Analytical solutions for some problems of optimum with applications in air traffic and economics

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Abstract: *Analytical solutions for some problems of optimum with applications in air traffic and economics are given. For air traffic minimal distances between commercial airplanes (flight corridors) are imposed considering various trajectories: straightlines, orthodromes and loxodromes. Some other applications are related to target functions submitted to linear or nonlinear restrictions. Although specialized numerical codes exist the analytical solutions are useful giving a more clear understanding, suggesting new ways of approach and providing fast tests for preconception.*

Key Words*: flight corridor, target function, linear/ nonlinear restrictions*

1. INTRODUCTION

Analytical solutions for some problems are in general useful, even when specialized numerical codes exist giving a more clear understanding, suggesting new ways of approach and providing fast tests for preconception.

2. OPTIMIZATIONS WITHOUT RESTRICTIONS

Let's consider two airplanes flying on two straight lines trajectories (D1) and (D2), given by the equations:

$$\left(D_1\right)\ \vec{r}_{D1}=\vec{r}_1+\lambda_1\vec{a}_1;\ \left(D_2\right)\ \vec{r}_{D2}=\vec{r}_2+\lambda_2\vec{a}_2, \tag{1}$$

$\vec{r}_i, \lambda_i, \vec{a}_i, i=\overline{1;2}$ being the position vectors of two fixed points fixe *M1* and *M2*, two variable parameters and the direction unit vectors of the two straight lines trajectories, respectively. It is required to determine the minimum distance between the two trajectories, in order to observe the flight corridors (Fig.1).

Solution

The vectors $\vec{r}_{D1}, \vec{r}_{D2}$ give the positions of two arbitrary points on trajectories. We shall look for a minimum for the module of the vector $\vec{r}_{D2}-\vec{r}_{D1}$, introducing the target function $F(\lambda_1, \lambda_2)$ equal to the module square as below:

$$F(\lambda_1,\lambda_2)=\left(\lambda_2\overrightarrow{a_2}-\lambda_1\overrightarrow{a_1}\right)^2+2\left(\overrightarrow{r_2}-\overrightarrow{r_1}\right).\left(\lambda_2\overrightarrow{a_2}-\lambda_1\overrightarrow{a_1}\right)+\left(\overrightarrow{r_2}-\overrightarrow{r_1}\right)^2 \quad , \tag{2}$$

Because $F(\lambda_1,\lambda_1)$ is pozitive it has always a minimum. In this case necessary extremum conditions are as well:

$$\begin{aligned}&\frac{\partial F}{\partial \lambda_1}=0\ ;\ \lambda_1 a_1^{\ 2}-\lambda_2\left(\overrightarrow{a_1}.\overrightarrow{a_2}\right)=\left(\overrightarrow{r_2}-\overrightarrow{r_1}\right).\overrightarrow{a_1}\\&\frac{\partial F}{\partial \lambda_2}=0;\ -\lambda_1\left(\overrightarrow{a_1}.\overrightarrow{a_2}\right)+\lambda_2 a_2^{\ 2}=-\left(\overrightarrow{r_2}-\overrightarrow{r_1}\right).\overrightarrow{a_2}\end{aligned} \tag{3}$$

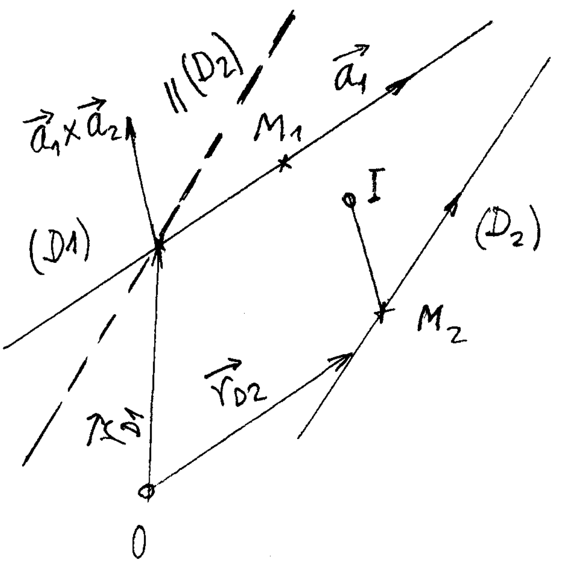

Fig. 1 Distance between two straight lines trajectories

The system (3) has the determinant:

$$\Delta=a^2b^2-\left(\overrightarrow{a_1}.\overrightarrow{a_2}\right)^2=\left|\overrightarrow{a_1}\times\overrightarrow{a_2}\right|^2 \quad , \tag{4}$$

