Aerodynamic Interference between Oscillating Lifting Surfaces and Fuselage. Part 6: A Panel Method for Non-Lifting Bodies Oscillating in Subsonic Flow

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Abstract: *In the sixth part of this series, we continue our study started in the previous article, i.e., we present the basis of the method for calculating the unsteady airforces on oscillating non-lifting bodies. The method proposed here is a panel one. As it is proved in the previous paper, the potential and velocity induced by a constant source planar panel is composed of two kinds of terms: some of them- the steady terms, can be expressed by the elementary functions; other terms- the unsteady terms cannot and should be calculated numerically. The present paper illustrates how the method proposed here can be used in conjunction with the preceding results.*

Key Words*: Flow about bodies, lifting surface, integral equation, harmonic oscillations, generalised airforces, flutter, oscillating bodies*

1. INTRODUCTION

This article is the sixth in a series that deals with the unsteady flow about wings, bodies and their combinations, [1]-[5]. This article brings some clarifications on the principles that were the basis of the foregoing one, [5]. It is also presented an immediate application of the panel method given in the same paper. The purpose is to obtain the pressure distribution and the velocity perturbation fields that are generated by an isolated fusiform body oscillating harmonically about a mean (equilibrium) position.

In the previous paper [5] we showed that the problem of estimating the unsteady aerodynamics of elongated bodies performing harmonic oscillations of small amplitude, has been the subject of several studies appeared over time. In 1923 Munk developed the concept of "flow in the transverse plane", [10] available for steady flow case; it applies to very elongated bodies. It assumes that in the transverse plane, the flow is mainly two-dimensional. For the case of the unsteady flow, Stewartson developed a method which also applies to the elogated bodies [11]. It implicitly assumes the same hypothesis of the transverse plane flow. Closer to our days, important contributions to the theory of unsteady flow about the revolution bodies had Wu, Garcia - Fogeda and Liu, [12], [13]. Their works exceed the narrow theory of the elongated bodies, so that they apply to the demands of modern configurations of missiles. However, their method cannot be applied to the bodies

having non-circular sections; they cannot be used to wing-body combination as well. This last case is presented in [14] and it will be analyzed in a further paper when we will discuss the unsteady aerodynamic interference problem.
The present article resumes and enhances some original ideas the author firstly treated in [6]-[9]. In the meanwhile, he has made some improvements regarding the numerical approach and mathematical presentation. The method allows the calculation of the distribution of pressure on non-lifting bodies of any aerodynamic shape. As we will see, the non-lifting case raises specific problems that are distinct from those found in the case of lifting surfaces. The resulting method can be extended to the applications concerning the combinations of wings and bodies that perform harmonic oscillations.

2. FORMULATION OF THE PROBLEM

Let us consider an elongated body such as an isolated fuselage of any cross section. The body is located in an infinite domain of fluid which flows with constant subsonic speed at infinity (fig. 1). On the other hand, the body performs small harmonic oscillations. Our purpose is to find the unsteady pressure on the body surface.

