A detailed laminar flow field within the normal shock wave considering variable specific heats, viscosity and Prandtl numbers

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DOI: 10.13111/2066-8201.2013.5.1.5

Abstract: *The gas flow field within an 1D normal shock wave at variable specific heats, viscosity and Prandtl numbers with temperature is considered. At* Pr = 0.75 *and constant specific heats and viscosity, the already known analytical solution in a somehow different form is found. At some distance from the wave, the flow is isoenergetical (constant total enthalpy). In order to see if the isoenergetical character of flow within the shock wave is maintained, a method to correct the solution for variable Prandtl number is developed. The obtained solution is close to an analytical one and proves that the deviation from the constant enthalpy hypothesis is less than 0.5%. An interesting thing pointed out is the coexistence of the supersonic and subsonic regimes within the shock wave. Examples of application for air at two Mach numbers are given.*

Key Words: *Prandtl number, dimensionless temperature, stagnation enthalpy, isoenergetical flow*

1. INTRODUCTION

The occurrence of a normal shock wave is possible if a supersonic flow is slowed down by an obstacle or by a counter pressure (Fig.1).

Although important quantitative aspects (pressure, density and velocity jumps) can be expressed without using directly the viscosity, the shock formation is due to it.

If the Navier-Stokes equations are used, more information is obtained, and the important role of the viscosity in such a narrow region is highlighted.

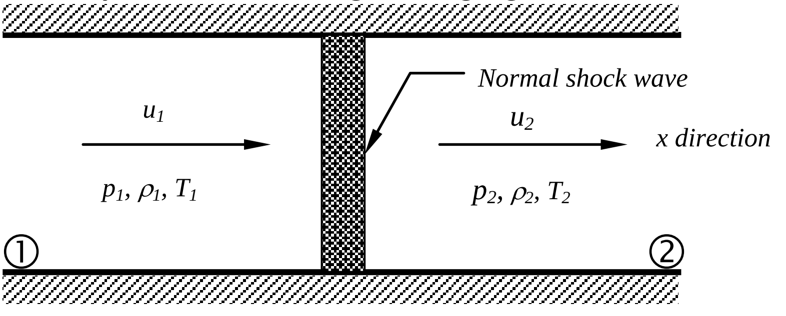

Fig. 1 – The normal shock wave

An analytical solution is known in case of constant specific heats and viscosity [1], for a fixed value of Prandtl number Pr = 0.75, when the total enthalpy is constant throughout the flow field. Due to the high value of the Reynolds number, the shock wave thickness is very small, which explains the good results obtained by treating it as a simple jump.

In case of considering this simple situation, the effect of variation of specific heats was first introduced numerically [2]. Then, the maintaining of the analytical expressions for pressure, temperature and density ratios etc. in terms of an equivalent Mach number was pointed out [3]. Now, along with the specific heats variable with temperature, the effect of viscosity (also variable) is introduced, correcting the assumption Pr = 0.75 as well.

2. THE BASIC EQUATIONS

One writes the conservation equations for steady viscous laminar flow in 1D. By adopting the usual notation *u*, ρ, *p*, *T* for velocity, density, pressure and temperature, one has [4], [5], [6]:

$$d(\rho u) = 0; \;\; \rho u = \text{const.}; \;\; \rho_1 u_1 = \rho_2 u_2 \;\; (\textit{continuity}) \tag{1-a}$$

$$\rho u \cdot du + d\left(p - \frac{4}{3}\mu\frac{du}{dx}\right) = 0 \;\; (\textit{momentum}) \tag{1-b}$$

$$\rho u \cdot dh^* = d\left[\lambda\frac{dT}{dx} + \frac{4}{3}\mu\frac{d}{dx}\left(\frac{u^2}{2}\right)\right] = 0; \;\; h^* = h + \frac{u^2}{2} \;\; (\textit{energy}) \tag{1-c}$$

where h, λ, μ are the enthalpy, the thermal conductivity and the dynamic viscosity respectively, all of them depending on temperature.

It is to be noted that in momentum and energy equations, instead of $4\mu/3$, the quantity $4\mu/3+\zeta$ is often considered (as in [4] and [6]) where ζ is the second coefficient of viscosity. In the following developments, the coefficient ζ will be neglected.