Denoting with $\vec{n}$ the unit vector attached to the vectorial product $\overrightarrow{a_1}\times\overrightarrow{a_2}$, one obtains the values of λ_1,λ_2, coresponding to the minimum distance:

$$\vec{n}=\frac{\overrightarrow{a_1}\times\overrightarrow{a_2}}{\left|\overrightarrow{a_1}\times\overrightarrow{a_2}\right|}\ ;\ \lambda_{1\min}=\frac{\left[\vec{n}\times\left(\vec{r}_2-\vec{r}_1\right)\right].\overrightarrow{a_2}}{\left|\overrightarrow{a_1}\times\overrightarrow{a_2}\right|};\ \lambda_{2\min}=\frac{\left[\vec{n}\times\left(\vec{r}_2-\vec{r}_1\right)\right].\overrightarrow{a_1}}{\left|\overrightarrow{a_1}\times\overrightarrow{a_2}\right|}. \tag{5}$$

The two planes positions for the minimum are:

$$\vec{r}_{D1\min}=\vec{r}_1+\lambda_{1\min}\vec{a}_1;\ \ \vec{r}_{D2\min}=\vec{r}_2+\lambda_{2\min}\vec{a}_2\,. \tag{6}$$

By working out some algebra one yelds a simple expression for the square distance

$$F(\lambda_1,\lambda_2)=\vec{n}\left[\vec{n}.\left(\vec{r}_2-\vec{r}_1\right)\right]\ \left(\vec{r}_{D2}-\vec{r}_{D1}\right)^2=\left[\vec{n}.\left(\vec{r}_2-\vec{r}_1\right)\right]^2\ . \tag{7}$$

In this way one has both the minimum distance and the plane positions.

Exemples

1. Let the points M_1, M_2 be given by the position vectors:

$\vec{r}_1=(12;16;30)^T$; $\vec{r}_2=(20;14;35)^T$; $\vec{r}_2-\vec{r}_1=8\overrightarrow{e_1}-2\overrightarrow{e_2}+5\overrightarrow{e_3}$,

and the direction unit vectors of the two straight lines trajectories:

$\overrightarrow{a_1}=(2;3;1.5)^T ; \overrightarrow{a_2}=(2;5;-0.5)^T$.

One calculates the vectorial product: $\overrightarrow{a_1}\times\overrightarrow{a_2}$, its unit vector $\vec{n}$ and some other necessary quantities:

$\overrightarrow{a_1}\times\overrightarrow{a_2}=-9\overrightarrow{e_1}+4\overrightarrow{e_2}+4\overrightarrow{e_3}$; $\left|\overrightarrow{a_1}\times\overrightarrow{a_2}\right|=\sqrt{113}$; $\vec{n}=\left(-9\overrightarrow{e_1}+4\overrightarrow{e_2}+4\overrightarrow{e_3}\right)/\sqrt{113}$,

$\sqrt{113}\,\vec{n}\times\left(\vec{r}_2-\vec{r}_1\right)=28\overrightarrow{e_1}+77\overrightarrow{e_2}-14\overrightarrow{e_3}$.

The minimum distance, $\lambda_{1min}, \lambda_{2min}$ and the plane positions are then:

$d_{min}=\left|\vec{n}.\left(\vec{r}_2-\vec{r}_1\right)\right|=\left|\dfrac{-60}{\sqrt{113}}\right|=5.64433 ; \lambda_{1min}=\dfrac{448}{113}=3.9646 ;\ \lambda_{2min}=\dfrac{266}{113}=2.35398 = 364.433\ m.$

$\vec{r}_{D1min}=\vec{r}_1+\lambda_{1min}\overrightarrow{a_1}$; $\vec{r}_{D2min}=\vec{r}_2+\lambda_{2min}\overrightarrow{a_2}$.

