Stability analysis for an UAV model in a longitudinal flight

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Abstract: *The paper presents the stability analysis of the equilibria in a longitudinal flight of an unmanned aircraft with constant forward velocity. The motion of the aircraft is described using delay differential equations with constant delays, the delay being considered in flight control compartment. The goal is to study the effects of the delays for the stability of the equilibrium points. It is eventually proved that a Hopf bifurcation appears.*

Key Words: *equilibrium point, Hopf bifurcation, delay-differential equations, stability*

1. INTRODUCTION

Due to various domains of application, unmanned aerial vehicles (UAVs) are the subject of intense research. One of the active areas for this type of aircraft, refers to the longitudinal flight with constant velocity. Since a human factor to react in real time for this type of vehicle is missing, a special attention is given to the landing phase and the final approach of the UAV, controlled by an automatic flight control system (AFCS). Due to the fact that the velocity variation of the state variables depends on their past and present values, the differential equations associated with this model are differential equations with time-lag argument. The mathematical modeling of delayed processes is relatively recent and has become necessary with increasing interest in the development of complex automated systems in some areas such as aerospace, robotics and telecommunications. It has also been used to understand complex phenomena in areas such as biology, medicine, ecology and economics [1]. In aerospace, the cause of these delays results from the high order system complexity and in case of digital systems, from the inherent sampling time. Digital control systems are attractive due to the high computing power, which enhances the complexity of the flight control system. The delays have a significant effect on the longitudinal and lateral-directional flight [1].

2. THE MODEL

Automatic Landing Flight Experiment (ALFLEX) is an unmanned aircraft built by NASDAQ, Japan.

This vehicle is a reduced scale model of an unmanned reusable orbiting spacecraft, H-II Orbiting Plane (HOPE) [2]. The existence of equilibrium points for the unmanned ALFLEX reentry vehicle when the automatic flight control system fails was studied in [2]. The Automatic Flight Control System helps the aircraft to have quick responses to commands.

For this research we focused on the case of a steady longitudinal flight of ALFLEX, with constant forward velocity, in case the automated control system is decoupled. Assuming that the angles of attack and sideslip, α and β, are small and the forward velocity V is constant we recall the following mathematical model of the ALFLEX from [2, 3, 4].

$$\begin{aligned}
\dot\beta &= p\sin\alpha - r\cos\alpha + \frac{Y}{mV}\\
\dot\alpha &= -p\beta + q + \frac{Z}{mV}\\
I_x\dot p - I_{xz}\dot r &= \left(I_y - I_z\right)qr + I_{xz}pq + L\\
I_y\dot q &= \left(I_z - I_x\right)pr - I_{xz}\left(p^2 - r^2\right) + M\\
I_z\dot r - I_{xz}\dot p &= \left(I_x - I_y\right)pq - I_{xz}qr + N\\
\dot\phi &= p + q\sin\phi\tan\theta + r\cos\phi\tan\theta\\
\dot\theta &= q\cos\phi - r\sin\phi.
\end{aligned} \tag{1}$$

The state vector consists of the angle of attack α, sideslip angle β, roll rate p, pitch rate q, yaw rate r, Euler pitch angle θ, and Euler roll angle ϕ.

The constants I_x, I_y, I_z where used to describe the moments of inertia about the $x-$, $y-$ and $z-$axes.

I_{xz} represents the product of inertia, g is the gravitational acceleration and m is the mass of the aerial vehicle.

