An analytical method for simulation of centrifugal compressors

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Abstract: *A pure analytical method for modeling and simulation of centrifugal compressors based on parametric analysis of governing equations is proposed. It will be shown that the main equations describing the compressor operation along with some new hypotheses allow to determine the most useful off-design characteristics of the machine for a given geometry and for given performances at design (nominal) regime. As example of application, a compressor map is determined along with other important features like surge and choke lines, optimal regimes line and constant efficiency curves (efficiency islands) without need of experimental data. The proposed method is intended for quick simulation of complex machines embedding centrifugal compressors, as an alternative of costly CFD analyses.*

Key Words: *centrifugal compressor, compressor map, surge line, choke line, optimal regime*

1. INTRODUCTION

The centrifugal compressors are frequently used in thermal machines like turbochargers, turboprops, turboshafts, aircraft APU's (Auxiliary Power Units) and various mini-jet engines for lab demos. As such, modeling and simulation of such machines cannot be done without a mathematical model for the compressor simulation.

Although the centrifugal compressor is a well-known solution for gases compression and the technical literature is full of methods for design and off-design analysis, so far no mathematical model was issued wherein only the compressor geometry and its design performances are used as input data. All the proposed methods use more or less experimental data acquired from some compressors already built or statistical results available for a range of similar machines. This paper is intended to completely remove the experimental measurements when estimate off-design properties of centrifugal compressors, leaving room only for analytical deductions. Of course, a comparison with experimental measurements will be necessary for the validation of proposed method.

As early as 1950's, when computers didn't exist, a series of simplified methods of estimation of compressors behavior were proposed. One of the simplifying assumptions used by these methods consists of considering the slip factor to be constant throughout all the operating regimes. The later experimental tests showed that this is not the case.

On the other hand, these methods were not able to provide all the characteristics like the surge limit and the efficiency islands based only on the proposed hypotheses. As such, experimental data were needed as well for a complete map construction [1]. Later, as the computation tools evolved, more methods for estimating the compressors characteristics were developed, mainly based on experimental data processing and combined with various simplifying assumptions applied to compressor equations. From the most recent ideas intended to solve analytically the problem of compressor simulation, we cite the compressor map regression modeling based on partial least squares proposed by Xu Li, Chuanlei Yang et al. in 2018 [2]. The authors propose two methods, one of them using a power function polynomial as the basis function while the other one uses a trigonometric function polynomial. However, the method is based on some available compressor characteristics maps. In another paper [3], a fitting method using elliptical curves was introduced. In this method, the compressor characteristics are expressed by means of the mathematical equation of an ellipse. The main drawback of the methods using experimental data consists of the need of performing interpolations or using regression functions in order to cover all the operating regimes of the compressor. Following this approach, one cannot estimate accurately some characteristics of compressors having a different design from those which have been measured (like surge limit, choke limit and efficiency islands).

In order to improve the accuracy, some researchers have recently introduced intelligent algorithms for various engineering models, including the off-design analysis of compressors. Among these algorithms, the artificial neural network (ANN) is an effective data-based modeling method [4] frequently used in many areas due to its capability of handling nonlinear processes along with storage of a high amount of experimental data. Theoretically, the ANN can handle any nonlinear model, developing the relationship between input and output variables without considering the involved physical processes.

Nowadays, the most used methods for estimating the off-design characteristics of compressors consist of CFD analyses, which have been continuously improved and have the ability of catching the unsteady effects as well; in exchange, they need a high volume of computer resources and time. Furthermore, if the simulation of a complex power unit containing centrifugal compressors is required, a CFD analysis becomes cumber some.

The advantage of using analytical models is the use of appropriate equations describing the real processes and the possibility of integrating the analysis into the simulation of complex power units. This paper presents such a method which needs no experimental measurements. The compressor analyzed herein uses air as working fluid, but the analysis can be extended for any compressible fluid obeying the perfect gas law.

2. BASIC ASSUMPTIONS

We will refer to the main hypotheses needed for the development of the mathematical model.

(i) The parameters defining the design point are known:

- air mass flow $\dot{M}_a$ [kg/s];
- overall pressure ratio $\pi_c = \frac{p_2^*}{p_1^*}$ where $[\ ]^*$ means "stagnation state", $[\ ]_1$ and $[\ ]_2$ means "inlet" and "exit", respectively;
- rotating speed n [rpm];

- isentropic efficiency η_c.

(ii) The main features defining the geometry of compressor are known:

- inlet area A_1 [m^2];
- mean radius of inducer R_{1m} [m], impeller radius R_2 [m] and diffuser (bladed) inlet radius R_3 [m] (notations according to fig.1 and fig. 2);
- angle of inducer absolute velocity at the mean radius, denoted by α_1;
- angle of inducer leading edge β_{1f};
- angle of impeller blade exit β_{2f};
- angle of difusser blade leading edge β_{1f};
- blade width at impeller exit b [m];
- blades number Z (including splitter blades).

