Using the gauge condition to simplify the elastodynamic analysis of guided wave propagation

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Abstract: *In this article, gauge condition in elastodynamics is explored more to revive its potential capability of simplifying wave propagation problems in elastic medium. The inception of gauge condition in elastodynamics happens from the Navier-Lame equations upon application of Helmholtz theorem. In order to solve the elastic wave problems by potential function approach, the gauge condition provides the necessary conditions for the potential functions. The gauge condition may be considered as the superposition of the separate gauge conditions of Lamb waves and shear horizontal (SH) guided waves respectively, and thus, it may be resolved into corresponding gauges of Lamb waves and SH waves. The manipulation and proper choice of the gauge condition does not violate the classical solutions of elastic waves in plates; rather, it simplifies the problems. The gauge condition allows to obtain the analytical solution of complicated problems in a simplified manner.*

Key Words: *Elastodynamics, Navier-Lame equations, Gauge condition, Lamb Wave, SH wave*

1. INTRODUCTION

In elastodynamics, the equations of motion for homogeneous isotropic linearly elastic solids are represented by the Navier-Lame equations, in vector form,

$$(\lambda+\mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})+\mu\nabla^2\vec{u}=\rho\ddot{\vec{u}} \tag{1}$$

where, $\vec{u}$ is the displacement vector, ρ is the density, λ and μ are the Lame constants.

To construct the solutions of Navier-Lame equations, the displacement fields can be considered as the superposition of the gradient of scalar potential Φ and the curl of the vector potential $\vec{H}$. Use the Helmholtz theorem (mentioned originally in ref. [1] and then in its translated version [2]) to write

$$\vec{u}=grad\,\Phi+curl\vec{H}=\vec{\nabla}\Phi+\vec{\nabla}\times\vec{H} \tag{2}$$

The potentials Φ and $\vec{H}$ satisfy the wave equation, i.e.,

$$c_p^2\nabla^2\Phi=\ddot{\Phi};\qquad c_s^2\nabla^2\vec{H}=\ddot{\vec{H}} \tag{3}$$

where, c_p and c_s are the pressure and shear wavespeeds, respectively.

It can be noted from Eq. (2), in three dimensions, the three components of displacement are represented by four components of potentials.
Thus, an additional unknown exists. In order to ensure the uniqueness of the solution, Eq. (2) is complemented by the gauge condition [1], i.e.,

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{4}$$

The gauge condition is needed to mitigate the requirement of the additional unknown in the potential formulation.
However, the formula given in Eq. (4) is not the only possible form of the gauge condition; in fact, a multitude of alternative forms exist [3] as used in elastodynamics [4] [5] (pg. 465), and electrodynamics [6] [7] [8].

1.1 General guided wave solution in terms of potentials

Meeker and Meitzler [9] developed the general solution for y-invariant straight-crested guided waves (Figure 1) using the Helmholtz potentials

$$\Phi = (A\cos\alpha z + B\sin\alpha z)e^{i(\xi x-\omega t)} \tag{5}$$

$$H_x = (C\cos\beta z + D\sin\beta z)e^{i(\xi x-\omega t)} \tag{6}$$

$$H_y = (E\cos\beta z + F\sin\beta z)e^{i(\xi x-\omega t)} \tag{7}$$

$$H_z = (G\cos\beta z + H\sin\beta z)e^{i(\xi x-\omega t)} \tag{8}$$

and the gauge condition

$$\vec{\nabla} \cdot \vec{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_z}{\partial z} = 0 \tag{9}$$

The constants A, B, C, D, E, F, G, H are eight unknowns to be determined from the six traction free boundary conditions on the top and bottom boundaries of the plate. Because the number of unknowns (8) is greater than the number of conditions (6), the gauge condition Eq. (9) is used to produce two additional conditions. This is done by evaluating the gauge condition at the top and bottom surfaces of the plate.
In order to produce the required additional equations, Graff [5] suggested to substitute the complex-valued H_x, H_z into equation Eq. (9) and to separate them into real and imaginary parts to produce four equations with four unknowns.
However, the traction-free boundary condition equations were not separated into real and imaginary parts.
This complication may explain why the solution of SH waves is usually expressed in terms of displacement although the Lamb waves are elegantly solved using potentials functions [5] [10] [11]. Thus, the gauge condition seems to remain a redundant condition in these classical solutions.