Y, Z, L, M, N, the external forces and moments are functions of the state variables. The control parameters are δ_a the aileron angle, δ_e the elevator angle and δ_r the rudder angle. For the external forces and moments we have the following expressions [2, 3, 4].

$$\begin{aligned}
Y &= mg\sin\phi\cos\theta + kV^2\left(C_{y\beta}\beta + C_{yr}r + C_{y\delta_r}\delta_r\right)\\
Z &= mg\left(\cos\phi\cos\theta - \cos\theta_0\right) + kV^2\left[C_{z\alpha}\left(\alpha - \alpha_0\right) + C_{y\delta_e}\delta_e\left(\delta_e - \delta_{e0}\right)\right]\\
L &= bkV^2\left(C_{l\beta}\beta + C_{lp}p + C_{lr}r + C_{l\delta_a}\delta_a + C_{l\delta_r}\delta_r\right)\\
M &= ckV^2\left[C_{m\alpha}\left(\alpha - \alpha_0\right) + C_{mq}q + C_{m\delta_e}\left(\delta_e - \delta_{e0}\right)\right]\\
N &= bkV^2\left(C_{n\beta}\beta + C_{np}p + C_{nr}r + C_{n\delta_a}\delta_a + C_{n\delta_r}\delta_r\right).
\end{aligned}$$

Assuming $\beta \equiv 0, p \equiv 0, r \equiv 0, \phi \equiv 0, \delta_a = \delta_r = 0$ in system (1), we obtain the system that governs the longitudinal flight of the unmanned ALFLEX reentry vehicle.

$$\begin{cases}
\dot\alpha = a_{11}\left(\alpha - \alpha_0\right) + q + \varepsilon\left(\cos\theta - \cos\theta_0\right) + b_1\left(\delta_e - \delta_{e0}\right)\\
\dot q = a_{21}\left(\alpha - \alpha_0\right) + a_{22}q + b_2\left(\delta_e - \delta_{e0}\right)\\
\dot\theta = q
\end{cases}$$

Here the state variables are α, q and θ, δ_e is the longitudinal control parameter. Consider for it the following form, with a delay τ [s].

$$\delta_e = \delta_{e0} + k_1[\cos\theta(t-\tau) - \cos\theta_0]$$

The following system of delay differential equations describes the longitudinal flight of an unmanned aircraft on the lines in [2].

$$\begin{cases} \dot{\alpha} = a_{11}(\alpha - \alpha_0) + q + \varepsilon(\cos\theta - \cos\theta_0) + b_1 k_1[\cos\theta(t-\tau) - \cos\theta_0] \\ \dot{q} = a_{21}(\alpha - \alpha_0) + a_{22}q + b_2 k_1[\cos\theta(t-\tau) - \cos\theta_0] \\ \dot{\theta} = q \end{cases} \tag{2}$$

3. THE STABILITY STUDY

Solving the equations $f_i(\alpha, q, \theta) = 0, i = \overline{1,3}$ we obtained the equilibrium points for system (2).

$$\begin{cases} a_{11}(\alpha - \alpha_0) + q + \varepsilon(\cos\theta - \cos\theta_0) + b_1(\delta_e - \delta_{e0}) = 0 \\ a_{21}(\alpha - \alpha_0) + a_{22}q + b_2(\delta_e - \delta_{e0}) = 0 \\ q = 0 \end{cases} \tag{3}$$

The equilibrium point for (1) has the form $(\alpha_0, 0, \theta_0)$.

Let $A = (a_{ij})_{i,j=\overline{1,3}}$ be the matrix of the first derivatives of the system with respect to α, q and θ calculated in the equilibrium point. For the same equilibrium point, we also consider the following matrix containing the derivatives with respect to $\theta(t-\tau)$, $B_{ij} = (b_{ij})_{i,j=\overline{1,3}}$.

For the equilibrium point $(\alpha_0, 0, \theta_0)$, the matrices A and B have the form:

$$A = \begin{pmatrix} a_{11} & 1 & -\varepsilon\sin\theta_0 \\ a_{21} & a_{22} & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -b_1 k_1 \sin\theta_0 \\ 0 & 0 & -b_2 k_1 \sin\theta_0 \\ 0 & 0 & 0 \end{pmatrix}$$

The characteristic equation associated with the linearization of the system around the equilibrium point is given by

$$\det(\lambda I_3 - A - Be^{-\lambda\tau}) = 0$$

The characteristic equation has the form

$$\lambda(\lambda - a_{11})(\lambda - a_{22}) - \lambda a_{21} + a_{21}\varepsilon\sin\theta_0 + e^{-\lambda\tau}[b_1 k_1 a_{21} + (\lambda - a_{11})b_2 k_1]\sin\theta_0 = 0 \tag{4}$$