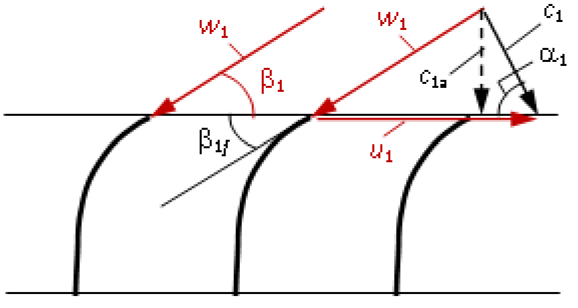

Fig. 1 – Inducer features at the inlet of a centrifugal compressor

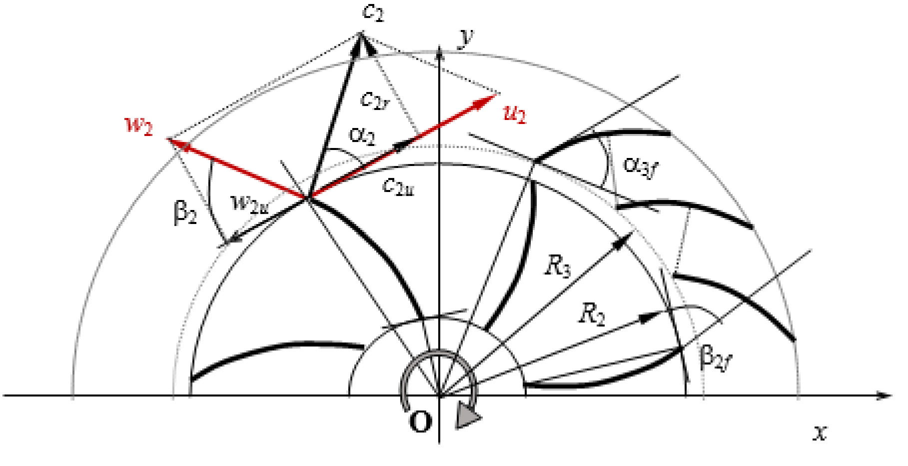

Fig. 2 – Impeller and diffuser features in a centrifugal compressor

(iii) The angle β_2 of the relative velocity at impeller exit w_2 will be considered constant for all operating regimes of compressor, while the slip factor φ_{2u} will be variable according to its definition formula ([5], [6]):

$$\varphi_{2u} = \frac{u_2 - c_{2r}\,\mathrm{ctg}\beta_2}{u_2 - c_{2r}\,\mathrm{ctg}\beta_{2f}} \tag{1}$$

This hypothesis replaces the old simplifying assumption φ_{2u} = const. because the correlations of φ_{2u} found in the literature are confirmed only at design regime, while for the rest of operating regimes, significant variations have been observed.

(iv) The maximum efficiency of compressor occurs at the design regime:

$$(\eta_c)_{\max} = (\eta_c)_n \tag{2}$$

where [ ]$_n$ means "design" (nominal) regime. In fact, in this paper, the design regime is defined as the regime which reaches the maximum efficiency.

(v) Since the efficiency depends on both speed and flow coefficient $\eta_c = f(\overline{c}_{1a}, n)$ where the flow coefficient is defined by

$$\overline{c}_{1a} = \frac{c_{1a}}{u_1} \tag{3}$$

a simple quadratic expression for the compressor efficiency will be adopted:

$$\eta_c = (\eta_c)_n \left[1 - K_\eta \left(1 - \frac{\overline{c}_{1a}}{(\overline{c}_{1a})_n} \right)^2 \right] \frac{n}{n_n} \left(2 - \frac{n}{n_n} \right) \tag{4}$$

where K_η is a dimensionless coefficient. It can be easily seen that for $n = n_n$ and $\overline{c}_{1a} = (\overline{c}_{1a})_n$, one gets $\eta_c = (\eta_c)_n$. The above formula will allow us to plot the efficiency islands.

In order to determine the coefficient K_η, the following judgment will be applied:

$$\pi_c = 1 \quad \Rightarrow \quad \eta_c = 0 \tag{5}$$

In addition, in this limit case, the exit air density is the same as the inlet one, since there is no compression, although an air flow does exist. As such, the following expressions hold:

$$\rho_1 = \rho_2 \quad \Rightarrow \quad c_{1a} A_1 = c_{2r} A_2 \quad \Rightarrow \quad \overline{c}_{2r} = \overline{c}_{1a} \frac{A_1}{A_2} \tag{6}$$

where

$$\overline{c}_{2r} = \frac{c_{2r}}{u_1} \tag{7}$$

Further, the Euler's formula for the compressor specific work (denoted as l_C) is needed:

$$l_C = u_2 c_{2u} - u_1 c_{1u} \tag{8}$$

By introducing the work coefficient $\overline{l}_C = l_C / u_1^2$ and by handling some geometric relations according to fig. 1 and 2, Euler's formula becomes:

$$\overline{l}_C = \overline{R}_2^2 - \overline{R}_2 \overline{c}_{2r} \mathrm{ctg}\beta_2 - \overline{c}_{1a} \mathrm{ctg}\alpha_1 \tag{9}$$

where $\overline{R}_2 = R_2 / R_{1m}$. Since $\pi_c = 1 \Rightarrow \overline{l}_C = 0$, from the above equation combined with (6) one yields:

$$\overline{c}_{1a} = \frac{\overline{R}_2^2}{\overline{R}_2 \frac{A_1}{A_2} \mathrm{ctg}\beta_2 + \mathrm{ctg}\alpha_1} \tag{10}$$

By substituting the last result in (4) for $\eta_c = 0$, the expression of coefficient K_η is found:

$$K_\eta = \left[1 - \frac{1}{(\bar{c}_{1a})_n} \frac{\bar{R}_2^2}{\bar{R}_2 \frac{A_1}{A_2} \text{ctg}\beta_2 + \text{ctg}\alpha_1}\right]^{-2} \tag{11}$$

which is constant since β_2 is constant according to one of the assumptions made.
This assumption simplifies dramatically the calculations, since it removes the need of modeling the various losses occurring over the entire domain of operating regimes.