We use the NASA data [7], [8] for specific heats in a slightly modified form [8], by introducing the dimensionless temperature θ, defined by:

$$\theta = \frac{T}{(\Delta T)_{\text{dim}}}; \;\; (\Delta T)_{\text{dim}} = 1000\text{K} \tag{2}$$

One starts with integrating once the energy equation (1- c) and writing it in a form containing the Prandtl number Pr, and the total enthalpy h^*:

$$\rho u\left(h^* - h_1^*\right) = \frac{4}{3}\mu\left[\frac{dh^*}{dx} + \left(\frac{3}{4\,\text{Pr}} - 1\right)\frac{d}{dx}\left(\frac{u^2}{2}\right)\right]; \;\; \text{Pr} = \frac{\mu c_p}{\lambda}; \tag{3}$$

h_1^* being the initial stagnation enthalpy.

Because the derivatives with respect to *x* are equal to zero at large distances from the shock wave (which is located around *x*=0), one yields:

$$h_1^* = h_2^*; \;\; h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{4}$$

By integrating the momentum equation (1-b) once one obtains:

$$\frac{4}{3}\mu\frac{du}{dx} = p + \rho u^2 = p_1 + \rho_1 u_1^2 = p_2 + \rho_1 u_2^2 \tag{5}$$

Again, because the derivatives with respect to x are equal to zero at large distances from the shock wave, the following momentum equality is derived:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_1 u_2^2 \tag{6}$$

3. THE METHOD OF SOLVING

The proposed method of solving consists in the following steps:
a) one finds the state 2 at large distances behind the shock wave, by solving the algebraic system of equations (1- a), (4) and (6); the solution for variable specific heats is taken from [3] and several values are given in Table 1, together with the stagnation temperature T^*;
b) one solves the energy equation (3) under the form:

$$\rho u\left(h^* - h_1^*\right) = \frac{4}{3}\mu\frac{dh^*}{dx}; \quad h^* = h_1^* = \text{const}. \tag{7}$$

where the term denoted as $h_1^* \cdot Trot(z)$ defined by:

$$h_1^* \cdot Trot(z) = \frac{4}{3}\frac{\mu}{\rho u}\left(\frac{3}{4\,\mathrm{Pr}} - 1\right)\frac{d}{dx}\left(\frac{u^2}{2}\right) \tag{8}$$

was neglected (the factor h_1^* being used for obtaining the dimensionless function *Trot* (z) defined in eq.12).

The solution (7) represents the isoenergetical flow, $h^* = h_1^* = \text{const}.$ which is exact at large distances from the shock wave. For constant c_p, μ, Pr, there is an analytical solution as follows:

$$\frac{\rho_1 u_1 x}{\mu_1} = \frac{1}{\beta_1(1-\alpha)}\ln\left[\left(\frac{2}{1-\alpha}\right)^{1-\alpha}\frac{z}{1-\alpha-z}\right]; \quad \beta_1 = \frac{3(k_1+1)}{8k_1}; \quad \alpha = \frac{u_2}{u_1}; \quad z = 1 - \frac{u}{u_1} \tag{9}$$

where z is a dimensionless velocity difference with respect to entrance, varying in the interval (0, 1 - α) and k_1 is the adiabatic exponent of air at the state "1". In case of variable c_p, μ, Pr, one uses the equations [5]:

$$\frac{\mu}{\mu_1} = \frac{\lambda}{\lambda_1} = \left(\frac{\theta}{\theta_1}\right)^{0.5}\frac{1 + C_S/\theta_1}{1 + C_S/\theta}; \quad C_S = 0.122; \quad \mathrm{Pr} = 0.72\left[\frac{c_p(\theta)}{c_p(\theta_1)}\right] \quad \text{(for air)} \tag{10}$$

and the solution is:

$$\frac{\rho_1 u_1 x}{\mu_1} = \frac{4}{3}\int_{z_0}^{z}\frac{F_p(\theta)}{Z(z)}\cdot dz; \quad Z(z) = z - \frac{R_{air}(\Delta T)_{\dim}}{u_1^2}\left(\frac{\theta}{1-z} - \theta_1\right);$$
$$z_0 = \frac{1-\alpha}{2}; \quad \theta = \theta(z) \tag{11}$$

R_{air} being the gas constant for air. The function $\theta = \theta(z)$ is obtained by interpolation (five nodes) using the energy equation (7) $h^* = h_1^* = \text{const}.$
c) by using the energy equation (7), one evaluates the neglected term *Trot* (z) - eq. (8) - under the form:

$$Trot(z) = \left(\frac{3}{4\,\mathrm{Pr}} - 1\right)\frac{u_1^2}{h_1^*} Z(z)(1 - z) \tag{12}$$

that introduced in the energy equation, gives the total enthalpy correction denoted as Csh (z), defined bellow:

$$h^* = h_1^*[1 - Csh(z)];\;\; Csh(z) - Trot(z) = Z(z)\frac{d}{dz}[Csh(z)] \tag{13}$$