Consider the case in which the characteristic equation has the following form:

$$\lambda - a - be^{-\tau\lambda} = 0 \tag{5}$$

The following result comes from [5], [6] and [7]:

Theorem 1.([6], page. 593) *All roots of equation* (5) *have negative real parts if and only if:*

(i) $a\tau < 1$

(ii) $a + b < 0$

(iii) $-b\tau < \sqrt{\sigma^2 + a^2\tau^2},$

where σ *is the unique root of* $\sigma = a\tau\tan\sigma,\ 0 < \sigma < \pi.$

Remark 1. *If* $b > 0,$ *then the conditions reduce to* $a\tau < 1$ *and* $a + b < 0$ *.*

Proposition 1. *Assume that the following conditions hold:*

$$\begin{gathered} a_{22} + a_{11} < 0 \\ -a_{21} + b_2 k_1 \sin\theta_0 + a_{11}a_{22} > 0 \\ +\varepsilon a_{21}\sin\theta_0 + b_1 k_1 a_{21}\sin\theta_0 - a_{11}b_2 k_1 \sin\theta_0 > 0 \\ (-a_{22} - a_{11})(-a_{21} + b_2 k_1 \sin\theta_0 + a_{11}a_{22}) - \varepsilon a_{21}\sin\theta_0 - b_1 k_1 a_{21}\sin\theta_0 \\ + a_{11}b_2 k_1 \sin\theta_0 > 0 \end{gathered} \tag{6}$$

Then equation (4) *is stable for* $\tau = 0.$

Proof: For $\tau = 0$ equation (3) becomes:

$$\begin{gathered} \lambda^3 + \lambda^2(-a_{22} - a_{11}) + \lambda(-a_{21} + b_2 k_1 \sin\theta_0 + a_{11}a_{22}) \\ + \varepsilon a_{21}\sin\theta_0 + b_1 k_1 a_{21}\sin\theta_0 - a_{11}b_2 k_1 \sin\theta_0 = 0 \end{gathered} \tag{7}$$

In order for the roots of equation (7) to be in the left half-plane, the Routh-Hurwitz criterion leads to (6). The proposition is proved.

To simplify the next calculations, we introduce the following notations:

$$a_1 = -a_{22} - a_{11}$$

$$a_2 = -a_{21} + a_{11}a_{22}$$

$$a_3 = \varepsilon a_{21}\sin\theta_0$$

$$a_4 = b_2 k_1 \sin\theta_0$$

$$a_5 = b_1 k_1 a_{21}\sin\theta_0 - a_{11}b_2 k_1 \sin\theta_0$$

The characteristic equation becomes:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + e^{-\lambda\tau}(a_4\lambda + a_5) = 0 \tag{8}$$

Consider the general form of the characteristic equation:

$$P(\lambda) + Q(\lambda)e^{-\tau\lambda} = 0 \tag{9}$$

Theorem 2. ([8], Theorem 1*) Consider equation (9), where* P *and* Q *are analytic functions in a right-half plane* $\operatorname{Re}\lambda > -\delta,\ \delta > 0,$ *which satisfy the following conditions.*

(i) $P(\lambda)$ *and* $Q(\lambda)$ *have no common imaginary zero.*

(ii) $\overline{P(-iy)} = P(iy),\ \overline{Q(iy)} = Q(iy),$ *for real* y

(iii) $P(0) + Q(0) \neq 0.$

(iv) There are at most a finite number of roots of (9) in the half-plane when $\tau = 0.$

(v) $F(y) = |P(iy)|^2 - |Q(iy)|^2$ *for real* y *, has at most a finite number of real zeros.*

Under these conditions, the following statements are true.

a. *Suppose that the equation* $F(y) = 0$ *has no positive roots. Then, if (9) is stable at* $\tau = 0$ *it remains stable for all* $\tau \geq 0$, *whereas if it is unstable at* $\tau = 0$, *it remains unstable for all* $\tau \geq 0$.