3. GOVERNING EQUATIONS OF CENTRIFUGAL COMPRESSOR

The basic equations which describe the operation of a centrifugal compressor consist mainly of mass and energy balance, along with the simplifying assumptions previously presented.

(a) Mass flow rate equation at the inducer inlet in terms of relative velocity [5], [6]

$$\dot{M}_a = \frac{p_{1w}^*}{\sqrt{\text{R}T_{1w}^*}} A_1 \sin\beta_1 q(\lambda_{1w}) K(k) \tag{12}$$

where λ_w = dimensionless relative velocity (Tchaplyguin number) defined by formula [7]

$$\lambda_w = \frac{w}{\sqrt{\frac{2k}{k+1}\text{R}T_w^*}} \tag{13}$$

$[\ ]_w^*$ = relative stagnation parameter, R = gas constant, k = adiabatic exponent (both for air)

$q(\lambda)$ = thermodynamic function of mass flow rate, defined by the expression [5], [7]

$$q(\lambda) = \lambda\left[\frac{k+1}{2}\left(1 - \frac{k-1}{k+1}\lambda^2\right)\right]^{\frac{1}{k-1}} \tag{14}$$

$K(k)$ = a notation for the expression [5]

$$K(k) = \sqrt{k\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \tag{15}$$

By denoting $D_{1m} = 2R_{1m}$ the mean diameter of the impeller eye (inducer), the transport velocity u_1 can be replaced by

$$u_1 = \omega R_{1m} = \frac{\pi D_{1m}}{60} n \tag{16}$$

where ω = angular velocity [rad/s].

Performing the analysis of velocities triangle (fig.1), the stagnation quantities in relative motion at the compressor inlet can be expressed as

$$\frac{T_{1w}^*}{T_1^*}=1+\frac{1}{2c_p}\left(\frac{\pi D_{1m}}{60}\right)^2\left(\frac{n}{\sqrt{T_1^*}}\right)^2\left(1-2\bar{c}_{1a}\,\mathrm{ctg}\alpha_1\right) \tag{17}$$

$$\frac{p_{1w}^*}{p_1^*}=\left[1+\frac{1}{2c_p}\left(\frac{\pi D_{1m}}{60}\right)^2\left(\frac{n}{\sqrt{T_1^*}}\right)^2\left(1-2\bar{c}_{1a}\,\mathrm{ctg}\alpha_1\right)\right]^{\frac{k}{k-1}} \tag{18}$$

where c_p = specific heat at constant pressure. By replacing in (12) the last two results, a final expression for the mass flow rate in terms of relative velocity can be given:

$$\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}=\left[1+\frac{1}{2c_p}\left(\frac{\pi D_{1m}}{60}\right)^2\left(\frac{n}{\sqrt{T_1^*}}\right)^2\left(1-2\bar{c}_{1a}\,\mathrm{ctg}\alpha_1\right)\right]^{\frac{k+1}{2(k-1)}}\frac{A_1}{\sqrt{\mathrm{R}}}\sin\beta_1 q(\lambda_{1w})K(k) \tag{19}$$

The dimensionless velocity λ_{1w} can be related to the flow coefficient $\bar{c}_{1a}$ by means of formula

$$\lambda_{1w}=\frac{\bar{c}_{1a}}{\sin\beta_1}\frac{\pi D_{1m}}{60}\sqrt{\frac{k+1}{2k\mathrm{R}}}\frac{n}{\sqrt{T_{1w}^*}} \tag{20}$$

so that relation (19) will contain only the flow coefficient as velocity parameter. The incidence angle β_1 can be also expressed as function of α_1 (which is constant) and $\bar{c}_{1a}$ by the expression:

$$\sin\beta_1=\frac{\bar{c}_{1a}\sin\alpha_1}{\sqrt{\bar{c}_{1a}^2-\bar{c}_{1a}\sin 2\alpha_1+\sin^2\alpha_1}} \tag{21}$$

Replacing this in (20) and using the equation (17) as well, one obtains for λ_{1w}:

$$\lambda_{1w}=\frac{1}{\sin\alpha_1}\frac{\pi D_{1m}}{60}\sqrt{\frac{k+1}{2k\mathrm{R}}}\frac{n}{\sqrt{T_1^*}}\sqrt{\frac{\bar{c}_{1a}^2-\bar{c}_{1a}\sin 2\alpha_1+\sin^2\alpha_1}{1+\frac{1}{2c_p}\left(\frac{\pi D_{1m}}{60}\right)^2\left(\frac{n}{\sqrt{T_1^*}}\right)^2\left(1-2\bar{c}_{1a}\,\mathrm{ctg}\alpha_1\right)}} \tag{22}$$

which exhibits the dependence of λ_{1w} on both the reduced speed and the flow coefficient.

By analyzing the above equation, one can observe that for a given speed of compressor, there exists a value of flow coefficient $\bar{c}_{1a}$ for which the flow becomes sonic ($\lambda_{1w}=1$).