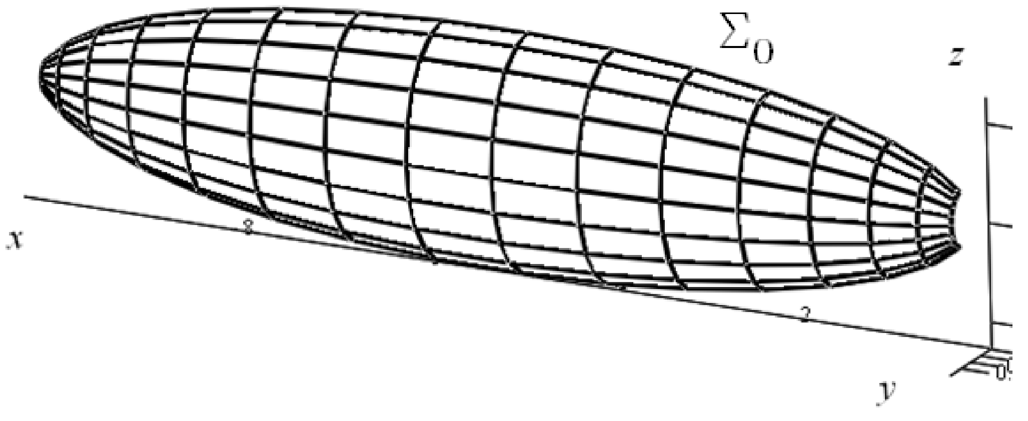

Fig. 1 Elongated body in subsonic stream that flows from right along Ox axis

Let us consider the surface of the body (Σ_0) when it is at rest with respect to its own frame and (Σ) the oscillating surface of the same body. Then we can write the body surface parametric equations in the two situations, at rest and when it performs oscillations of small amplitude,

$$\begin{aligned} &(\Sigma_0) \qquad \vec{P} = \vec{P}_0(U,V), \\ &(\Sigma) \qquad \vec{P} = \vec{P}_0(U,V) + \delta \cdot \vec{P}_1(U,V) e^{i\omega t} \end{aligned} \tag{1}$$

Here, $\vec{P} = \begin{pmatrix} x & y & z \end{pmatrix}^T$ is a point on (Σ_0) or (Σ), the parameter $0 < \delta << 1$ is the local non-dimensional amplitude factor, ω is the angular frequency of the oscillations and $i = \sqrt{-1}$; $\frac{|\vec{P}_1|}{l} = O(\delta_F)$ where l is a reference length, here the body length and δ_F = body max. diameter /l.

Consider the velocity potential of the form:

$$\Phi(x, y, z; t) = U_\infty x + \varphi_0(x, y, z) + \delta \cdot \varphi_1(x, y, z) e^{i\omega t} \tag{2}$$

So, the velocity field is given by:

$$\vec{V}(x, y, z; t) = U_\infty \vec{i} + \vec{v}_0(x, y, z) + \delta \cdot \vec{v}_1(x, y, z) e^{i\omega t} \tag{3}$$

The velocity fields $\vec{v}_0$ and $\vec{v}_1$ are given by:

$$\vec{v}_0(x, y, z) = \mathrm{grad}\varphi_0(x, y, z) \ , \ \vec{v}_1(x, y, z) = \mathrm{grad}\varphi_1(x, y, z) \tag{4}$$

If $\vec{n}_0$ is the unit normal vector to the (Σ_0), we can write the ”quasi-unit” normal vector to (Σ) as

$$\vec{n} = \vec{n}_0(U, V) + \delta \cdot \vec{n}_1(U, V) e^{i\omega t} \tag{5}$$

The demonstration for this formula and expressions for $\vec{n}_0$ and $\vec{n}_1$ are given in Appendix A; see also [1]).
The boundary condition is obtained from (3) and (5). It splits in two equations, one for steady and the other for unsteady motions. They can be written as:

$$\begin{cases} \vec{v}_0 \cdot \vec{n}_0 + n_{0x} = 0 \\ \vec{v}_1 \cdot \vec{n}_0 + n_{1x} - ik\vec{P}_1 \cdot \vec{n}_0 = 0 \end{cases} \tag{6}$$

In the above equations, $\vec{n}_0$, $\vec{n}_1$, $\vec{P}_1$ are calculated at the same point P_0 on (Σ_0), and

$$n_{0x} = \vec{n}_0 \cdot \vec{i} \ , n_{1x} = \vec{n}_1 \cdot \vec{i} \ .$$

On the other hand, $\vec{v}_0, \vec{v}_1$ are calculated at a point Q, in the neighborhood of the point P_0 (fig. 2):

$$\vec{Q} = \vec{P}_0 + \varepsilon \cdot \vec{n}_0, \quad \text{with } \varepsilon > 0 \text{ and } \varepsilon \to 0$$