1.2 General solution in terms of displacements

Alternatively, Achenbach [12] [13] proposed a guided wave solution using an ingenious definition of the displacements that utilizes the solution of membrane wave equation, i.e.,

$$\begin{aligned} u_x^n &= \frac{1}{k_n} V^n(z) \frac{\partial \varphi}{\partial x}(x,y) \\ u_y^n &= \frac{1}{k_n} V^n(z) \frac{\partial \varphi}{\partial y}(x,y) \qquad \text{(Lamb wave)} \\ u_z^n &= W^n(z)\varphi(x,y) \end{aligned} \tag{10}$$

$$\begin{aligned} u_x^n &= \frac{1}{l_n} U^n(z) \frac{\partial \psi}{\partial y}(x,y) \\ u_y^n &= -\frac{1}{l_n} U^n(z) \frac{\partial \psi}{\partial x}(x,y) \qquad \text{(SH wave)} \\ u_z^n &= 0 \end{aligned} \tag{11}$$

where k and l are wavenumber-like quantities and the functions φ, ψ satisfies the membrane wave equation. (The details can be found in ref. [12] [13])
However, this approach does not involve the wave equation since the displacement satisfies the Navier-Lame equation but not the wave equation.

1.3 The scope of this article

In this article, we propose a unified potential-based solution to the guided wave propagation that is simpler (and has fewer unknowns) than that of ref. [4] and [5]. We will show that it is possible to reduce the eight unknowns of Eqs. (5)-(8) to only six unknowns by the proper utilization and manipulation of the gauge condition and thus, produce a much simpler solution of the guided wave propagation problem.
The origin of the gauge condition in elastodynamics is discussed in Section 0; it will be shown that the gauge condition can be chosen arbitrarily within certain limits. The different forms of the gauge condition in electrodynamics are discussed is Section 0.
The proper choice and manipulation of the gauge condition of elastodynamics is discussed in Section 0; our manipulation on the gauge condition does not violate the fundamental elastodynamics assumptions. The use of the proposed method is demonstrated on two classical problems, i.e., the straight crested and the circular crested guided wave propagation in a uniform plate.

2. GOVERNING EQUATIONS AND ORIGIN OF GAUGE CONDITION IN ELASTODYNAMICS

2.1 Origin of the gauge condition in elastodynamics

The backbone of classical elastodynamics is the Navier-Lame equations [5]. The origin of the gauge condition can be traced to the Navier-Lame equations as follows:
Substitute Eq. (2) into Eq. (1) to get

$$(\lambda+\mu)\vec{\nabla}\left\{\vec{\nabla}\cdot(\vec{\nabla}\Phi+\vec{\nabla}\times\vec{H})\right\}+\mu\nabla^2(\vec{\nabla}\Phi+\vec{\nabla}\times\vec{H})=\rho(\vec{\nabla}\ddot{\Phi}+\vec{\nabla}\times\ddot{\vec{H}}) \tag{12}$$

Upon rearrangement,

$$(\lambda+2\mu)\vec{\nabla}(\nabla^2\Phi)+\mu\vec{\nabla}\times(\nabla^2\vec{H})+(\lambda+\mu)\vec{\nabla}\left\{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})\right\}=\rho(\vec{\nabla}\ddot{\Phi}+\vec{\nabla}\times\ddot{\vec{H}}) \tag{13}$$

Recall the general vector property $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})=0$ (divergence of any curl is zero); hence, the third term of Eq. (13) drops out. Combining the similar potential functions, Eq. (13) can be written as

$$\vec{\nabla}\left\{(\lambda+2\mu)\nabla^2\Phi-\rho\ddot{\Phi}\right\}+\vec{\nabla}\times(\mu\nabla^2\vec{H}-\rho\ddot{\vec{H}})=0 \tag{14}$$

Eq. (14) is separated into two wave equations

$$(\lambda+2\mu)\nabla^2\Phi-\rho\ddot{\Phi}=0 \quad \text{(scalar wave equation)} \tag{15}$$

$$\mu\nabla^2\vec{H}-\rho\ddot{\vec{H}}=0 \quad \text{(vector wave equation)} \tag{16}$$

Assuming harmonic time variation with circular frequency ω and defining $c_p=\sqrt{(\lambda+2\mu)/\rho}$, $c_s=\sqrt{\mu/\rho}$, Eq. (15) and (16) become

$$\nabla^2\Phi+\frac{\omega^2}{c_p^2}\Phi=0 \tag{17}$$

$$\nabla^2\vec{H}+\frac{\omega^2}{c_s^2}\vec{H}=0 \tag{18}$$

Eq. (17) indicates that the scalar potential Φ propagates with the pressure wave speed c_p, whereas Eq. (18) indicates that the vector potential $\vec{H}$ propagates with the shear wave speed c_s. It can be shown that the pressure waves are *irrotational waves* i.e., have zero rotation, whereas the shear waves are *equivolume waves*, i.e., they have zero dilatation and are known as *distortional waves* [5]. From now on, we call the scalar potential Φ as *pressure potential* and the vector potential $\vec{H}$ as *shear potential*.