The maximum flow rate (choking flow) occurs when the velocity incidence is β_{1f} (i.e. leading edge angle of inducer). Replacing in (19) $q(\lambda_{1w})=1$ and $\beta_1=\beta_{1f}$, the expression of choking flow rate at inlet is found:

$$\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)_{cr}^{(1)}=\left[1+\frac{1}{2c_p}\left(\frac{\pi D_{1m}}{60}\right)^2\left(\frac{n}{\sqrt{T_1^*}}\right)^2\left(1-2(\bar{c}_{1a})_{cr}\,\mathrm{ctg}\alpha_1\right)\right]^{\frac{k+1}{2(k-1)}}\frac{A_1}{\sqrt{\mathrm{R}}}\sin\beta_{1f}K(k) \tag{23}$$

where $(\bar{c}_{1a})_{cr}$ can be determined by applying the condition $\beta_1 = \beta_{1f}$ in (21); one yields

$$(\bar{c}_{1a})_{cr} = \frac{\sin\alpha_1 \sin\beta_{1f}\left[-\sin\beta_{1f}\cos\alpha_1 + \sqrt{\sin^2\beta_{1f}\cos^2\alpha_1 + 4\left(\sin^2\alpha_1 - \sin^2\beta_{1f}\right)}\right]}{\sin^2\alpha_1 - \sin^2\beta_{1f}} \tag{24}$$

(b) Mass flow rate equation at the inducer inlet in terms of absolute velocity:

$$\dot{M}_a = \frac{p_1^*}{\sqrt{\mathrm{R}T_1^*}} A_1 \sin\alpha_1 q(\lambda_1) K(k) \tag{25}$$

where the dimensionless absolute velocity has the expression

$$\lambda_1 = \frac{\bar{c}_{1a}}{\sin\alpha_1}\frac{\pi D_{1m}}{60}\sqrt{\frac{k+1}{2k\mathrm{R}}}\frac{n}{\sqrt{T_1^*}} \tag{26}$$

Taking into account the choking condition (23), the final equation of the mass flow rate at the inlet of compressor will have the form:

$$\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*} = \min\left[\frac{A_1}{\sqrt{\mathrm{R}}}\sin\alpha_1 q(\lambda_1)K(k),\ \left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)_{cr}^{(1)}\right] \tag{27}$$

(c) Mass flow rate equation at the impeller exit:

$$\dot{M}_a = \frac{p_2^*}{\sqrt{\mathrm{R}T_2^*}} A_2 \sin\alpha_2 q(\lambda_2) K(k) \tag{28}$$

where

$$p_2^* = \pi_c p_1^* \quad \text{and} \quad A_2 = 2\pi R_2 b \tag{29}$$

The overall energy balance of the compressor yields the following relation between the stagnation temperatures at inlet and exit:

$$T_2^* = T_1^*\left[1 + \frac{1}{\eta_c}\left(\pi_c^{\frac{k-1}{k}} - 1\right)\right] \tag{30}$$

From the analysis of velocities triangle at the impeller exit (fig.2), the following relation can be derived:

$$\sin\alpha_2 = \frac{\bar{c}_{2r}\sin\beta_2}{\sqrt{\bar{c}_{2r}^2 - \bar{R}_2\bar{c}_{2r}\sin 2\beta_2 + \bar{R}_2^2\sin^2\beta_2}} \tag{31}$$

On the other hand, by using (30) and (31), the dimensionless velocity λ_2 gets the expression:

$$\lambda_2 = \sqrt{\frac{k+1}{2k\mathrm{R}}}\frac{\pi D_{1m}}{60}\frac{n}{\sqrt{T_1^*}}\frac{1}{\sqrt{1 + \frac{1}{\eta_c}\left(\pi_c^{\frac{k-1}{k}} - 1\right)}}\sqrt{\frac{\bar{c}_{2r}^2}{\sin^2\beta_2} - 2\bar{R}_2\bar{c}_{2r}\mathrm{ctg}\beta_2 + \bar{R}_2^2} \tag{32}$$

As in the case of inducer, for a given compressor speed and pressure ratio, there exists a regime for which $\lambda_2 = 1$ on the direction of diffuser blade, i.e. α_{3f}; this is the exit choking flow. By processing equation (28) applied to the radius R_3, the expression of the maximum exit flow rate will be:

$$\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)_{cr}^{(2)} = \frac{\pi_c}{\sqrt{1+\frac{1}{\eta_c}\left(\pi_c^{\frac{k-1}{k}}-1\right)}}\frac{A_3\sin\alpha_{3f}}{\sqrt{\mathrm{R}}}K(k) \tag{33}$$

One deduces thus that the flow through the compressor is possible only if the following condition is satisfied:

$$\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*} \le \left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)_{cr}^{(2)} \tag{34}$$

where the left hand side is given by (27).

On a compressor map, the locus of all states for which the above relation becomes equality defines the so-called "choking line".