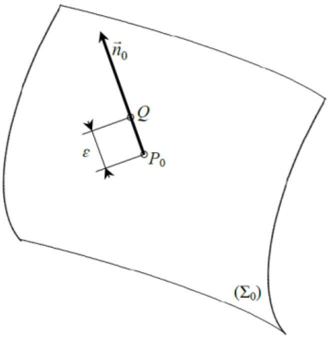

Fig. 2 For the boundary condition

The pressure coefficient is calculated in Appendix B. One founds that

$$c_p = c_{p0} + \delta \cdot c_{p1} e^{i\omega t} \tag{7}$$

For elongated bodies, c_{p0} and c_{p1} assume, within the framework of linearization, the appoximate forms

$$\begin{cases} c_{p0} = -2\dfrac{1}{U_\infty}\dfrac{\partial\varphi_0}{\partial x} \\ c_{p1} = -2\dfrac{1}{U_\infty}\dfrac{\partial\varphi_1}{\partial x} - 2i\dfrac{\omega}{U_\infty}\dfrac{\varphi_1}{U_\infty} \end{cases} \tag{8}$$

Let us write the steady and oscillatory potentials, $\varphi_0(x,y,z)$ and $\varphi_1(x,y,z)$.
The flow about this single body can be represented (see Appendix B) by two simple layers (or source distributions) potentials of intensities $q_0(\xi,\eta,\zeta;M)$ and $q(\xi,\eta,\zeta;M,\kappa)$.

$$\begin{aligned} \varphi_0(x,y,z) &= -\frac{1}{4\pi}\int_{(\Sigma_0)}\frac{1}{R}q_0(\xi,\eta,\zeta;M)d\Sigma_0 \\ \varphi_1(x,y,z;M;\omega) &= -\frac{1}{4\pi}\int_{(\Sigma_0)}\frac{e^{i\kappa(Mx_0-R)}}{R}q_1(\xi,\eta,\zeta;M,\kappa)d\Sigma_0 \end{aligned} \tag{9}$$

Here,

$$\begin{aligned} &(\Sigma_0) = \text{body surface}; (\xi,\eta,\zeta)\in(\Sigma_0); \\ &x_0 = x-\xi \;;\; y_0 = y-\eta \;;\; z_0 = z-\zeta \;;\; (\xi,\eta,\zeta)\in(\Sigma_0); \\ &R = \sqrt{x_0^2+\beta^2 r^2} \;;\; r^2 = y_0^2+z_0^2 \;;\; \kappa = \frac{\omega}{U_\infty}\frac{M}{\beta^2}; \\ &U_\infty = \text{wind speed at infinity}; \\ &M_\infty = \frac{U_\infty}{a_\infty}, \text{where } a_\infty = \text{sound speed at infinity}; \\ &\beta^2 = 1-M_\infty^2. \end{aligned} \tag{10}$$

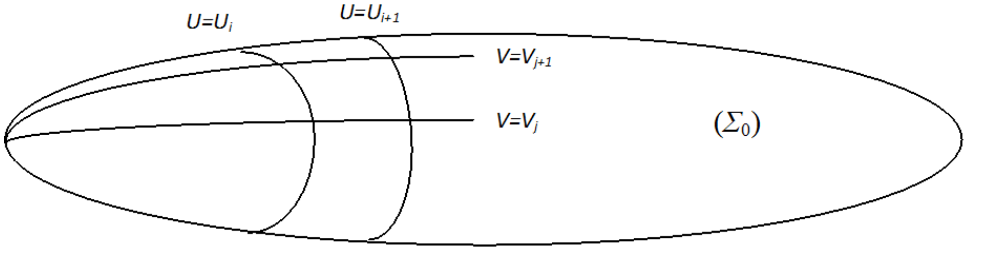

Fig. 3 An elongated body (fuselage) and a curvilinear panel generated by the co-ordinate lines $U=U_i$, $U=U_{i+1}$ and $V=V_j$, $V=V_{j+1}$ on the body surface, (Σ_0)

The next property holds true:

$$\lim_{\omega\to 0}\varphi_1(x,y,z;M;\omega) = \varphi_1(x,y,z;M;0) = \varphi_0(x,y,z;M)$$

That is the oscillatory potential converts itself into the steady one when $\omega\to 0$. The body surface $\vec{P}_0(U,V)$ is known; and so are M and ω. The body oscillation mode $\delta\cdot\vec{P}_1(U,V)$ is given as well. We have reduced our aerodynamic problem to the Neumann problem (4)+(6)+(9) with q_0 and q_1 unknown functions.