b. *Suppose that the equation* $F(y) = 0$ *has at least one positive real root and that each positive root is simple. As* τ *increases, stability switches may occur. There exists a positive number* τ^* *such that the equation (9) is unstable for all* $\tau > \tau^*$. *As* τ *varies from 0 to* τ^*, *at most a finite number of stability switches may occur.*

In order to study the equation (8) we use Theorem 2. We define

$$\begin{aligned} P(z) &= z^3 + a_1 z^2 + a_2 z + a_3 \\ Q(z) &= a_4 z + a_5 \end{aligned} \tag{10}$$

Note that conditions *(i)-(v)* from the theorem are satisfied.

The stability of equation (8) depends on the roots of the equation:

$$|P(iy)|^2 = |Q(iy)|^2 \tag{11}$$

If equation (11) has no $y > 0$ as a root then, if (9) is stabile with $\tau = 0$, it will be stable for all $\tau > 0$. If equation (11) has at last one positive root and all the positive roots are simple, as τ increases there might be stability switches. Thus, if (9) is stable at $\tau = 0$, it may become unstable when $\tau > 0$. Define

$$F(y) = |P(iy)|^2 - |Q(iy)|^2 \tag{12}$$

We have:

$$F(y) = y^6 + (a_1^2 + a_2^2 - 2a_2)y^4 - (2a_1a_3 - a_4)y^2 + a_3^2 - a_5^2 = 0 \tag{13}$$

We have to find the positive roots of (13).

4. HOPF BIFURCATION

Consider the case $\tau > 0$ again. We recall that the characteristic equation is:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + e^{-\lambda\tau}(a_4\lambda + a_5) = 0 \tag{14}$$

Let $P(iy) = P_R(y) + iP_I(y)$ and $Q(iy) = Q_R(y) + iQ_I(y)$, with P_R, P_I, Q_R, Q_I real valued.

Proposition 2. Assume that (13) has a root $y > 0$. Then equation (8) will have a pair of purely imaginary roots $\lambda = \pm iy$ that cross the imaginary axis at $\tau = \tau^*$ from left to right if $s > 0$ and from right to left if $s < 0$, where

$$s = sign\{F'(y)\} \tag{15}$$

τ^* is given by:

$$\begin{aligned} \cos y\tau^* &= -\frac{P_R(y)Q_R(y) + P_I(y)Q_I(y)}{Q_R^2(y) + Q_I^2(y)} \\ \sin y\tau^* &= \frac{P_I(y)Q_R(y) - P_R(y)Q_I(y)}{Q_R^2(y) + Q_I^2(y)} \end{aligned} \tag{16}$$

In the first case, if we have stability for $\tau = 0$, a Hopf bifurcation will appear when τ exceeds τ^* (see [9])

Proof: We prove the proposition by using Theorem 2 stated earlier.

Equation $|P(iy)|^2 = |Q(iy)|^2$ has at least one simple positive real root and thus equation (14) will have a pair of purely imaginary roots. This means that there is a value $\tau = \tau^*$ at which there might be a change in stability. The value τ^* at which stability switches might occur results from the fact that equation (14) has iy as a solution if and only if:

$$\begin{aligned} Q_R(y)\cos y\tau + Q_I(y)\sin y\tau &= -P_R(y) \\ Q_I(y)\cos y\tau - Q_R(y)\sin y\tau &= -P_I(y) \end{aligned} \tag{17}$$

From which we get:

$$\cos y\tau^* = -\frac{P_R(y)Q_R(y) + P_I(y)Q_I(y)}{Q_R^2(y) + Q_I^2(y)}$$

$$\sin y\tau^* = \frac{P_I(y)Q_R(y) - P_R(y)Q_I(y)}{Q_R^2(y) + Q_I^2(y)}$$

According to [8], the sign of s determines the direction of crossing the imaginary axis. The proposition is proved. Remark (see [8]) that

$$s = sign\{F'(y)\} = sign\{-2a_1y(a_3 - a_1y^2) + (a_2y - y^3)(a_2 - 3y^2) - a_4^2y\} \tag{18}$$

5. NUMERICAL SIMULATIONS

Equations (3) give the equilibrium point (8.18; 0; 6.123313307151168). For the configuration of parameters found in [2, 4] we notice that, by (6), the equilibrium point is stable for $\tau = 0$, see Figure 1.