2.2 Inception of the gauge condition

Now let's take a look at the dropped out term in Eq. (13), i.e.,

$$(\lambda+\mu)\vec{\nabla}\left\{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})\right\}=\vec{0} \tag{19}$$

Using the vector property $\vec{\nabla}\left\{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})\right\}=\vec{\nabla}\times\left\{\vec{\nabla}(\vec{\nabla}\cdot\vec{H})\right\}$, Eq. (19) can be written as

$$(\lambda+\mu)\vec{\nabla}\times\left\{\vec{\nabla}(\vec{\nabla}\cdot\vec{H})\right\}=\vec{0} \tag{20}$$

But $(\lambda+\mu)\neq 0$, hence, Eq. (20) can be written as

$$\vec{\nabla}\times\left\{\vec{\nabla}(\vec{\nabla}\cdot\vec{H})\right\}=\vec{0} \tag{21}$$

Let $\vec{\nabla}\cdot\vec{H}=\Gamma$, a scalar quantity; then, Eq. (21) becomes

$$\vec{\nabla} \times \left\{ \vec{\nabla} \Gamma) \right\} = \vec{0} \tag{22}$$

Eq. (22) represents the vector property that curl of any gradient field is zero. Thus, Γ can be chosen arbitrarily without affecting the generality of the solution; this is called *gauge invariance*. This is similar to the *gauge invariance* used in solving the Maxwell's equations in electrodynamics through the potential approach (see section 2.5 of chapter 2 of ref. [3]). Owing to the uniqueness of the physical problem, any solution that satisfies the Navier-Lame equations be the unique solution to the problem, regardless of the value assumed by Γ.

The selection of the gauge depends on the nature of the problem. The simplest gauge condition may be selected as $\Gamma = \vec{\nabla} \cdot \vec{\mathrm{H}} = 0$ which is similar to the Coulomb gauge [7] in electrodynamics. The physical quantities such as displacements and stresses do not depend on the choice of the gauge for a problem with unique solution. However, the proper choices of gauge make the problems easier to solve. As an example, Gazis used $\Gamma = F(\vec{r}, t)$ [4] in order to simplify the shear potentials when developing the solution of wave propagation in a hollow cylinder. To avoid any confusion on the gauge condition, we can quote from ref. [14] a statement on the gauge condition used in electrodynamics. "As a rule, one should keep in mind that there are no 'right' or 'wrong' admissible gauge choices. Any proper gauge will lead to the same values of gauge invariant quantities. But, depending on an actual problem, a certain gauge can be more appropriate than others."

Therefore, the gauge condition may be used to simplify the problem. It is noted that the gauge condition does not depend on the pressure potential Φ; rather, it depends only on the shear potential $\vec{H}$. The proper choice and manipulation of the gauge condition should simplify complicated wave problems.

3. DIFFERENT FORMS OF GAUGE CONDITION IN ELECTRODYNAMICS

Helmholtz theorem gained its popularity for simplifying the problems in numerous fields of physics: hydrodynamics, elastodynamics, electrodynamics etc. In electrodynamics, Maxwell's equations are solved using Helmholtz potential functions with a gauge condition that is not necessarily to be zero; rather, actual fields are invariant of the gauge condition. The choice of gauge is arbitrary and does not change the physical quantities, and the potential functions are adjusted according to the choice of gauge [6]. However, a certain gauge may be more appropriate than a random choice and may make the problem easier to solve analytically. Researchers in electrodynamics have taken advantage of this by utilizing various forms of the gauge condition to solve various problems in classical electrodynamics and quantum electrodynamics.

Different choices of the gauge condition have already been used to solve different problems in electrodynamics. For example, the gauge invariance of classical field theory applied to electrodynamics allows one to consider the vector potential $\vec{A}$ with various gauge conditions [7], i.e.,

$$\vec{\nabla} \cdot \vec{A} = 0 \text{ (Coulomb gauge)} \tag{23}$$

$$\partial_\mu A^\mu = 0 \text{ (Lorenz gauge with } \mu = 0,1,2,3) \tag{24}$$

$$n_\mu A^\mu = 0 \text{ (Light cone gauge with } n^2 = 0) \tag{25}$$

$$x_\mu A^\mu = 0 \text{ (Fock-Schwinger gauge)} \tag{26}$$

$$A_0 = 0 \text{ (Hamiltonian or temporal gauge)} \tag{27}$$

etc, where $\vec{A}$ is the vector potential, n_μ is the time axis, ∂_μ is the four gradient, x_μ is the position four-vector. Each of the gauge condition mentioned here were used to solve particular types of electrdynamics problem and the appropriate choice of gauge simplified the calculations. However, in elastodynamics, very few variations of the gauge condition has been observed so far. In this article, the proper choice and manipulation of the gauge condition will be demonstrated for two problems: (a) Straight crested guided waves in a plate (Lamb waves and shear horizontal, SH waves) and (b) Circular crested guided waves in a plate (Lamb waves and shear horizontal, SH waves). Both problems will be solved by the potential approach.

