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DOI: 10.13111/2066-8201.2014.6.3.3

Abstract: In the fourth article of our series we continue to deal with the calculation of the aerodynamic unsteady forces on lifting surfaces. Now we present some applications of the theory discussed in the previous papers to the study of flapping wings.

Key Words: Lifting surface, integral equation, harmonic oscillations, generalised airforces, flutter, flapping wings, Fourier series, Fourier transform.

1. INTRODUCTION

In our previous two articles [2], [3] we presented several applications of the harmonic oscillating lifting surface theory [1] to the study of non-harmonic oscillations of the wing, or even in the case of some non oscillatory wing motions.

We now present the study of the flapping wing by using the theories developed before. So the purpose of this work is to study the straight flight aerodynamics of the micro-air vehicle (MAV).

It should be emphasized that we consider that the flapping oscillations are of small amplitude. The wake is therefore plane.

The influence of the leading edge suction and the induced drag are not yet included in our study.

2. A SHORT PRESENTATION OF THE OSCILLATING LIFTING SURFACE THEORY IN RECTILINEAR MOTION

We present below the main results from [1] and [2].

As usually, the lifting surface is denoted here by the symbol (W) and is represented by the next parametric form:

\[ \tilde{M} = \tilde{M}_0(\lambda, s) + \tilde{O}(\lambda, s; t) \]  (1)

We remind that \( \lambda \) is a chordwise parameter (= 0 for the leading edge and 1 for the trailing edge) and \( s \) a spanwise parameter (\(-1 \leq s \leq 1\) for symmetrical wings).
If $\tilde{O}(\lambda, s; t) = 0$ we have a steady lifting surface denoted here by $(W_0)$. The wing performs oscillations of small amplitude,

$$0 < \left| \tilde{O}(\lambda, s; t) \right| \ll 1.$$ 

The oscillating surface is represented here by the symbol $(W)$. We always consider that

$$\tilde{O}(\lambda, s; t) = l \cdot \tilde{\sigma}(\lambda, s) \cdot q(t)$$  \hspace{1cm} (2)

Here $l$ is a reference length, $\tilde{\sigma}(\lambda, s)$ is a function describing the non-dimensional displacement mode, and $q(t)$ it represents the generalised displacement. The variable $t \in (-\infty, \infty)$.

We will use the normalwash to $(W)$:

$$\tilde{n}(\lambda, s) = \tilde{n}_0(\lambda, s) + \tilde{n}_1(\lambda, s)q(t) + \tilde{n}_2(\lambda, s)q^2(t).$$  \hspace{1cm} (3)

In the above equation $\tilde{n}_0$ is the unit normal vector to $(W_0)$, while $\tilde{n}_1$ is infinitely small of the first order and $\tilde{n}_2$ is infinitely small of the second order. The last normal component in (3) will be neglected from now on.

There are two cases of interest:

1) **Steady lifting surface case**: $q(t) = 0$. The steady normalwash

$$w_0(\lambda, s) = -n_0(\lambda, s)$$  \hspace{1cm} (4)

and the pressure coefficient jump $p^*_0$ are linked by the following integral equation

$$\frac{1}{8\pi} \int_{(W_0)} K_0(\lambda, s; \mu, \sigma; M) p^*_0(\mu, \sigma) dS = w_0(\lambda, s),$$

$$dS = \sqrt{EG - F^2} d\mu d\sigma$$  \hspace{1cm} (5)

2) **Unsteady lifting surface case**: $q(t) = e^{i\omega t}$. Between the oscillatory normalwash

$$w_1(\lambda, s; \omega, t) = U_\infty \left\{ - n_{1x}(\lambda, s) + i \cdot \frac{\omega l_{ref}}{U_\infty} \cdot \left[ \tilde{\sigma}(\lambda, s) \cdot \tilde{n}_0(\lambda, s) \right] \right\} e^{i\omega t} =

= U_\infty \left\{ - n_{1x}(\lambda, s) + i \cdot k \cdot \left[ \tilde{\sigma}(\lambda, s) \cdot \tilde{n}_0(\lambda, s) \right] \right\} e^{i\omega t} = U_\infty \tilde{w}_1(\lambda, s; k) e^{i\omega t}$$  \hspace{1cm} (6)

and the pressure coefficient $p_1^*$, the following integral equation holds true:

$$\frac{1}{8\pi} \int_{(W_0)} \tilde{K}_1(\lambda, s; \mu, \sigma; M; k) p_1^*(\mu, \sigma) dS = \tilde{w}_1(\lambda, s; \mu, \sigma; k)$$  \hspace{1cm} (7)

In the above equation (6) $l_{ref}$ represents a reference length, while $k$ is the reduced frequency.

$$k = \frac{\omega l_{ref}}{U_\infty}$$  \hspace{1cm} (8)

If $k=0$ (i.e $\omega=0$), the kernel $\tilde{K}_1$ turns into $K_0$. The kernel is given in [1] but it is too complicated to be reproduced here.
However, \( \tilde{w}_1(\lambda, \sigma_0) = -n_{1x} \) while \( w_0(\lambda, s) = -n_{0x}(\lambda, s) \).