(d) Energy balance equation:

$$l_C = \frac{c_p T_1^*}{\eta_c}\left(\pi_c^{\frac{k-1}{k}}-1\right) \tag{35}$$

Taking into account Euler's equation (9) and formula (16) for the inducer transport velocity, the equation of energy gets the form:

$$\frac{c_p}{\eta_c}\left(\pi_c^{\frac{k-1}{k}}-1\right) = \left(\frac{\pi D_{1m}}{60}\right)^2\left(\frac{n}{\sqrt{T_1^*}}\right)^2\left(\overline{R}_2^2 - \overline{R}_2\overline{c}_{2r}\,\mathrm{ctg}\beta_2 - \overline{c}_{1a}\,\mathrm{ctg}\alpha_1\right) \tag{36}$$

The complete non-linear algebraic system governing the centrifugal compressor operation comprises the following 7 equations: (4), (26), (27), (28), (31), (32) and (36). The vector of unknowns contains 9 components:

$$\vec{X} = \left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}, \frac{n}{\sqrt{T_1^*}}, \pi_c, \eta_c, \overline{c}_{1a}, \overline{c}_{2r}, \lambda_1, \lambda_2, \alpha_2\right) \tag{37}$$

As such, two of above unknowns will be chosen as "free" parameters while the other ones will be expressed as functions of these parameters.

Usually, the selected "free" parameters are the reduced mass flow rate and the reduced speed while the quantities expressed as functions of these parameters are the pressure ratio π_c and the isentropic efficiency η_c:

$$\pi_c, \eta_c = f\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}, \frac{n}{\sqrt{T_1^*}}\right) \tag{38}$$

4. PARAMETRIC ANALYSIS OF GOVERNING EQUATIONS

In order to plot the compressor map, a parametric analysis will be performed on the algebraic system of governing equations. Thus, by applying various conditions on the governing equations, the number of "free" parameters reduces from two to one and thus, the most relevant lines on the compressor map can be plotted.

(i) Lines of constant speed

From (38), one derives:

$$\frac{n}{\sqrt{T_1^*}}=\text{const.} \;\Rightarrow\; \pi_c,\eta_c=f\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)\Bigg|_{n/\sqrt{T_1^*}} \tag{39}$$

(ii) Surge line

By "surge line" we understand a curve in the compressor map which separates the stable regimes form the unstable ones. Since the unstable regimes occur on the positive slope of characteristic lines, the surge line will be defined as the locus of π_c = max on the characteristic lines of constant speed (39). Using equations (4) and (36), the condition which has to be applied in order to find the equation of surge line can be written as:

$$\left[1-K_\eta\left(1-\frac{\bar{c}_{1a}}{(\bar{c}_{1a})_n}\right)^2\right]\left(\bar{R}_2^2-\bar{R}_2\bar{c}_{2r}\text{ctg}\beta_2-\bar{c}_{1a}\text{ctg}\alpha_1\right)=\max. \tag{40}$$

In order to simplify the calculations, an additional assumption will be made: the compression process will be considered a polytropic one, according to the equation:

$$\frac{\rho_1}{\rho_2}=\left(\frac{p_1}{p_2}\right)^{\frac{1}{\nu}}\cong\left(\frac{1}{\pi_c}\right)^{\frac{1}{\nu}} \tag{41}$$

where the exponent ν has an accepted average value of 1.5 (see [1]). From the mass balance equation, it follows that:

$$\bar{c}_{2r}=\bar{c}_{1a}\frac{A_1}{A_2}\frac{1}{\pi_c^{\;2/3}} \tag{42}$$

Introducing this in (40), the function which needs to be maximized gets the expression:

$$f(\bar{c}_{1a})=\left[1-K_\eta\left(1-\frac{\bar{c}_{1a}}{(\bar{c}_{1a})_n}\right)^2\right]\left[\bar{R}_2^2-\bar{c}_{1a}\left(\bar{R}_2\frac{A_1}{A_2}\frac{1}{\pi_c^{\;2/3}}\text{ctg}\beta_2+\text{ctg}\alpha_1\right)\right] \tag{43}$$

In the next, the following notations will be used:

$$X=\frac{\bar{c}_{1a}}{(\bar{c}_{1a})_n},\; B=\bar{R}_2\frac{A_1}{A_2}\frac{1}{\pi_c^{\;2/3}}\text{ctg}\beta_2+\text{ctg}\alpha_1 \tag{44}$$

so that the function (43) becomes:

$$f(X)=\left[1-K_\eta(1-X)^2\right]\left(\bar{R}_2^2-B(\bar{c}_{1a})_n X\right) \tag{45}$$

One applies the condition

$$\frac{\mathrm{d}f\left(X\right)}{\mathrm{d}X}=0 \tag{46}$$

observing that $\mathrm{d}\pi_c / \mathrm{d}X = 0$. The convenient solution of resulted equation is:

$$X_P = \frac{\left(\bar{c}_{1a}\right)_p}{\left(\bar{c}_{1a}\right)_n} = 1 - \frac{B_p\left(\bar{c}_{1a}\right)_n}{2K_\eta \bar{R}_2^2} \tag{47}$$

where the subscript [$]_p$ means “surge” (from the French “pompage”). By replacing in (36), the following equation is found:

$$\begin{aligned}\left(\pi_c\right)_P^{\frac{k-1}{k}} - 1 = \frac{\left(\eta_c\right)_n}{c_p}\left[1 - \frac{1}{4K_\eta}\left(\frac{B_P\left(\bar{c}_{1a}\right)_n}{\bar{R}_2^2}\right)^2\right]\frac{n}{n_n}\left(2 - \frac{n}{n_n}\right)\left(\frac{\pi D_{1m}}{60}\right)^2\left(\frac{n}{\sqrt{T_1^*}}\right)^2 \times \\ \times\left\{\bar{R}_2^2 - B_P\left(\bar{c}_{1a}\right)_n\left[1 - \frac{B_P\left(\bar{c}_{1a}\right)_n}{2K_\eta \bar{R}_2^2}\right]\right\}\end{aligned} \tag{48}$$

where

$$B_P = \bar{R}_2 \frac{A_1}{A_2}\frac{1}{\left(\pi_c\right)_P^{2/3}}\,\mathrm{ctg}\beta_2 + \mathrm{ctg}\alpha_1 \tag{49}$$