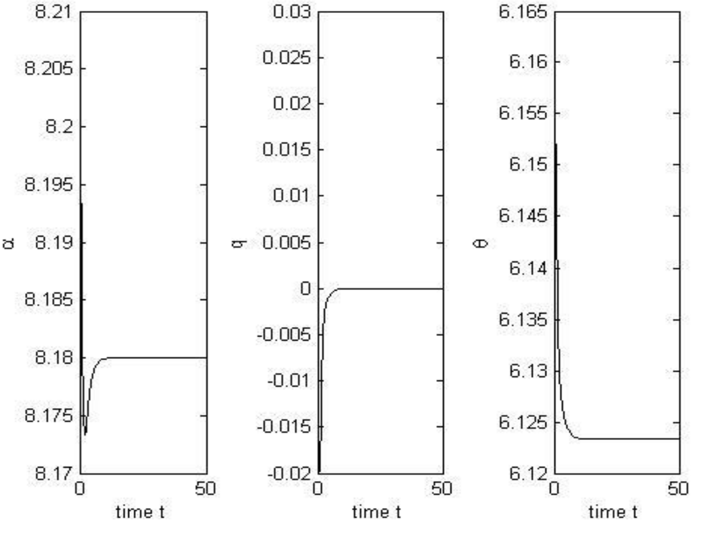

Figure 1. Behavior of solutions starting near equilibrium point

Equation (14) is stable for $\tau = 0$ and is changing stability at $\tau^* = 2.559$. As $s > 0$, the pair of purely imaginary roots cross the imaginary axis from left to right. The equation switches from a stable state to an unstable state end oscillations will appear. This can be seen in Figure 2 and Figure 3.