3. THE NUMERICAL SOLUTION

In order to solve numerically our problem, we must be able to calculate the boundary condition (6) and and the pressure coefficients (8). In other words, we have to:

- Calculate the normalwash over the body surface (Σ_0) - for (6)
- Calculate the potential and the *Ox* projection of the induced velocity- for (8).

Consequently, we need a method to calculate numerically both the potential and the velocity field in space.
Consider a certain parametric equation that describes the elongated body surface:

$$\begin{aligned} x &= l \cdot U \\ y &= l \cdot R(U,V)\cos V \\ z &= l \cdot R(U,V)\sin V \end{aligned} \tag{11}$$

where $U \in (0,1)$ and $V \in [0,2\pi)$.
We consider that our surface (Σ_0) is symmetrical with respect to the *xOz* plane. This is the situation of the majority of the applications in aviation and rocketry.
Let us consider a number of $M+1$ and respectively $2N+1$ sections through the body surface, i.e. $\{U_i\}_{i=1,M+1}$, $\{V_j\}_{j=1,2N+1}$

They are in fact coordinate lines on (Σ_0) (figs. 1, 3). We choose the planes $x = l \cdot U_1$, $x = l \cdot U_{N+1}$ very close to the body tips (fig. 1). The coordinate lines V_1, V_{2N+1} have the property that, for a certain *U*, they define the same point on the ventral position. The curvilinear panels are counted in rows between two sections *i* and *i*+1, from the ventral (V_1) to the dorsal position (V_N) of the right semi-body, then the next row, between *i*+1 and *i*+2. Knowing the curvilinear panel vertices $\mathrm{N}_k, k = 1,2,3,4$, one can find the planar panel vertices N'_k, using the familiar rule of panneling the aerodynamic surfaces, [15],[16].
Consider a panel (Π_i). We define a point $Q \in (\Pi_i)$ as the point at which we impose the boundary condition. For the steady case, it is recommended to use the null point of the tangetial velocity. As for the oscillatory case there is no such point, one can use one of the two ways:

- To impose a position which has been found previously using numerical experiments;
- To use the panel gravity centre, as some authors recommend for the steady case.

In what follows, we will refer only to the oscillatory case. The steady case is interpreted as a particular one of the unsteady case, as we have previously noted.
Consider the potential and induced velocity fields at a point *Q*:

$$\begin{aligned} \varphi_1(Q) &= -\tfrac{1}{4\pi}\sum_{j=1}^{2N_{pan}} q_{1j} \int\limits_{(\Pi_j)} \frac{\exp[i\kappa(Mx_0 - R)]}{R} dS \\ \vec{v}_1(Q) &= -\tfrac{1}{4\pi}\sum_{j=1}^{2N_{pan}} q_{1j} \int\limits_{(\Pi_j)} \vec{W}(x_0, y_0, z_0; M; \omega) dS \end{aligned} \tag{12}$$

where N_{pan} is the total number of panels over the half of the symmetrical surface (Σ_0), and $\vec{W}$ is the induced velocity given by:

$$\vec{W}(x_0, y_0, z_0; M; \omega) = \left\{-\frac{\vec{R}}{R^3} + i\kappa\left(M\vec{i} - \frac{\vec{R}}{R}\right)\frac{1}{R}\right\}e^{i\kappa(Mx_0 - R)} \tag{13}$$

where

$$\vec{R} = x_0 \cdot \vec{i} + \beta \cdot \vec{r} \ , \ \vec{r} = y_0 \cdot \vec{j} + z_0 \cdot \vec{k}$$