The last two equations are equivalent to an implicit expression for

$$\left(\pi_c\right)_P = f\left(\frac{n}{\sqrt{T_1^*}}\right) \tag{50}$$

On the other hand, the mass flow rate on surge line has the expression:

$$\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)_P = \frac{A_1}{\sqrt{\mathrm{R}}}\sin\alpha_1 q\left(\lambda_1\right)_P K\left(k\right) \tag{51}$$

where

$$\left(\lambda_1\right)_P = \frac{\left(\bar{c}_{1a}\right)_n}{\sin\alpha_1}\frac{\pi D_{1m}}{60}\sqrt{\frac{k+1}{2k\mathrm{R}}}\frac{n}{\sqrt{T_1^*}}\left[1 - \frac{B_P\left(\bar{c}_{1a}\right)_n}{2K_\eta \bar{R}_2^2}\right] \tag{52}$$

The last two equations are equivalent to the implicit relation

$$\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)_P = f\left(\frac{n}{\sqrt{T_1^*}}\right) \tag{53}$$

The expressions (50) and (53) represent the parametric equations of surge line, where the parameter is $n/\sqrt{T_1^*}$, so that the surge line can be plotted.

(iii) Choke line

By “choke line” one understands a curve in the compressor map which limits the regimes of low pressure.

It refers to the regimes where choking occurs at the diffuser inlet. As such, the condition which determines the choke line can be written as:

$$\min\left[\frac{A_1}{\sqrt{R}}\sin\alpha_1 q(\lambda_1)K(k),\ \left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)_{cr}^{(1)}\right]=\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)_{cr}^{(2)} \tag{54}$$

which is the same as (34) where the inequality is replaced by equality and the left hand side is replaced by (27).

(iv) Line of optimal regimes

It is considered as line of optimal regimes the locus of the states lying on the curves $n/\sqrt{T_1^*}=\text{const.}$ for which the isentropic efficiency is maximum. By considering the efficiency formula (4), the aforementioned condition reduces to:

$$\bar{c}_{1a}=\left(\bar{c}_{1a}\right)_n \tag{55}$$

so that, when applied to governing equations, one allows the line of optimum regimes to be plotted.

(v) Line of constant efficiency (efficiency islands)

By analyzing the proposed formula of the isentropic efficiency (4), one can derive the values of flow coefficient corresponding to a given value of efficiency at a given speed:

$$\frac{\bar{c}_{1a}}{\left(\bar{c}_{1a}\right)_n}=1\pm\sqrt{\frac{1}{K_\eta}\left[1-\frac{\eta_c}{\left(\eta_c\right)_n}\frac{1}{\bar{n}\left(2-\bar{n}\right)}\right]} \tag{56}$$

where $\bar{n}=n/n_n$. This condition along with the governing equations allows to plot the efficiency islands.

5. EXAMPLE OF APPLICATIONS

In the next, the off-design parameters are computed for a centrifugal compressor having the following design performances and geometric features:

$$\dot{M}_a=16\ \text{kg/s};\ \ \pi_c=3{,}893;\ \ n=11000\ \text{rpm};\ \ \eta_c=0{,}85;$$

$$A_1=0{,}103\ \text{m}^2;\ \ R_{1\text{m}}=0{,}123\ \text{m};\ \ \alpha_1=90°;\ \ \beta_{1f}=47°;\ \ Z=20\ \text{blades};$$

$$b=0{,}035\ \text{m};\ \ R_2=0{,}38\ \text{m};\ \ \beta_{2f}=90°;\ \ \alpha_{3f}=30°.$$

Table 1 shows the main parameters on the surge line. Tables 2 and 3 present the main computed parameters on choke line and optimal regimes line, respectively.

Table 4 illustrates the main parameters computed on a line of selected constant speed, that is *n* = 9000 r.p.m and Fig. 3 shows the complete map plotted for the given centrifugal compressor.

Fig. 4 presents a map of efficiency variation over off-design regimes.

Table 1. – Computed parameters on the surge line

n [rpm]	B_P	$\left(\pi_c\right)_P$	$\left(\lambda_1\right)_P$	$\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)_P$
11000	1,0214	4,2871	0,2827	0,00180
10500	1,1016	3,8277	0,2578	0,00165
10000	1,1860	3,4264	0,2334	0,00150
9500	1,2744	3,0763	0,2097	0,00135
9000	1,3663	2,7711	0,1868	0,00121
8500	1,4614	2,5051	0,1649	0,00107
8000	1,5590	2,2736	0,1440	0,00094
7500	1,6584	2,0721	0,1243	0,00081
7000	1,7590	1,8970	0,1059	0,00069
6500	1,8597	1,7450	0,0890	0,00058
6000	1,9597	1,6131	0,0735	0,00048
5500	2,0579	1,4991	0,0596	0,00039
5000	2,1532	1,4007	0,0474	0,00031
4500	2,2444	1,3162	0,0368	0,00024
4000	2,3303	1,2441	0,0278	0,00018