4. APPLICATION OF GAUGE CONDITION TO STRAIGHT CRESTED GUIDED WAVES IN A PLATE

The wave equations Eq. (17) and (18) can be expanded in Cartesian coordinates (Figure 1) as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} + \frac{\omega^2}{c_p^2}\Phi = 0 \tag{28}$$

$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} + \frac{\omega^2}{c_s^2} H_x = 0 \tag{29}$$

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} + \frac{\omega^2}{c_s^2} H_y = 0 \tag{30}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} + \frac{\omega^2}{c_s^2} H_z = 0 \tag{31}$$

The gauge condition takes the form

$$\Gamma = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = f(\vec{r} \cdot \omega) \tag{32}$$

where, $f(\vec{r} \cdot \omega)$ may be chosen differently depending on the nature of the problem.
Expansion of Eq. (2) gives the displacement components in terms of pressure and shear potentials as

$$u_x = \frac{\partial \Phi}{\partial x} + \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \tag{33}$$

$$u_y = \frac{\partial \Phi}{\partial y} + \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \tag{34}$$

$$u_z = \frac{\partial \Phi}{\partial z} + \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \tag{35}$$

4.1 Solution for y -invariant straight crested Lamb + SH waves in a plate

In the case of y -invariant straight crested guided waves ($\partial / \partial y \equiv 0$), we state that the manipulation of the gauge condition yields

$$H_x = 0 \tag{36}$$

The rationale for Eq. (36) will be discussed in Section 0.
The application of the y -invariant condition, $\partial / \partial y \equiv 0$, and Eq. (36) into Eqs. (28)-(31) allows us to group the equations into Lamb waves and SH waves, i.e.,
Lamb waves:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} + \frac{\omega^2}{c_p^2} \Phi = 0 \tag{37}$$

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} + \frac{\omega^2}{c_s^2} H_y = 0 \tag{38}$$

SH waves:

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial z^2} + \frac{\omega^2}{c_s^2} H_z = 0 \tag{39}$$

The displacement equations Eqs. (33)-(35), can also be grouped, i.e., Lamb waves:

$$u_x = \frac{\partial \Phi}{\partial x} - \frac{\partial H_y}{\partial z} \tag{40}$$

$$u_z = \frac{\partial \Phi}{\partial z} + \frac{\partial H_y}{\partial x} \tag{41}$$

SH waves:

$$u_y = -\frac{\partial H_z}{\partial x} \tag{42}$$

Note that only three potentials Φ, H_y, H_z are involved in Eqs. (37)-(42), since $H_x = 0$ according to Eq. (36). The Lamb waves are represented by two potentials Φ, H_y, and the SH waves are represented by a single potential H_z.
The stress components can also be grouped as follows.

Lamb waves:

$$\sigma_{xx} = -\lambda \frac{\omega^2}{c_p^2}\Phi + 2\mu(\frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2 H_y}{\partial x \partial y}) \tag{43}$$

$$\sigma_{yy} = -\lambda \frac{\omega^2}{c_p^2}\Phi \tag{44}$$

$$\sigma_{zz} = -\lambda \frac{\omega^2}{c_p^2}\Phi + 2\mu(\frac{\partial^2\Phi}{\partial z^2} + \frac{\partial^2 H_y}{\partial x \partial z}) \tag{45}$$

$$\sigma_{zx} = \mu(2\frac{\partial^2\Phi}{\partial x \partial z} + \frac{\partial^2 H_y}{\partial x^2} - \frac{\partial^2 H_y}{\partial z^2}) \tag{46}$$

SH waves:

$$\sigma_{xy} = \mu(-\frac{\partial^2 H_z}{\partial x^2}) \tag{47}$$

$$\sigma_{yz} = \mu(-\frac{\partial^2 H_z}{\partial x \partial z}) \tag{48}$$

The solution for the Lamb waves is obtained by applying plate boundary conditions $\sigma_{zz}\big|_{z=\pm d} = 0$ and $\sigma_{xz}\big|_{z=\pm d} = 0$ that yields the Rayleigh-Lamb characteristic equation for the wavenumbers [11]. The Lamb wave solution is the classical solution [10] and will not be repeated for the sake of brevity.