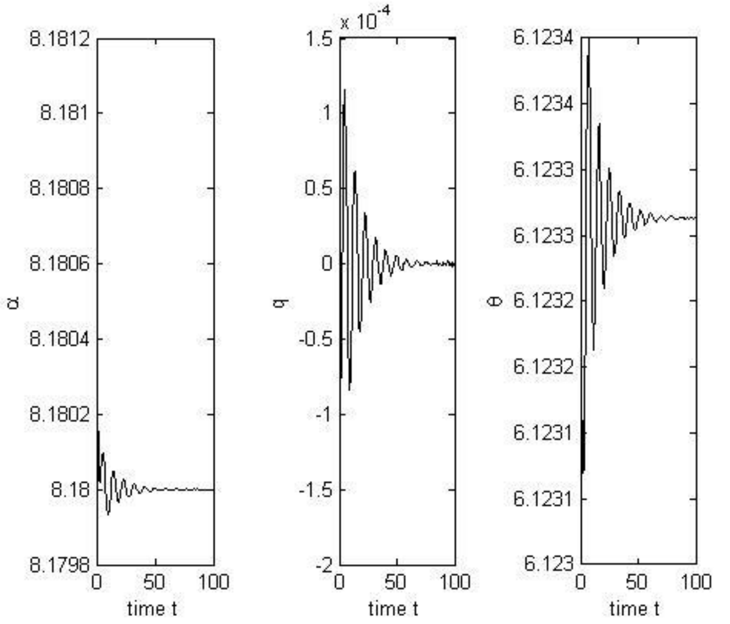

Figure 2. Behavior of solutions starting near equilibrium point for $\tau = 2.3$

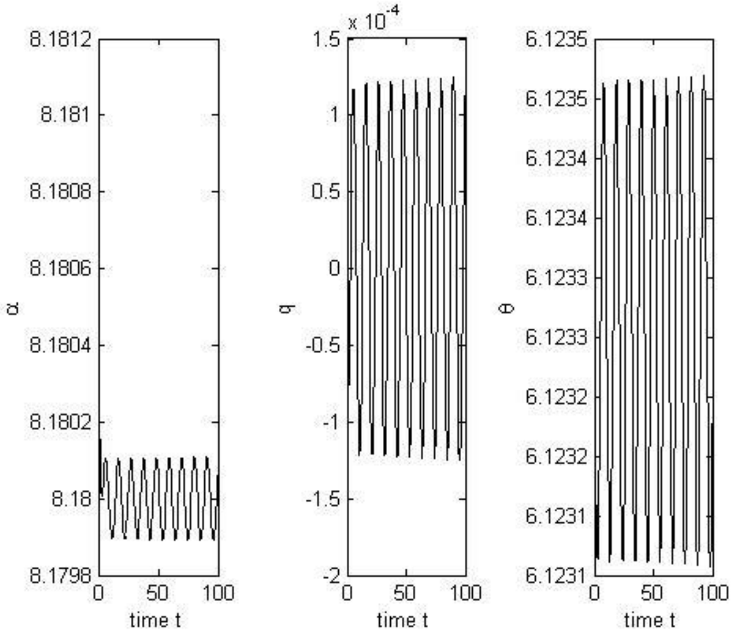

Figure 3. Behavior of solutions starting near equilibrium point for $\tau = 2.56$

We remark the appearance of a limit cycle due to a Hopf bifurcation.

6. CONCLUSIONS

In this paper we used models that describe the motion of an unmanned vehicle with constant forward velocity to study the effect of the delay introduced in the control parameter.

The stability of the equilibrium point was studied through the characteristic equation, and it was proved that a Hopf bifurcation appears. Numerical simulations show the oscillatory nature of the solutions. In future work we will study the stability of the limit cycle that appears due to the Hopf bifurcation.

7. NUMERICAL DATA

The following data were taken from [2] and [4].

$m = 760$	$\delta_{e0} = 3^0$	$C_{y\beta} = -0.6849$	$C_{np} = 0.0032$
$S = 9.45$	$\alpha_0 = 8.18^0$	$C_L = 0.2387$	$C_{yr} = 0$
$g = 9.81$	$\theta_0 = -9.16^0$	$C_{l\beta} = -0.1774$	$C_{L\delta e} = 0.6355$
$b = 3.295$	$I_x = 407$	$C_{n\beta} = -0.0657$	$C_{lr} = 0.004$
$c = 3.154$	$I_y = 1366$	$C_{y\delta r} = 0.1907$	$C_{nr} = -0.006$
$\rho = 1.156$	$I_z = 1634$	$C_{L\alpha} = 2.016$	$C_{m\alpha} = -0.0134$
$k = \rho S / 2$	$I_{xz} = 10.4$	$C_{lp} = -0.007$	$C_D = 0.0745$
$V = 73.84$			
$C_{l\delta a} = 0.1488$	$C_{mq} = -0.0474$	$C_{l\delta r} = 0.0788$	$C_{m\delta e} = -0.2152$
$C_{n\delta a} = -0.0266$	$C_{D\alpha} = 0.2714$	$C_{n\delta r} = -0.099$	$C_{D\delta e} = 0.1019$

$$C_{z\alpha} = -C_{L\alpha}\cos\alpha_0 + C_L \sin\alpha_0 - C_D\cos\alpha_0 - C_{D\alpha}\sin\alpha_0$$
$$C_{z\delta e} = -C_{L\delta e}\cos\alpha_0 - C_{D\delta e}\sin\alpha_0$$

$$a_{11} = \frac{kV}{m} C_{z\alpha} = -1.10058 \quad a_{21} = ckV^2 C_{m\alpha} = -0.92142 \quad a_{22} = ckV^2 Cmq = -3.25936$$

$$b_1 = \frac{kV}{m} C_{z\delta e} = -0.34151 \quad b_2 = ckV^2 C_{m\delta e} = -14.79777 \quad \varepsilon = \frac{g}{V} = 0.13285$$

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