Table 2. – Computed parameters on the choke line

n [rpm]	$\overline{c}_{1a}$	η_c	λ_1	$\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}$	π_c	λ_2	$\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)^2_{cr}$	$\overline{c}_{2r}$	$\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)^2_{cr}$
11000	2,3318	0,6119	1,0639	0,00414	2,421	0,9541	0,00337	1,26	0,00337
10500	2,2064	0,6525	0,9609	0,00416	2,365	0,9240	0,00334	1,27	0,00334
10000	2,0317	0,6996	0,8427	0,00404	2,311	0,8926	0,00331	1,28	0,00331
9500	1,7644	0,7549	0,6952	0,00370	2,259	0,8597	0,00329	1,29	0,00329
9000	1,4759	0,7903	0,5509	0,00318	2,154	0,8255	0,00326	1,31	0,00318
8500	1,4196	0,7817	0,5005	0,00295	1,975	0,7899	0,00324	1,33	0,00295
8000	1,3764	0,7673	0,4567	0,00275	1,811	0,7530	0,00322	1,36	0,00275
7500	1,3468	0,7477	0,4190	0,00255	1,663	0,7147	0,00319	1,39	0,00255
7000	1,3317	0,7232	0,3867	0,00238	1,533	0,6751	0,00317	1,44	0,00238
6500	1,3324	0,6939	0,3592	0,00223	1,420	0,6345	0,00316	1,49	0,00223
6000	1,3509	0,6597	0,3362	0,00211	1,323	0,5931	0,00314	1,56	0,00211
5500	1,3901	0,6207	0,3171	0,00200	1,241	0,5513	0,00312	1,65	0,00200
5000	1,4541	0,5761	0,3016	0,00191	1,174	0,5096	0,00311	1,77	0,00191
4500	1,5493	0,5253	0,2892	0,00183	1,119	0,4688	0,00310	1,91	0,00183
4000	1,6856	0,4668	0,2797	0,00178	1,076	0,4299	0,00309	2,10	0,00178

Table 3. – Computed parameters on the line of optimal regimes

n [rpm]	η_c	λ_1	$\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}$	π_c	λ_2	$\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)^2_{cr}$	$\overline{c}_{2r}$
11000	0,6119	1,0639	0,00414	2,421	0,9541	0,00337	1,2692
10500	0,6525	0,9609	0,00416	2,365	0,9240	0,00334	1,2746
10000	0,6996	0,8427	0,00404	2,311	0,8926	0,00331	1,2831
9500	0,7549	0,6952	0,00370	2,259	0,8597	0,00329	1,2954
9000	0,7903	0,5509	0,00318	2,154	0,8255	0,00326	1,3120
8500	0,7817	0,5005	0,00295	1,975	0,7899	0,00324	1,3338
8000	0,7673	0,4567	0,00275	1,811	0,7530	0,00322	1,3619
7500	0,7477	0,4190	0,00255	1,663	0,7147	0,00319	1,3975
7000	0,7232	0,3867	0,00238	1,533	0,6751	0,00317	1,4424
6500	0,6939	0,3592	0,00223	1,420	0,6345	0,00316	1,4988
6000	0,6597	0,3362	0,00211	1,323	0,5931	0,00314	1,5696
5500	0,6207	0,3171	0,00200	1,241	0,5513	0,00312	1,6589
5000	0,5761	0,3016	0,00191	1,174	0,5096	0,00311	1,7723
4500	0,5253	0,2892	0,00183	1,119	0,4688	0,00310	1,9181
4000	0,4668	0,2797	0,00178	1,076	0,4299	0,00309	2,1086

Table 4. – Computed parameters on the line of constant speed n = 9000 rpm

$\overline{c}_{1a}$	η_c	λ_1	$\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}$	$\dot{M}_a$ [kg/s]	π_c	λ_2	$\left(\frac{\dot{M}_a\sqrt{T_1^*}}{p_1^*}\right)^2_{cr}$	$\overline{c}_{2r}$
0,4750	0,7911	0,1773	0,00115	6,860	2,610	0,8878	0,00373	0,4403
0,7500	0,8157	0,2800	0,00178	10,620	2,546	0,8593	0,00367	0,6709
0,8000	0,8182	0,2986	0,00189	11,277	2,529	0,8549	0,00365	0,7125
0,8500	0,8200	0,3173	0,00200	11,923	2,510	0,8508	0,00363	0,7542
0,9000	0,8212	0,3360	0,00210	12,559	2,490	0,8470	0,00360	0,7960
0,9500	0,8218	0,3546	0,00221	13,185	2,467	0,8434	0,00358	0,8381
1,0000	0,8218	0,3733	0,00231	13,799	2,444	0,8401	0,00355	0,8804
1,0500	0,8212	0,3920	0,00241	14,400	2,419	0,8371	0,00352	0,9231
1,1000	0,8199	0,4106	0,00251	14,990	2,392	0,8343	0,00348	0,9663
1,1500	0,8180	0,4293	0,00261	15,566	2,365	0,8319	0,00345	1,0100
1,2000	0,8155	0,4480	0,00270	16,129	2,335	0,8298	0,00341	1,0542
1,2500	0,8123	0,4666	0,00279	16,677	2,305	0,8279	0,00337	1,0991
1,3000	0,8085	0,4853	0,00288	17,211	2,273	0,8265	0,00333	1,1448
1,3500	0,8042	0,5040	0,00297	17,731	2,241	0,8254	0,00329	1,1913
1,4000	0,7991	0,5226	0,00305	18,234	2,207	0,8247	0,00325	1,2388
1,4500	0,7935	0,5413	0,00314	18,722	2,172	0,8244	0,00320	1,2873
1,4600	0,7923	0,5450	0,00315	18,818	2,165	0,8244	0,00319	1,2971
1,4700	0,7911	0,5488	0,00317	18,913	2,157	0,8244	0,00318	1,3070
1,4800	0,7898	0,5525	0,00318	19,007	2,150	0,8245	0,00317	1,3169

6. CONCLUSIONS

This paper presents a pure analytical method for determining the off-design behavior of centrifugal compressors.