For SH waves, the governing equation Eq. (39) can be solved for the shear potential H_z by using the separation of variables as

$$H_z = (C_1 \sin\eta_s z + C_2 \cos\eta_s z)e^{i\xi x} \tag{49}$$

where C_1, C_2 are constants, and ξ is the wavenumber in x direction, and $\eta_s^2 = \omega^2 / c_s^2 - \xi^2$. Substituting Eq. (49) into Eq. (42), (47), (48), the expressions of displacement and stress components become

$$u_z = -i\xi(C_1 \sin\eta_s z + C_2 \cos\eta_s z)e^{i\xi x} \tag{50}$$

$$\sigma_{xy} = \mu\xi^2(C_1 \sin\eta_s z + C_2 \cos\eta_s z)e^{i\xi x} \tag{51}$$

$$\sigma_{yz} = -i\mu\eta_s\xi(C_1 \cos\eta_s z - C_2 \sin\eta_s z)e^{i\xi x} \tag{52}$$

The zero-traction boundary conditions apply at the top and bottom of the plate, i.e.,

$$\sigma_{yz}\big|_{z=\pm d} = 0 \tag{53}$$

Substitution of Eq. (52) into boundary conditions Eq. (53) yields

$$C_1 \cos \eta_s d - C_2 \sin \eta_s d = 0 \tag{54}$$

$$C_1 \cos \eta_s d + C_2 \sin \eta_s d = 0 \tag{55}$$

Subtracting Eq. (54) from Eq. (55), the symmetric SH wave modes can be obtained and the characteristic equation corresponding to the symmetric modes becomes

$$\sin \eta_s d = 0 \tag{56}$$

Adding Eq. (54) and Eq. (55), the antisymmetric SH wave modes can be obtained and the characteristic equation corresponding to the antisymmetric modes becomes

$$\cos \eta_s d = 0 \tag{57}$$

The characteristic equations Eq. (56) and (57) obtained through the potential approach are the same as the solution of SH waves in terms of u_y [11] and subsequently the solutions for displacements and stresses should be the same.

4.2 Manipulation of the gauge condition in Cartesian coordinates

In this section, we give the rationale for taking $H_x = 0$ in Eq. (36) of the previous Section 0. At first we discuss the general case and then concentrate on the y-invariant case. Examination of Eq. (33), (34), (35) yields the following observations:

a) u_x does not depend on shear potential H_x

b) u_y does not depend on shear potential H_y

c) u_z does not depend on shear potential H_z

d) u_x, u_y, u_z depend on pressure potential Φ

We notice that the pressure potential Φ contributes to all the displacement components. However, the shear waves may be divided into vertically polarized shear waves (SV waves) contained in the xz plane and horizontally polarized shear waves (SH waves) contained in the xy plane (Figure 1). SV and SH waves may depend on all three shear potentials if coupling between them is expected in a physical problem. However, SV waves have u_z particle motion that does not depend on H_z whereas SH waves have u_y particle motion that does not depend on H_y.

Since only two types of shear waves exist, it is apparent that SV waves must depend on H_y and SH waves must depend on H_z. Therefore, the wave equations Eqs. (30) and (31) may be associated with SV and SH waves, respectively.

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} + \frac{\omega^2}{c_s^2} H_y = 0 \quad \text{(SV waves)} \tag{58}$$

$$\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}+\frac{\partial^2 H_z}{\partial z^2}+\frac{\omega^2}{c_s^2}H_z=0 \quad \text{(SH waves)} \tag{59}$$

The third potential H_x has contributions to both u_y and u_z particle motions as indicated by Eq. (34) and (35).

In straight crested guided wave propagation through plate-like structures, SH and SV waves may be treated separately; SV waves combine with pressure waves to form Lamb waves [5] whereas SH waves remain independent. Depending on the initial and boundary conditions, the Lamb waves and SH waves can either exist alone or coexist in the elastic body. The problem where only one of these waves exists is much easier to deal with. In a problem where both waves coexist, they can be treated separately and then superimpose on each other. The gauge condition of Eq. (32) may be considered as a superposition of Lamb wave gauge (Γ_{LW}) and SH wave gauge (Γ_{SH}); hence, it can be resolved into two parts as follows

$$\Gamma=\Gamma_{LW}+\Gamma_{SH}=f(\vec{r},\omega) \tag{60}$$

This partition of the gauge condition does not violate the classical solution; rather, it simplifies the problem. Considering that H_x is part of Lamb waves (eventually, H_x becomes zero for the y-invariant case), the gauge condition Eq. (32) may be written as

$$\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}=f_{LW}(\vec{r},\omega) \quad \text{(Lamb wave gauge)} \tag{61}$$

$$\frac{\partial H_z}{\partial z}=f_{SH}(\vec{r},\omega) \quad \text{(SH wave gauge)} \tag{62}$$

where $f(\vec{r},\omega)=f_{LW}(\vec{r},\omega)+f_{SH}(\vec{r},\omega)$, with $f_{LW}(\vec{r},\omega)$ and $f_{SH}(\vec{r},\omega)$ being responsible for Lamb wave gauge and SH wave gauge, respectively.

