis can be applied.
4. The applicability of the Briggs-Bers stability analysis is not sufficient for the solutions built up by this method to satisfy the linearized Euler equation in a classical sense.
5. When theoretical results are tested against experimental results it is crucial that the computed mathematical variable expresses exactly the measured quantity. This has to be a criterion when the functional framework is chosen. For instance, in this paper there are three different mathematical variables for expressing that the perturbation is small. Since $x \in R'$ (the tube is infinitely long) which one of them can be measured experimentally and how?
6. Since continuous dependence on the initial data is an expression of stability on a finite interval of time $[0,T]$ a natural question occurs: what is really important in practice? Is the linear stability on $[0,+\infty)$ or the fact that the problem is well-posed on any interval $[0,T]$?

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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