2.1 The general case: the minimum distance between two curves in space

Let the trajectories of the airplanes A_1, A_2, be given by the equations:

$$\vec{r}_{A1}=\vec{r}_{A1}(\lambda_1);\ \vec{r}_{A2}=\vec{r}_{A2}(\lambda_2), \tag{8}$$

the parameter λ_1, λ_2 being independent. The problem to be solved is:

$$d=\left|\vec{r}_{A2}(\lambda_2)-\vec{r}_{A1}(\lambda_1)\right|=min., or\ d^2=\left|\vec{r}_{A2}(\lambda_2)-\vec{r}_{A1}(\lambda_1)\right|^2=F(\lambda_1,\lambda_2)=min. \tag{9}$$

$$d^2=F(\lambda_1,\lambda_2)=r_{A1}^2+r_{A2}^2-2(\overrightarrow{r_{A1}}.\overrightarrow{r_{A2}}), \tag{10}$$

The necessary and sufficient conditions for minimum are:

$$\frac{\partial F(\lambda_1,\lambda_2)}{\partial\lambda_1}=0;\ \frac{\partial F(\lambda_1,\lambda_2)}{\partial\lambda_2}=0, \tag{11}$$

leading to the system of equations:

$$r_{A1}\frac{dr_{A1}}{d\lambda_1}-\overrightarrow{r_{A2}}.\frac{d\overrightarrow{r_{A1}}}{d\lambda_1}=0; r_{A2}\frac{dr_{A2}}{d\lambda_2}-\overrightarrow{r_{A1}}.\frac{d\overrightarrow{r_{A2}}}{d\lambda_2}=0. \tag{12}$$

The vectors $\overrightarrow{r_{A1}}, \overrightarrow{r_{A2}}$ being given, the nonlinear system (12) provides the unknowns $\lambda_{1min}, \lambda_{2min}$ and, with them,the minimum position.

2.2 Orthodrome. Loxodrome. Minimum distance

1. For the orthodrome we have by definition an intersection of a sphere of radius (R_m+h) (R_m the local mean radius of earth); h-the height of flight with a plane passing through its the center:

$$\begin{cases} R = R_m + h = const = R_{flight}(=R_f); \\ ax + by + cz = 0 \quad ; \end{cases} \tag{13}$$

$$a=y_D z_A-y_A z_D \quad ; \quad b=z_D x_A-z_A x_D \quad ; \quad c=x_D y_A-x_A y_D. \tag{14}$$

We use the spherical coordinates to write:

$$x = R_f \sin\theta\cos\omega;\ y = R_f \sin\theta\sin\omega; z = R_f \cos\theta; \tag{15}$$

From (13) one yields $(R_f \neq 0)$:

$$a \sin\theta\cos\omega + b \sin\theta\sin\omega + c \cos\theta = 0\ . \tag{16}$$

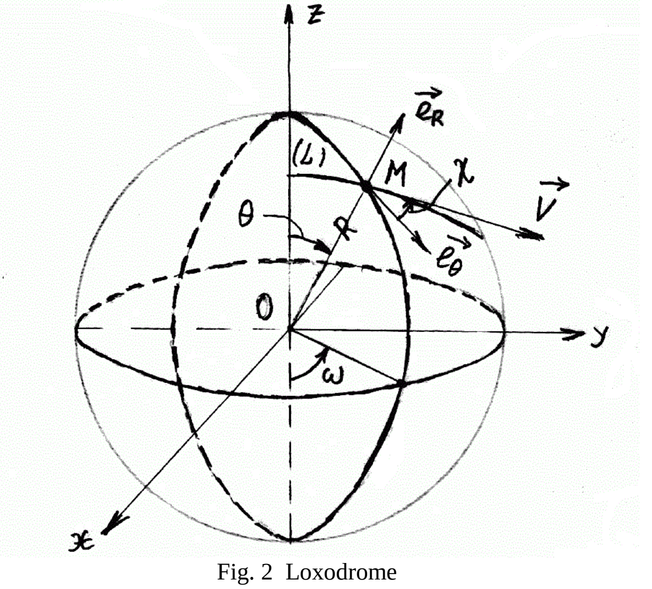

Fig. 2 Loxodrome

The relation (16) can be used to express one of the coordinates ω, θ as function of the remaining one, for exemple:

- for $c = 0$: $(a \cos\omega + b \sin\omega)=0$, $(\theta\neq 0)$, (17)

$$tg\omega_0 = -\frac{a}{b} = const. \quad (\theta\neq 0), \tag{18}$$

one obtains the flight along a meridian $(\omega_0=-arctg\frac{a}{b})$.

The orthodrome is:

$$x=R_f \cos\omega_0 \sin\theta \quad ; \tag{19-a}$$

$$y=R_f \sin\omega_0 \sin\theta \;;\quad z=R_f \cos\theta\,, \tag{19-b}$$

written as a space curve function of one parameter θ.

• for $c\neq 0$ one can express simpler θ as a function of ω:

$$ctg\,\theta = -\frac{a}{c}\cos\omega - \frac{b}{c}\sin\omega \;;\theta = atan\left(\frac{-c}{a\cos\omega + b\sin\omega}\right), c\neq 0 \tag{20}$$

2. For a **loxodrome** one imposes an angle χ between the flight velocity and the local meridian:

$$\frac{\vec{V}\cdot\vec{e}_\theta}{\left|\vec{V}\right|} = \cos\chi = const. \;;\; V_\theta = \left|\vec{V}\right|\cos\chi\,;\chi = \left(\overrightarrow{e_\theta},\vec{V}\right)\,, \tag{21}$$

because the direction of the meridian is given by the unitvector $\vec{e}_\theta$.

Considering the velocity constant in module, by (21) the velocity component V_θ is imposed which represents a relation between the two parameters ω, θ. The simplest way is to consider the spherical coordinates. The velocity is then written as follows:

$$\vec{V} = R_f\left(\frac{d\omega}{dt}\sin\theta\,\vec{e}_\omega + \frac{d\theta}{dt}\vec{e}_\theta\right);\, V^2 = R^2\left(\left(\frac{d\omega}{dt}\sin\theta\right)^2 + \left(\frac{d\theta}{dt}\right)^2\right);\, V_R = 0; \tag{22}$$

$$\left|\vec{V}\right| = R_f\sqrt{\left(\frac{d\omega}{dt}\right)^2\sin^2\theta + \left(\frac{d\theta}{dt}\right)^2}\;;\, V_\omega = R_f\frac{d\omega}{dt}\sin\theta;\; V_\theta = R_f\frac{d\theta}{dt}\;. \tag{22-a}$$

One obtains a differential equation of the first degree:

$$\frac{d\theta}{\sin\theta}\tan\chi = d\omega\;;\;\tan\chi = \frac{V_\omega}{V_\theta} = const. \tag{23}$$

The equation (23) can be solved analyticaly starting from $\omega=\omega_D$,$\theta=\theta_D$; χ is taken as parameter to meet the arrival point $\omega=\omega_A$,$\theta=\theta_A$.

The solution is:

$$\omega = \omega_0 + \tan\chi\ln\left(\tan\frac{\theta}{2}\right)\;. \tag{24}$$

For an **orthodrome along a meridian**, one has a loxodrome with $\chi = 0$ and the equation (24) becames: $\omega = \omega_0$, ω_0 being calculated from the initial conditions. The velocity is $V = V_\theta$, the flight velocity on the orthodrome. Therefore the parametrical equations are:

- for orhodrome:

$$R=R_f\;;\;\; \omega=\omega;\; \theta = atan\left(\frac{-c}{a\cos\omega + b\sin\omega}\right) + j\pi, c\neq 0, j = (0,1), \theta\in(0;\pi), \tag{25}$$

the parameter being ω. For $c = 0$, one obtains a flight on a vertical, ω being undetermined;

- for loxodrome:

$$R = R_f \quad ; \ \omega = \omega_0 + tan\chi.lntan\frac{\theta}{2}; \ \theta = \theta, \tag{26}$$

the parameter being θ.