For the y-invariant problem, we may choose the simplest gauge $f_{LW}(\vec{r},\omega)=0$. Hence, Eq. (61) becomes

$$\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}=0 \tag{63}$$

Since the problem is y-invariant, Eq. (63) yields

$$\frac{\partial H_x}{\partial x}=0 \tag{64}$$

Integrating Eq. (64) gives

$$H_x=C \tag{65}$$

where C is a constant or a function of z. The simplest selection is $C=0$. Thus, Eq. (65) becomes

$$H_x=0 \tag{66}$$

and Eq. (36) is thus justified.
We have thus seen that the application of gauge condition in this simple y-invariant problem has yielded the shear potential H_x to be zero. This illustrates how the gauge condition has made the problem much simpler. Our result is similar to the solution of elastic waves in rods by Gazis [5] where one of the potentials was made zero using the gauge invariance property.

5. APPLICATION OF GAUGE CONDITION TO CIRCULAR CRESTED GUIDED WAVES IN A PLATE

The governing equations in cylindrical coordinates (Figure 2) are much more involved than in Cartesian coordinates. We can simplify the governing equations using the axisymmetric assumption, $\partial / \partial \theta \equiv 0$. For the present analysis, we will consider the axisymmetric problem and will demonstrate that the proper choice and manipulation of the gauge condition yields a much simpler formulations. Our solution will be compared with existing classical solutions [5] [11].

5.1 Axisymmetric circular crested guided waves in a plate

Under axisymmetric assumption $\partial / \partial \theta \equiv 0$, the governing equations Eqs. (17), (18) in can be written in cylindrical coordinates as

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} + \frac{\omega^2}{c_p^2}\Phi = 0 \tag{67}$$

$$\frac{\partial^2 H_r}{\partial r^2} + \frac{1}{r}\frac{\partial H_r}{\partial r} + \frac{\partial^2 H_r}{\partial z^2} - \frac{1}{r^2}H_r + \frac{\omega^2}{c_s^2}H_r = 0 \tag{68}$$

$$\frac{\partial^2 H_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial H_\theta}{\partial r} + \frac{\partial^2 H_\theta}{\partial z^2} - \frac{1}{r^2}H_\theta + \frac{\omega^2}{c_s^2}H_\theta = 0 \tag{69}$$

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r}\frac{\partial H_z}{\partial r} + \frac{\partial^2 H_z}{\partial z^2} + \frac{\omega^2}{c_s^2}H_z = 0 \tag{70}$$

The gauge condition takes the form

$$\frac{\partial H_r}{\partial r} + \frac{1}{r}H_r + \frac{\partial H_z}{\partial z} = f(\vec{r} \cdot \omega) \tag{71}$$

Upon manipulation of gauge condition we make $H_r = 0$ (see Section 0) and the governing equations Eqs. (67)- (70) can be grouped as follows.
Lamb waves:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} + \frac{\omega^2}{c_p^2}\Phi = 0 \tag{72}$$

$$\frac{\partial^2 H_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial H_\theta}{\partial r}+\frac{\partial^2 H_\theta}{\partial z^2}-\frac{1}{r^2}H_\theta+\frac{\omega^2}{c_s^2}H_\theta=0 \tag{73}$$

SH waves:

$$\frac{\partial^2 H_z}{\partial r^2}+\frac{1}{r}\frac{\partial H_z}{\partial r}+\frac{\partial^2 H_z}{\partial z^2}+\frac{\omega^2}{c_s^2}H_z=0 \tag{74}$$

The displacement equations may also be grouped as follows.
Lamb waves:

$$u_r=\frac{\partial\Phi}{\partial r}-\frac{\partial H_\theta}{\partial z} \tag{75}$$

$$u_z=\frac{\partial\Phi}{\partial z}+\frac{\partial H_\theta}{\partial r}+\frac{H_\theta}{r} \tag{76}$$