We usually know the normal wash \( \tilde{w}_1 \) (or for the steady flow, \( w_0 \)) and we need \( p_1^* \) (or, \( p_0^* \), respectively).

Thus, we have reduced the problem to that of solving the integral equation (7) for unsteady flow, or (5) for the steady flow.

There are several methods for solving these integral equations. The author uses the so-called doublet-lattice method (DLM).

All the above theory is well documented, but it has been presented here to facilitate understanding of the text that follows.

### 3. PERIODIC WING OSCILLATIONS

In the following, we present some results obtained in our previous work \[2\]. We shall use the results of the unsteady lifting surface theory presented above. Suppose that the integral equation (7) has been solved using the DLM method. For this purpose the author has available a Fortran code. The following cases are of interest:

#### 3.1 Harmonic oscillation of the wing- type 1, cosinus time-law

\[
\tilde{O}_i(\lambda, s, t) = l \cdot \tilde{\sigma}(\lambda, s) \cos \omega \cdot t
\]

The pressure-coefficient jump response will be:

\[
P_i(\lambda, s, t) = 2 \left[ \text{Re}(p_1^*) \cos \omega \cdot t - \text{Im}(p_1^*) \sin \omega \cdot t \right]
\]

In the above equation, \( \tilde{p}_1^* \) corresponds to the following normal wash

\[
\tilde{O}_i(\lambda, s, t) = l \cdot \tilde{\sigma}(\lambda, s) e^{i\omega t}
\]

#### 3.2 Harmonic oscillation of the wing- type 2, sinus time-law

\[
\tilde{O}_s(\lambda, s, t) = l \cdot \tilde{\sigma}(\lambda, s) \sin \omega \cdot t
\]

Now, the pressure-coefficient jump for the same normal wash (11) will be:

\[
P_i(\lambda, s, t) = 2 \left[ \text{Re}(p_1^*) \sin \omega \cdot t + \text{Im}(p_1^*) \cos \omega \cdot t \right]
\]

#### 3.3 General periodic oscillations. The motion law is given by:

\[
\tilde{O}(\lambda, s, t) = l \cdot \tilde{\sigma}(\lambda, s) q(t)
\]

The time-function \( q(t) \) is assumed to be periodic, so that it can be represented as a Fourier series:

\[
q(t) = q_0 \left[ \frac{a_0}{2} + \sum_{n=1}^{N} \left( a_n \cos \frac{2\pi}{T} nt + b_n \sin \frac{2\pi}{T} nt \right) \right] = q_0 \left[ \sum_{m=-N}^{N} c_m e^{i\frac{2\pi mt}{T}} \right]
\]

\[
q_0 \sum_{m=-N}^{N} c_m e^{i\omega_m t}
\]
Here \( a_n \) and \( b_n \) are the Fourier coefficients and

\[
\omega_m = \frac{2\pi}{T} m, \ m \in \{-N, N\}, \text{or } m = \pm n
\]  

(16)

We remind that the complex form of the Fourier expansion has the complex coefficients:

\[
\begin{align*}
    c_n &= \frac{a_n - i b_n}{2} \\
    c_0 &= a_0 \\
    c_{-n} &= \frac{a_n + i b_n}{2} = \bar{c}_n
\end{align*}
\]  

(17)

Then the pressure-coefficient jump is given by:

\[
P(\lambda, s, t) = 2 \sum_{n=1}^{N} \left[ \text{Re}(c_n \cdot p_1^{*\,(n)}) \cos \omega_n t - \text{Im}(c_n \cdot p_1^{*\,(n)}) \sin \omega_n t \right] + p_0^*
\]  

(18)

Here, the pressure coefficients \( p_1^{*\,(n)} \) and \( p_0^* \) correspond to the next normalwashes:

\[
\begin{align*}
    \vec{w}^{(n)} &= -n_{1x} + i k^{(n)} \cdot \vec{\phi} \cdot \vec{n}_0 \\
    w_0 &= -n_{0x}
\end{align*}
\]  

(19)

The \( n \)-th reduced frequency is calculated as:

\[
k^{(n)} = \frac{\omega_n^{ref}}{U_\infty}
\]  

(20)

One can see that both \( \omega_n \) and \( k_n \) are positive, while \( \omega_m = \pm \omega_n \) and \( k_m = \pm k_n \).