In the above relations, the parameters *a,b,c* are given by relations (14).

The loxodrome angle can also be expessed using the coordinates of the departure and arrival points *D, A:*

$$tan\chi = \frac{\omega_D - \omega_A}{ln\left(tan\frac{\theta_D}{2}/tan\frac{\theta_A}{2}\right)} \ ; \ \omega_0 = \omega_D - tan\chi.lntan\frac{\theta_D}{2} \ ; \tag{27}$$

$$tan\frac{\omega_{D,A}}{2} = \frac{y_{D,A}}{x_{D,A}}; \ \cos\theta_{D,A} = \frac{z_{D,A}}{R_f}. \tag{28}$$

Problem 2. Find the minimal distance and its location for: a) two orthodromes; b) one orthodrome and a loxodrome; c) two loxodromes, in horizontal flight.

Solution.

Let the orthodrome and the loxodrome be given by equations (25) and (26) in the forms:

$$R = R_m + h_{io}; \ \ \omega = \omega; \ \theta = atan\left(\frac{-c_i}{a_i\cos\omega + b_i\sin\omega}\right) + j\pi, c_i \neq 0, \ i = 1;2; \ j = 0;1; \tag{29}$$

$$R = R_m + h_{il} \ \ ; \ \omega = \omega_{0i} + tan\chi_i.ln\left(tan\frac{\theta}{2}\right); \ i = 1;2, \tag{30}$$

the altitude *h* and and the navigation angles being diferent.

In horizontal flight the altitudes are constant for each trajectory, and the minimal distances are the altitude differences:

$$d_{o,min} = \left|h_{2o} - h_{1o}\right| \ ; \ \ d_{l,min} = \left|h_{2l} - h_{1l}\right| \ ,; \ \ i = 1;2, \tag{31}$$

for orthodromes and for loxodromes. As regards the minimum location, one needs to satisfy the conditions:

$$\overrightarrow{e_{R1}} = \overrightarrow{e_{R2}}; \theta_1 = \theta_2; \omega_1 = \omega_2, \tag{32}$$

giving the minimum distance: according to (2.3), the distance function $F(\lambda_1, \lambda_2)$, for $\overrightarrow{r_{A1}} = const., \overrightarrow{r_{A2}} = const.$ is minimal when the scalar product $\overrightarrow{r_{A1}}.\overrightarrow{r_{A2}}$ is maximum for (32).

Case a). The equality conditions of coordinates $\omega \ and \ \theta$ (320 lead to:

$$\begin{aligned} &\frac{-c_1}{a_1\cos\omega_{min} + b_1\sin\omega_{min}} = \frac{-c_2}{a_2\cos\omega_{min} + b_2\sin\omega_{min}}; \\ &tan\omega_{min} = \frac{a_2c_1 - a_1c_2}{c_2b_1 - c_1b_2} \ ; \ \theta_{min} = atan\left(\frac{-c_1}{a_1\cos\omega_{min} + b_1\sin\omega_{min}}\right) \end{aligned} \tag{33}$$

Case b). The equality conditions of coordinates ω *and* θ lead to:

$$\omega_{min} = \omega_{02} + tan\chi_2 .ln\left| tan\frac{1}{2} atan(\frac{-c_1}{a_1 \cos\omega_{min} + b_1 \sin\omega_{min}})\right| . \tag{34}$$

After the angle ω_{min} is obtained from (3.22), one calculates the angle θ_{min} from relation:

$$\theta_{min} = atan(\frac{-c_1}{a_1 \cos\omega_{min} + b_1 \sin\omega_{min}}) . \tag{35}$$

Case c). The equality conditions of coordinates ω *and* θ (32) lead to:

$$\begin{gathered} \omega_{01} + tan\chi_1 .lntan\frac{\theta_{min}}{2} = \omega_{02} + tan\chi_2 .lntan\frac{\theta_{min}}{2} ; \\ tan\frac{\theta_{min}}{2} = exp(\frac{\omega_{02} - \omega_{01}}{tan\chi_1 - tan\chi_2}) ; \omega_{min} = \frac{\omega_{02} tan\chi_1 - \omega_{01} tan\chi_2}{tan\chi_1 - tan\chi_2} . \end{gathered} \tag{36}$$

Example. Find the location of the possible minimal distance between the trajectories of the flights: 1) London-Rio de Janeiro; 2) New York- Cape Town.