Using the symmetry property of the body, we consider two such symmetrical panels with respect to the *xOz* plane: $(\Pi_j), (\Pi'_j)$. Then

$$\begin{aligned}
\varphi_1(Q) &= -\tfrac{1}{4\pi}\sum_{j=1}^{N_{pan}} q_{1j}\left(\int_{(\Pi_j)} \frac{\exp[i\kappa(Mx_0 - R)]}{R}dS \pm \int_{(\Pi'_j)} \frac{\exp[i\kappa(Mx_0 - R')]}{R'}dS'\right)\\
\vec{v}_1(Q) &= -\tfrac{1}{4\pi}\sum_{j=1}^{N_{pan}} q_{1j}\left[\int_{(\Pi_j)} \vec{W}(x_0, y_0, z_0; M; \omega)dS \pm \int_{(\Pi'_j)} \vec{W}(x_0, y'_0, z_0; M; \omega)dS'\right]
\end{aligned} \tag{14}$$

In the above equations,

$$y'_0 = y + \eta \ ; \ R' = \sqrt{x_0^2 + \beta^2 \cdot (y_0'^2 + z_0^2)}$$

and the sign + is available for symmetrical oscillation modes with respect to *xOz* plane, while – is taken for antisymmetrical modes.
On the other hand all the expressions like

$$-\tfrac{1}{4\pi}\int_{(\Pi_j)} \frac{\exp[\ldots]}{R}dS; -\tfrac{1}{4\pi}\int_{(\Pi_j)} \vec{W}dS$$

can be calculated using *the methods given in our previous article* [5].
Consider now $Q \in (\Pi_i)$ and let $\vec{n}_{0i}$ be the normal unit vector to (Π_i). We can write the boundary condition

$$\sum_{i=1}^{N_{pan}} q_{1j}C_{i\,j} = w_i \ ; \quad j = 1,2,\ldots N_{pan} \tag{15}$$

In the above equation,

$$\begin{aligned}
C_{i\,j} &= \vec{n}_{0i} \cdot \left[\left(-\tfrac{1}{4\pi}\right)\int_{(\Pi_j)} \vec{W}dS \pm \left(-\tfrac{1}{4\pi}\right)\int_{(\Pi'_j)} \vec{W}'dS'\right];\\
w_i &= (-n_{1x} + ik\vec{P}_1 \cdot \vec{n}_0)_i
\end{aligned} \tag{16}$$

The matrix $C_{i\,j}$ is called the influence coefficients matrix. As formulated here, at certain pair of numbers *M* and *k*, the problem has two coefficients matrices $C_{i\,j}{}^{+}$ and $C_{i\,j}{}^{-}$ one for symmetrical oscillation modes and other for antisymmetrical oscillation modes.

Usually we are interested in several oscillation modes, both symmetrical or antisymmetrical. Therefore we prefer to solve the system (15) by inverting the matrix:

$$q_{1i} = \sum_{i=1}^{N_{pan}} D_{i\,j} w_j, \qquad \text{where } D_{i\,j} = C_{i\,j}^{-1} \text{ and } i = 1,\ldots N_{pan} \tag{17}$$

One can now write the pressure coefficient as

$$c_{p_i} = -2 \sum_{j=1}^{N_{pan}} \mathcal{K}_{i\,j} q_j \tag{18}$$

In the above equation,

$$\mathcal{K}_{i\,j} = (-\tfrac{1}{4\pi}) \int_{(\Pi_j)} W_x dS \pm (-\tfrac{1}{4\pi}) \int_{(\Pi'_j)} W'_x dS' + ik \left[\int_{(\Pi_j)} \frac{\exp(\ldots)}{R} dS \pm \int_{(\Pi'_j)} \frac{\exp(\ldots)'}{R'} dS' \right] \tag{19}$$

So we found the pressure distribution on the oscillating body surface. We have a pressure distribution after each oscillation mode.

Adopting the method already used in case of the lifting surface [2], we can calculate the matrix of generalized air forces acting on the body $Q_{k,m}$, where the load is calculated with the mode *m* and with displacement after the mode *k*.

4. A NUMERICAL EXAMPLE

In the next article we will present a number of examples of flows about steady and oscillating bodies. Here we will limit ourselves to one simple case: an ellipsoid of revolution with a thickness ratio $\delta_R = \dfrac{diam}{l} = 0.1$. It oscillates harmonically after a rigid mode of pitchig about its centre,

$$\vec{P}_1 = -(x - 0.5\,l)\vec{k}$$

Consider now k=0.1 at M=0.