The method removes the need of experimental data and uses only the geometrical features of the compressor and its design performances.

The numerical example shows the advantage of this analytical method which uses a low volume of computing resources and time, proving an acceptable accuracy and the possibility of extending the simulation over a full power unit.

Some new assumptions were proposed for the off-design analysis:

1) A simple parabolic expression for the isentropic efficiency was introduced, which removes the need of losses estimation over off-design regimes; also, this expression allows to plot the efficiency islands;
2) A physical condition was applied for the analytical determination of the surge line;
3) The hypothesis of constant exit angle for relative velocity has replaced the assumption of constant slip factor.

Based on these new assumptions, the proposed method can be easily extended to other kinds of compressors, like double aspiration centrifugal compressors, multistage centrifugal compressors, combined axial + centrifugal compressors etc.

The method can be also successfully implemented for the simulation of complex power units which have embedded centrifugal compressors.

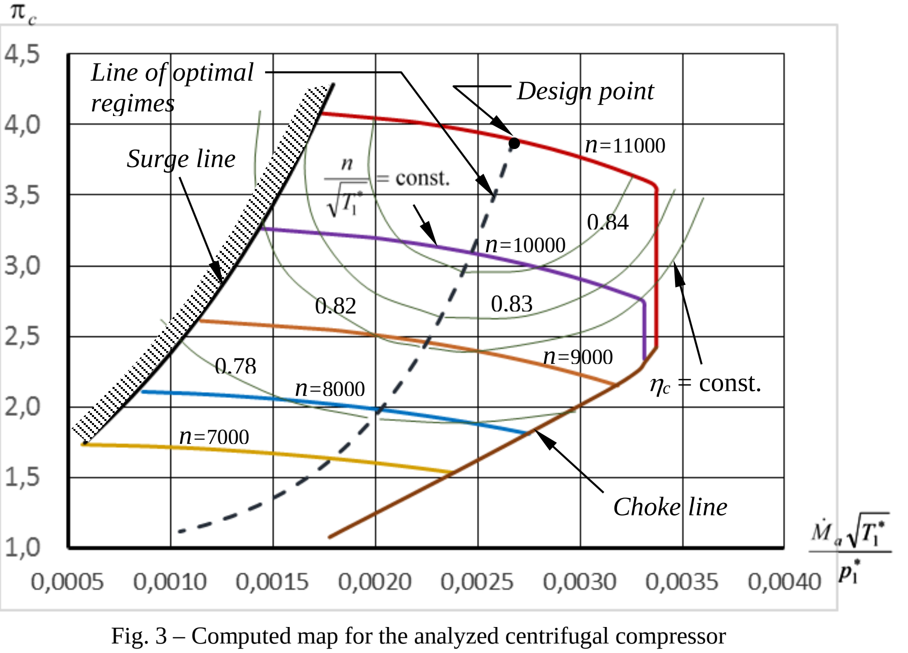

Fig. 3 – Computed map for the analyzed centrifugal compressor

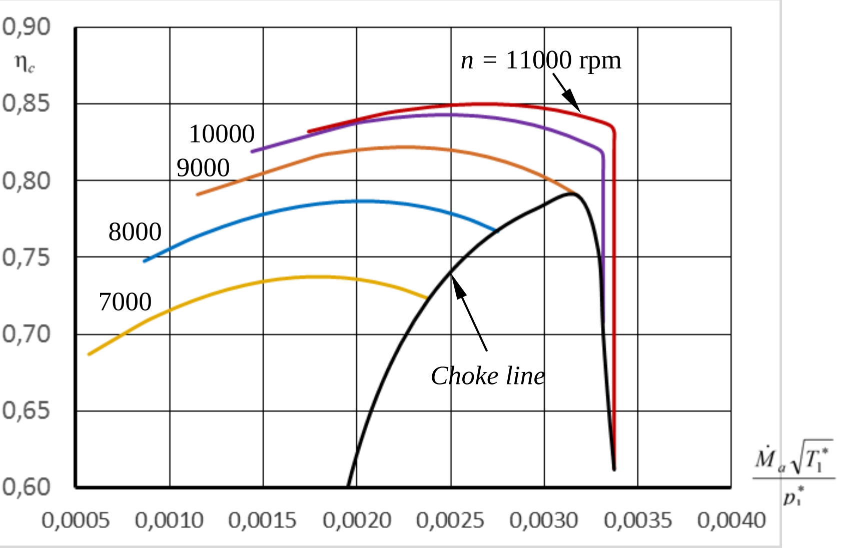

Fig. 4 – Variation of efficiency for the analyzed centrifugal compressor

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