Solution. The polar and carthesian coordinates and *a,b,c* are:

a) If the two trajectories are orthodromes, the location of the minimal distance is: $\omega_{\min ort} = -29.172 \deg .;$

$\theta_{\min ort} = 78.893 \deg .$ a point above the Atlantic Ocean (latitude 90- 78.893= 11.107 deg.).

If the two trajectories are loxodromes, the coordinates for the minimum distance are: $\omega_{\min lox} = -28.515 \deg .;\ \theta_{\min ort} = 79.285 \deg .$ very close to orthodromes. The loxodrome directions are:

$$\tan\chi 1 = -0.60501; \tan\chi 2 = 1.007;\ \chi 1 = -31.174 \deg .;\ \chi 2 = 45.225 \deg .$$

$$\begin{pmatrix} & L & RJ & NY & CT \\ R\,km & 6400 & 6400 & 6400 & 6400 \\ \omega \deg & 0 & -50 & -63 & 20 \\ \theta \deg & 40 & 114 & 49 & 125 \\ x\,km & 4113.8 & 3758.2 & 2192.8 & 4926.4 \\ y\,km & 0.000 & -4478.8 & -4303.7 & 1793.1 \\ z\,km & 4902.7 & -2603.1 & 4198.8 & -3672.9 \end{pmatrix} ; \begin{pmatrix} & L-RJ & NY-CT \\ a.10^{-7} & 2.1958 & 0.82696 \\ b.10^{-7} & 2.9132 & 2.8734 \\ c.10^{-7} & -1.8422 & 2.5135 \end{pmatrix} \tag{37}$$

3. OPTIMIZATIONS WITH LINEAR TARGET FUNCION AND NON LINEAR RESTICTION

Let the target function $F : \mathbb{R}^3 \to \mathbb{R}$, be linear:

$$F=\sum \alpha_i\, x_j\ ;\ \alpha_j,\ \ j=\overline{1;\,3,,}\,. \tag{38}$$

$a_j,\ j=\overline{1;3}$ are known coefficients.

One looks for the minimum (maximum) with the restriction (ellipsoid):

$$(E)\left(\frac{x_1-w_1}{a_1}\right)^2+\left(\frac{x_2-w_2}{a_2}\right)^2+\left(\frac{x_3-w_3}{a_3}\right)^2=1, \tag{39}$$

$$a_i=\frac{x_{iM}-x_{im}}{2};i=\overline{1;3},\ w_i=\frac{x_{iM}+x_{im}}{2};i=\overline{1;3}, \tag{40}$$

where $x_{im},x_{iM};i=\overline{1;3}$, are minimum (maximum) values, for x_i, and $w_i,i=\overline{1;3}$, are coordinateles of the ellipsoid centre, respectively.

In particular, in a problem of economics, $x_i,\ i=\overline{1;3}$, represent the cost prices for material, labour and overhead on various markets in different locations.

After an interval of time by the mecanism of supply and demand the minimum prices in the poor regions and the maximum prices in the reach regions defining a paralellepiped are rounded on an ellipsoid (39), whereas $\alpha_j,\ j=\overline{1;\,3}$, giving the amounts for a product, are constant.

Thus one will look for a minimum possible price for production and for a maximum possible price for sale.

Solution.