The differential equation from (13) is linear and has the solution:

$$Csh(z) = \mathrm{expi}(z)\frac{u_1^2}{h_1^*}\int_z^{z_{max}}\left(\frac{3}{4\,\mathrm{Pr}(w)} - 1\right)\frac{1-w}{\mathrm{expi}(w)}\cdot dw;$$
$$z_{max} = 1 - \alpha;\;\; \mathrm{expi}(z) = e^{\int_{z0}^{z}\frac{dw}{Z(w)}} \tag{14}$$

w being an integration variable.

The condition to determine the integration constant has to be imposed at $z = z_{max}$.

4. EXAMPLES OF APPLICATION

In Table 1, the state 2 far behind the shock wave is given, as well as the stagnation temperature for two initial Mach numbers M_1 = 2 and M_1 = 3 and the initial temperature (dimensionless), θ_1 = 0.300, for variable c_p, μ, Pr .

Table 1 (air; $\theta_1 = 0.300$)

M_1	$u_1.10^{-3}$	u_2 / u_1	θ_2	*Wave thickness* ($\times 10^7$ m)	θ^*	Pr (*interval*)
2	0.6959	0.3727	0.5045	5.992	0.5369	0.720 - 0.738
3	1.0844	0.2517	0.7874	3.491	0.8187	0.720 - 0.785

The shock wave form is given in Fig.2 and 3. The continuous curve (xv (z) – for variable c_p, μ, Pr) indicates a thicker wave, as the viscosity increases with temperature:

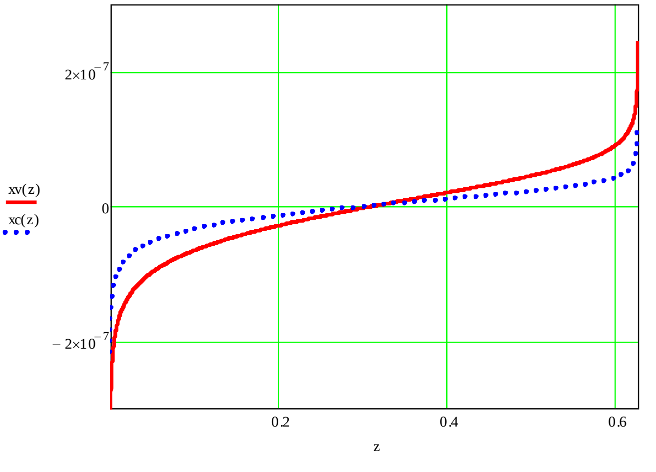

Fig. 2 - M_1= 2; Shock wave form (air)
xv(z) - c_p and μ = variable with temp., xc(z) - c_p and μ = const.

One has defined the wave thickness from the condition for velocities to reach 0.9999 of the values u_1, u_2; the resulted values are given in Table1.

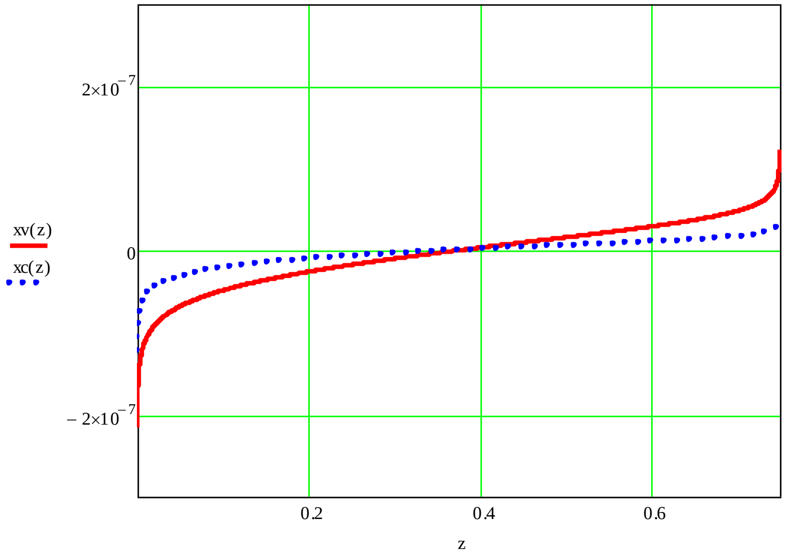

Fig. 3 - M_1 = 3; Shock wave form (air)
xv(z) - c_p and μ = variable with temp., xc(z) - c_p and μ = const.