SH waves:

$$u_\theta=-\frac{\partial H_z}{\partial r} \tag{77}$$

The stress components can also be grouped as follows.
Lamb waves:

$$\sigma_{rr}=-\lambda\frac{\omega^2}{c_p^2}\Phi+2\mu\left(\frac{\partial^2\Phi}{\partial r^2}-\frac{\partial^2 H_\theta}{\partial r\partial z}\right) \tag{78}$$

$$\sigma_{\theta\theta}=-\lambda\frac{\omega^2}{c_p^2}\Phi+2\mu\left(\frac{1}{r}\frac{\partial\Phi}{\partial r}-\frac{1}{r}\frac{\partial H_\theta}{\partial z}\right) \tag{79}$$

$$\sigma_{zz}=-\lambda\frac{\omega^2}{c_p^2}\Phi+2\mu\left(\frac{\partial^2\Phi}{\partial z^2}+\frac{\partial^2 H_\theta}{\partial r\partial z}+\frac{1}{r}\frac{\partial H_\theta}{\partial z}\right) \tag{80}$$

$$\sigma_{rz}=\mu\left(2\frac{\partial^2\Phi}{\partial r\partial z}+\frac{\partial^2 H_\theta}{\partial r^2}-\frac{H_\theta}{r^2}+\frac{1}{r}\frac{\partial H_\theta}{\partial r}-\frac{\partial^2 H_\theta}{\partial z^2}\right) \tag{81}$$

SH waves:

$$\sigma_{r\theta}=-2\mu\left(\frac{\partial^2 H_z}{\partial r^2}-\frac{1}{r}\frac{\partial H_z}{\partial r}\right) \tag{82}$$

$$\sigma_{\theta z}=-\mu\frac{\partial^2 H_z}{\partial r\partial z} \tag{83}$$

The Lamb wave expressions for displacements and stresses are the same as for the classical solution [11].
The Rayleigh-Lamb equation is obtained from the plate boundary conditions; the complete Lamb wave solution will not be repeated for the sake of brevity. The SH wave solution in terms of potentials is discussed next.
The governing equation Eq. (74) can be solved for shear potential H_z by using separation of variables, i.e.,

$$H_z = (A_1 \sin \eta_s z + A_2 \cos \eta_s z) H_0^1(\xi r) \tag{84}$$

where A_1, A_2 are constants, ξ is the wave number in the r direction, $\eta_s^2 = \omega^2 / c_s^2 - \xi^2$, and $H_0^1(\xi r)$ is the Hankel function of the first kind and order zero.
The traction free boundary conditions apply at the top and bottom of the plate i.e.

$$\sigma_{\theta z}\big|_{z=\pm d} = 0 \tag{85}$$

Substituting Eq. (84) into Eq. (83) gives

$$\sigma_{\theta z} = -\mu \eta_s (A_1 \cos \eta_s z - A_2 \sin \eta_s z)(H_0^1(\xi r))' \tag{86}$$

Substitution of Eq. (86) into boundary conditions Eq. (85) yields

$$A_1 \cos \eta_s d - A_2 \sin \eta_s d = 0 \tag{87}$$

$$A_1 \cos \eta_s d + A_2 \sin \eta_s d = 0 \tag{88}$$

Subtraction and addition of Eq. (87) and (88), yields the symmetric and antisymmetric SH wave modes; the characteristic equations corresponding to the symmetric and antisymmetric modes are

$$\sin \eta_s d = 0 \quad \text{(symmetric)}; \qquad \cos \eta_s d = 0 \quad \text{(antisymmetric)} \tag{89}$$

The characteristic equations indicated by Eq. (89) are the same as the classical solutions of SH waves; subsequently, the solutions for displacements and stresses are also same. Hence, the potential approach has been shown to be easily implemented through the manipulation of the gauge condition.

5.2 Manipulation of the gauge condition in cylindrical coordinates

In this section, we give the rationale for taking $H_r = 0$ in Section 0. We follow similar analogy that has already been discussed in Section 0 to allocate the potential functions to the Lamb waves and the SH waves.
In cylindrical coordinates, pressure waves must depend on Φ; SV waves must depend on H_θ; and SH waves must depend on H_z.
The third shear potential, H_r may have contributions to both SV and SH waves. However, when solving Eq. (68) and (69) by separation of variable methods, we notice that the shear potentials H_r and H_θ follow the same order (order 1) of Hankel function.
On the other hand, the shear potential H_z has a different order (order 0) of Hankel function. The pressure waves and SV waves group up to generate Lamb waves whereas SH waves

remain as independent. Separating the gauge condition Eq. (71) into Lamb waves and SH waves yields:

$$\frac{\partial H_r}{\partial r} + \frac{1}{r} H_r = f_{LW}(\vec{r}, \omega) \quad \text{(Lamb wave gauge)} \tag{90}$$

$$\frac{\partial H_z}{\partial z} = f_{SH}(\vec{r}, \omega) \quad \text{(SH wave gauge)} \tag{91}$$

where, $f(\vec{r}, \omega) = f_{LW}(\vec{r}, \omega) + f_{SH}(\vec{r}, \omega)$.