We look for an analytical solution also useful for a numerical code testing.
To this aim one transforms the ellipsoid (39) in a sphere,by introducing the new coordinates $X_i,i=\overline{1;3}$, defined as bellow:

$$X_i=\frac{x_i-w_i}{a_i},\ i=\overline{1;\,3;}\quad (S)\ \sum_{i=1}^{3}X_i^2=1\ , \tag{41}$$

The target function becomes:

$$F=\sum_{j=1}^{3}\left(\alpha_j\,a_j\,X_j+\alpha_j\,w_j\right);\ grad_x\,F=g_x=(\alpha_j\,a_j)^T \tag{42}$$

Restriction (39) led to a sphere with the centre in origine (S), of the radious unity (41), whereas the target function represents a plane (*P*); it decreases in the gradient g_X (relation (30)) direction.

The normal *(N)* to the plan (P), passing through the sphere centre *O* (Fig.1) has the equation:

$$(N)\ \sum_{j=1}^{3}\left(\alpha_j\,a_j\,X_j\right)=0 \tag{43}$$

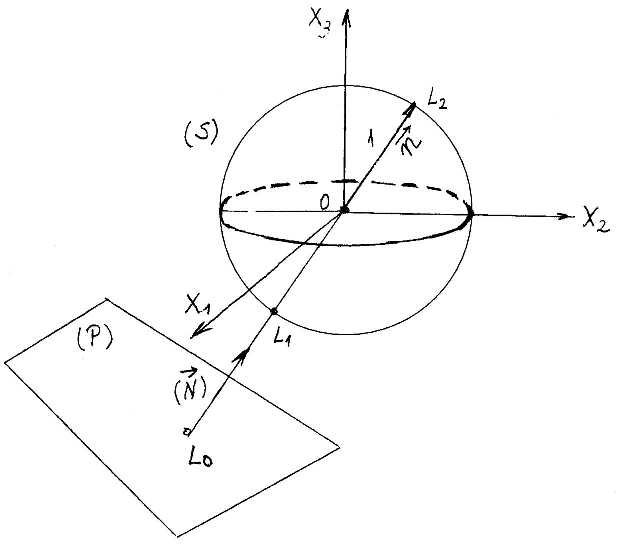

Fig. 3 Minimum and maximum with one restriction *(S)*

The normal *(N)* intersects the sphere *(S)* in two points: *L1* and *L2*; these points represent the looked for the minimum and for the máximum, respectively. Their coordinates are:

$$X_{jL1} = -\frac{g_{Xj}}{|g_X|} = -\frac{\alpha_j a_j}{|g_X|}\ ;\ X_{jL2} = -X_{jL1}\ ;\ |g_X| = \left(\sum_{j=1}^{3}(\alpha_j a_j)^2\right) \tag{44}$$

By replacing these coordinates in the target function (3), one gets:

$$min F = \sum_{j=1}^{3}(\alpha_j w_j) - |g_X|;\ max F = \sum_{j=1}^{3}(\alpha_j w_j) + |g_X|\ . \tag{45}$$

The coordinates coresponding to *L1*, *L2*, on the ellipsoid *(E)*, are:

$$x_{jLi} = a_j X_{jLi} + w_j\ ;\ j = \overline{1;3};\ i = \overline{1;2}\ . \tag{46}$$

Numerical results. The following vectors are given α, x_m, x_M (UM/product/per hour - labour, kg - materials and overhead: $\alpha = (0.3; 1.2; 1.0)^T$; $x_m = (1.0; 0.5; 0.2)^T$; $x_M = (8.0; 3.0; 0.8)^T$. The values are presented in Table 1

Table 1

a_1	a_2	a_3	w_1	w_{21}	w_3	F_{L1}	F_{L2}
3.50	1.25	0.30	4.50	1.75	0.50	2.0946	5.8034

4. CONCLUSIONS

Analytical solutions for several problems of optimum with/without restictions are given. For the minimum distance between two stright lines trajectories, as well as for the position of minimum explicite compact formulas are obtained. The general formulation for the arbitrary curve trajectories is also presented. Aplications for flight on orthodromes and loxodromes are solved by using spherical coordinations and original parametrical representations proposed by the authors.

An application to price optimization for both production and sale considering material, labour and overhead costs is developed, and treated in conditions of market competition.

Numerical examples are also presented.

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