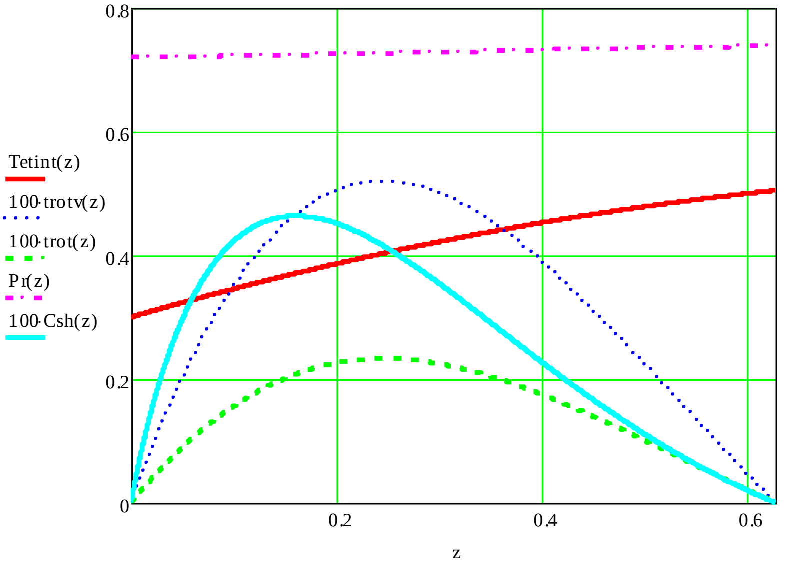

Fig. 4 - Solution corrections; M_1 = 2; $\theta_1 = 0.300$

The wave thickness decreases with flow Mach number. Therefore, one confirms that the “far distance” starts at microns from the wave.

The solution corrections are represented in Fig.4 and 5 for the two Mach numbers for the neglected term in the energy equation (*trot* (*z*)) and for the total enthalpy (*Csh* (*z*)).

Both are small as compared to the initial gas stagnation enthalpy and do not exceed 0.5%. The correction *Csh*(*z*) is smaller at M_1 = 3, because the Prandtl number goes through the value Pr = 0.75 where the correction is zero.

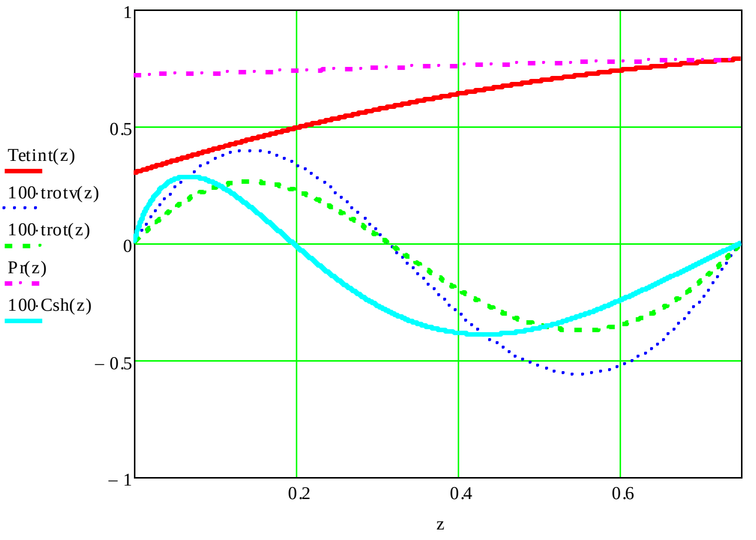

Fig. 5 - Solution corrections; M_1 = 3; $\theta_1 = 0.300$

- Pr (z)- first curve from above;
- Tetint(z)- dimensionless temp.as function of z (interpolation with 5 nodes); sec. curve;
- trotv(z)- evaluation of Prandtl number term neglected in isoenergetical approx. with variable c_p, μ,
 and Pr; (δshv = 3.491E-7 m); - third curve from above;
- trot(z)- evaluation of Prandtl number term neglected in isoenergetical approx. with const. c_p, μ, and
 Pr; (δshc = 1.689E-7 m); - fourth curve from above;
- Csh(z)- stagn. enthalpy correction