### 4. PERIODIC WING OSCILLATIONS

We consider the case of an MAV that performs a straight flight with the constant speed \( U_\infty \). Simultaneously, its wings\(^1\) execute two kinds of motions:

- plunging motion;
- pitchig motion.

We will give some examples of this kind of motions. The two wings assembly \( W_0 \) lay on a plan \( xOy \), fig 1. Due to each wing movements of flapping and pitching, its instantaneous position will be \( W \).

However \( W \) will be considered very close to \( W_0 \), so we will be able to apply the lifting surface theory presented before.

Firstly, we will present some symmetrical wing motions:

#### 4.1 Flapping (plunging) motion

One of the simplest example could be the next one:

---

\(^1\) Usually in the flapping flight theory authors call ‘wing’ each of the two (left and right) lifting surfaces. So a bird has two wings, although for aeroplanes we call ‘wing’ the assembly of the two ‘semi-wings’.
In the above equation, $A_f$ is the plunging (flapping) amplitude and $2s$ is the wings assembly span. $b_f(t)$ is the usual time-function, depending on the plunging mechanism. It is a periodic function. A simple example could be:

$$b_f(t) = \cos \omega t$$

One can easily see that (21) describes a rigid motion. Of course, we can have elastic modes as well.

### 4.2 Pitching motion

We will give an example of a pitching rigid mode motion:

$$z_2(x, t) = A_p \frac{x - x_{ac}}{l_{ref}} b_p(t)$$

Here $A_p$ is the pitching amplitude and $x_{ac}$ is the aerodynamic centre abscissa. One can see that the wings oscillate about the $y=x_{ac}$ (aerodynamic centre axis). The time-function $b_p(t)$ is, of course, a periodic function. We can have another pitching motion described by:

$$z_3(x, t) = A_p \frac{x - x_{ac}}{l_{ref}} \frac{y}{s} b_p(t)$$

Equation (24) contains an elastic twist mode. Figure 2 presents a typical $b_p(t)$ function: the pitching mechanism changes rapidly the value of $b_p$ in the proximities of $t=0; T/2; T;...$, but maintains some kind of plateaux between these values. This allows the maintaining of almost constant angles of attack during long periods of time.
As we will see later, the MAV’s use a flapping motion, for example (21) and a pitching motion (23) or (24) simultaneously. We can also imagine some antisymmetric modes:

### 4.3 Antisymmetric flapping motion

The next example contains a *flapping rigid mode motion*:

\[
z_d(y, t) = A_f \frac{y}{s} b_f(t)
\]  

(25)

This rigid mode is an antisymmetric flaping mode used by some MAV builders for its simplicity (see, for example the ’X-Wing’ entomopter, [4]).

### 4.4 Antisymmetric pitching motion

The next example contains a *flapping rigid mode*:

\[
z_5(x, t) = \begin{cases} 
z_2(x, t) & \text{if } y \geq 0 \\
-z_2(x, t) & \text{if } y < 0
\end{cases}
\]  

(26)

The two wings are rigid bodies and they oscillate in phase opposition. A very simple elastic mode of oscillation could be the next one:

\[
z_6(x, y, t) = A_p \frac{x - x_{ac}}{l_{ref}} \frac{y}{s} b_p(t)
\]  

(27)

These antisymmetric pitching modes can be used in combination with the symmetric modes to control the MAV flight (roll, yaw). The antisymmetric flapping mode can also be used in combination with the antisymmetric pitching modes to obtain the straight flight (case never found in nature, but preferred sometimes for its simplicity of flapping-pitching mechanism).

### 5. CASE STUDY: SYMMETRIC FLAPPING AND PITCHING

We shall apply the theory presented before to the study of the symmetric flapping and pitching wings. Figure 3 presents a typical wing-section of a flapping wing. We can write the aerodynamic resultant

\[\begin{align*}
\vec{R} &= \vec{L} + \vec{D} = \vec{N} + \vec{T} \\
\end{align*}\]

Fig. 3 Typical wing-section in flapping and pitching motions.

*R*-aerodynamic resultant (including LE suction); *L*-lift; *D*-drag; *N*-normal force; *T*-thrust; *w*-relative wind due to wing motion.
So, the aerodynamic resultant $\vec{R}$ is made of lift and drag. $\vec{R}$ includes tacitly the leading edge suction.

Firstly we observe that the above theory does not allow the calculation of the drag and leading edge suction (the last, not represented in fig. 3, but included in $\vec{R}$). We presented before a way to calculate the normal forces on the lifting surfaces.