For the axisymmetric problem, we may choose the simplest gauge $f_{LW}(\vec{r}, \omega) = 0$. Hence, Eq. (90) becomes

$$\frac{\partial H_r}{\partial r} + \frac{1}{r} H_r = 0 \tag{92}$$

Using the differential product rule, Eq. (92) yields

$$\frac{1}{r} \frac{\partial}{\partial r} (r H_r) = 0 \tag{93}$$

Integrating Eq. (93) gives

$$H_r = C_1 \tag{94}$$

where C_1 can be a constant or a function of z. The simplest selection may be $C_1 = 0$. Thus, Eq. (94) becomes

$$H_r = 0 \tag{95}$$

This is similar to the y-invariant problem discussed for Cartesian coordinates in Section 0. Its application simplifies the axisymmetric guided wave propagation solution as discussed in Section 0.

6. CONCLUSION

The gauge condition originated in elastodynamics from the Navier-Lame equations upon application of Helmholtz theorem. The proper choice and manipulation of the gauge condition may simplify the problem and permits straight forward analytical solution. The gauge condition provides the necessary conditions for the complete solution of the elastic waves in plates by the potential function approach. The gauge condition may be considered as the superposition of separate gauge conditions for Lamb waves and SH waves, respectively. Each gauge condition contains a different combination of the shear vector potential components.

The gauge condition established a bridge between Lamb waves and SH waves. The gauge condition may decouple for the physical problems in which the Lamb and SH waves are expected to decouple. The decoupling of the gauge condition does not violate the classical Lamb wave and SH wave solutions; rather, it simplifies the problem. The gauge condition plays a vital role in the separation of Lamb waves and SH waves; thus, it transforms a complicated problem into two simpler problems.

In this article, the manipulation of the gauge condition has been illustrated on two well-known problems of guided waves in plates in which the Lamb waves and SH waves can be physically decoupled. The next challenge for this approach would be to address a coupled problem (Lamb waves + SH waves) such as the non-axisymmetric guided wave propagation in a plate.

FUTURE WORK

The gauge condition may be further explored more to analyze more complicated non-axisymmetric problems. The proper choice and manipulation of the gauge condition may be utilized to decouple the Lamb waves and SH waves in the non-axisymmetric problem and obtain the analytical solution by potential function approach.

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Figure Captions

Figure 1: Problem definition and displacement components in Cartesian coordinate system
Figure 2: Problem definition in cylindrical coordinate system

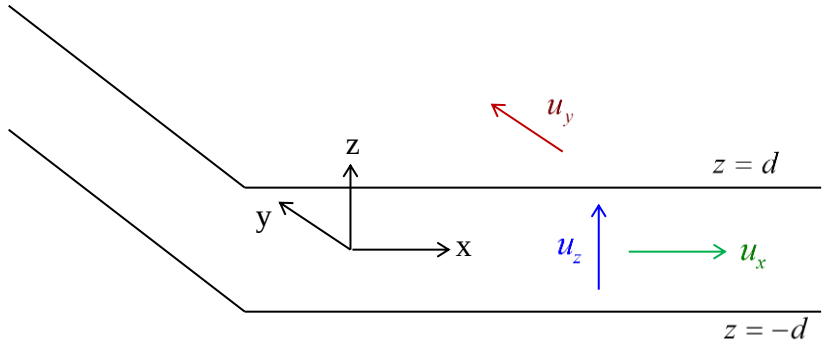

Figure 1: Problem definition and displacement components in Cartesian coordinate system

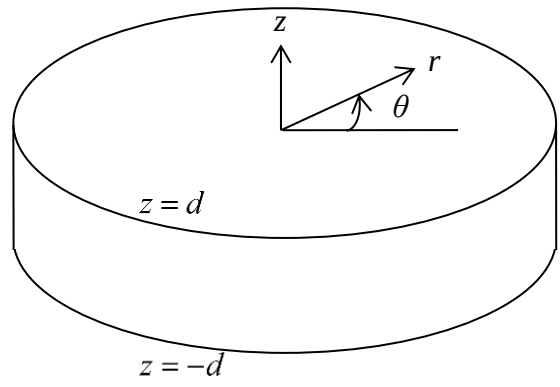

Figure 2: Problem definition in cylindrical coordinate system