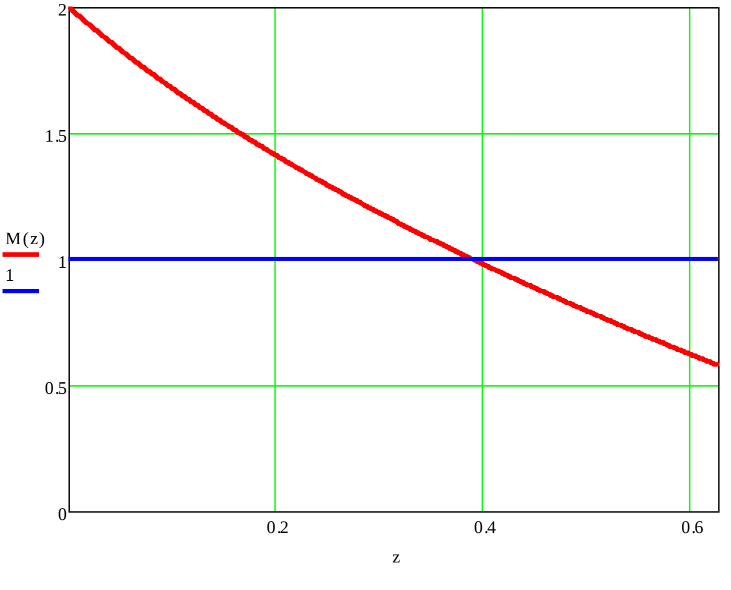

Fig. 6 - The variation of the Mach number through shock wave; the subsonic and supersonic regimes coexist (M_1 = 2; air)

An interesting thing is highlighted in Fig. 6 and 7, namely the coexistence of the supersonic and subsonic regimes within the shock wave.

The passing from the supersonic to subsonic flow is continuous, although very rapid, pointing out the effect of viscosity to make jumps smooth.

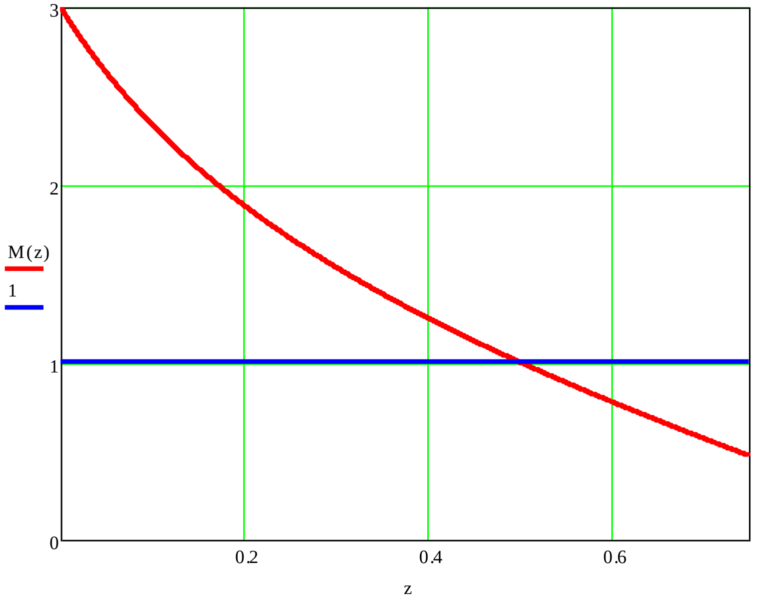

Fig.7 - The variation of the Mach number through shock wave; the subsonic and supersonic regimes coexist (M_1 = 3; air)

5. CONCLUSIONS

An almost analytical solution for the flow within a normal shock wave in laminar regime considering the specific heat, viscosity and Prandtl number variations with temperature was given. The isoenergetical flow is proved to be valid even inside the shock wave with a small error not exceeding 0.5% of the initial stagnation enthalpy.

The wave thickness is very small (tens of microns) and decreases with the Mach number of the incident flow. The increasing of viscosity and specific heats with temperature leads to a thickening of the shock wave.

This fact, together with the coexistence of the supersonic and subsonic regimes inside the shock, highlights the role of the viscosity to make smooth flow jumps. A more complex problem could be related to the oblique shock wave, in a 2D flow, using the existing solutions for the simple jump [9].

As the viscosity increases when the flow is turbulent, it is to be expected a shock wave thickening in turbulent flows [10], [11].

AKNOWLEDGEMENTS

This work has been co-funded by the Sectoral Operational Programme Human Resources Development 2007-2013 of the Romanian Ministry of Labour, Family and Social Protection through the Financial Agreement POSDRU/89/1.5/S/62557.

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