The calculation of the lift-induced drag and suction force require the use of the integral form of the theorem of momentum.

This problem will be solved later. On the other part, $\vec{N}$ is the normal force to the flight path and $\vec{T}$ is the thrust.

So consider an assembly of two flapping wings (fig. 4). Consider that at rest the two wings lie on the $xOy$ plane.

The position vector of a point on the resting wings is:

$$\vec{M}_0 = x\hat{i} + y\hat{j} + 0\hat{k}$$  \hbox{(28)}

![Diagram of two flapping wings at rest with pitching hinges](image)

Fig. 4 The two flapping wings at rest; at $x_{ac}=0.25c$ are the pitching hinges

The wing chord is $c = l_{ref} = 0.025m$ and its length is $R = 0.1m$ (=semi-span of their combination).

The two wings are set in motion after the two modes of oscillation:

$$\begin{align*}
  z_1(y, t) &= A_f \frac{|y|}{s} \cos \omega t \\
  z_2(x, t) &= A_p \frac{x - x_{ac}}{l_{ref}} b_p(t)
\end{align*}$$  \hbox{(29)}
The two modes of oscillation are represented in fig. 4. The pitching mode from fig. 2 is approximated by:

\[ b_p(t) = 1.25\sin\omega_t + 0.3829\sin\alpha_3t + 0.2213\sin\alpha_2t + 0.1543\sin\omega_t; \]

\[ \omega_p = \frac{2\pi}{T}, \quad T = 0.1s, \quad p = 1; 3; 5; 7 \]  

(30)

We calculate the reduced frequencies: \( k_p = 0.157; 0.471; 0.785; 1.1. \) For the moment, we put in (30) \( A_f = 1 \) and \( A_p = 1 \) (Normally, for our case \( A_f = O(c), A_p = O(0.1rad) \)).

With the geometry of the whole wing (fig. 4), the two oscillating modes and each of the four reduced frequencies we calculate the pressure coefficients:

- \((p_1^f)^f\) for mode 1, flapping (plunging);
- \((p_1^p)^p\) \( p = 1, 3, 5, 7 \) for mode 2, pitching.

The pressure distributions are given by the Fortran code. We used a lattice of \( NC = 10 \) boxes in chord and \( NS = 15 \) boxes in span.

To illustrate the pressure distributions we used here the representative section \( y/R = 0.5 \).
The real and imaginary parts of the pressure distributions, pitching modes 1, 3, 5, 7 and at \( y/R=0.5 \).

Fig. 6

The pressure distribution after the plunging mode (30-I) is represented in fig. 7 and it was calculated with (10).

We can see the pressure evolution at certain values of the time in the first quadrant \( t=0, T/16, 2T/16, 3T/16, 4T/16 \).

Fig. 7

The same evolution of the pressure distribution, this time for the pitching mode (30-II) is presented in fig. 8.
Fig. 8 Pressure coefficient variation with time, after mode 2: pitching; $A_p=1$

$t=0, T/16, 2T/16, 3T/16, 4T/16$

The next picture (fig. 9) gives the spanwise variation of the wing load,

$$p_f = \frac{\sum_{i=1}^{NC} (p_i^*) \Delta A_i}{\sum_{i=1}^{NC} \Delta A_i}$$

Fig. 9 Spanwise load due to flapping mode, $A_f=1, A_p=1$. 

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One can see that:
- The extremum occurs at \( y/R = 0.75 \)
- the values are spread equally over the entire range for all values of \( t \).

The next figure presents the similar curve, but for the pitching mode. The load distribution is now similar to that in the steady flow.

However the curves gather in a "bunch" for \( t = T/16 \ldots 6T/16 \). This behaviour is not unexpected, due to the special shape of the \( b_p(t) \) curve.

**Fig 10** Spanwise load due to pitching mode, \( A_p = 1 \)

The last curve presents the global aerodynamic resultants for the two modes, flapping and pitching (fig. 11). One can see that the pitching resultant presents the "waves" that characterise the Fourier approximation (30).

**Fig. 11** Global force normal to the wings, \( A_f = 1, A_p = 1 \)
6. CONCLUSIONS

1. This is the first step towards the study of the aerodynamics of the flapping wings. There are several parts of the problem to be clarified. For example, we mention the suction force and wing induced drag. As we know the suction force plays a crucial role in the thrust generation. For these calculations, one can use the momentum theorem (to be included in a further paper).

2. It is known that the ornithopteres use frequently gliding flight combined with periods of flapping flight. This kind of flight can be approached by the method presented in [3]. The suction and induced drag will be calculated in a similar manner as presented at point 1.

REFERENCES