For our simple example, we have used a small number of panels NPAN=4×18=72. We must mention that the diagram represents the real and imaginary parts of the function $\dfrac{C_p}{\sin V}$ at different positions along *x*-axis. We took this function because, for elongated bodies of revolution, at U=const, $C_p(V)$ is a sinusoid.

Although the number of panels is small, the results are good when compared with Stewartson's analytical method [11], represented as dot lines in fig. 4.

The real parts of the pressure coefficient are so close that the curves practically coincide. Small differences occur only for the imaginary parts (dot lines on fig. 4).

Much more examples will be shown in our further paper.

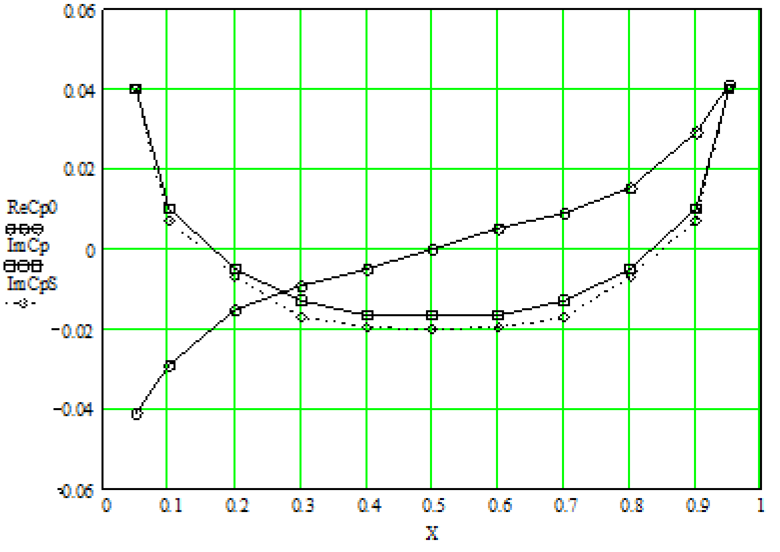

Fig. 4 Pressure coefficient distribution along the *x*-axis of an ellipsoid of revolution oscillating in pitch, *k*=1, *M*=0.

5. CONCLUSIONS

The paper describes a method for calculating the pressure distribution on non-lifting bodies, rocket bodies, fuel stores, bombs, airships.

The method used here is based on the linearised solutions of the unsteady flow in subsonic regime. Moreover, the unsteady flow is assumed to be hamonic. In these conditions, the velocity potential equation can be brought to the Helmholtz equation whose solution is known. It is expressed as a sum of two types of solutions: a simple layer potential and a double layer potential. The double layer potential is used in the theory of lifting surfaces. In our case, we use only the simple layer potential. The paper presents the method by which the previous mathematical solution is applied to the flow about the elongated bodies. Using a panneling scheme and a previous article of this series, the problem of the flow about the oscillating bodies is solved.

APPENDIX A
UNIT NORMAL VECTOR TO THE OSCILLATING SURFACE (Σ)

Consider the body surface equation at rest in one of its parametric form:

$$(\Sigma_0): \quad \vec{P} = \vec{P}_0(u,v), \tag{A1}$$

Similarly, consider the displacement of a point of parametric coordinates (u,v) as

$$\vec{P} = \delta \cdot \vec{P}_1(u,v)e^{i\omega t}. \tag{A2}$$

The parameter $\delta > 0$ but $\delta << 1$.

Then the instantaneous equation of the oscillating body is:

$$(\Sigma): \vec{P} = \vec{P}_0(u,v) + \delta \cdot \vec{P}_1(u,v)e^{i\omega t}. \tag{A3}$$

The velocity of a point on the body surface is given by

$$\vec{v}_\Sigma = i\omega \cdot \vec{P}_1(u,v)e^{i\omega t}. \tag{A4}$$

The normal vector of (Σ) is then

$$\begin{gathered}\vec{N} = \left(\frac{\partial \vec{P}_0}{\partial u} + \delta \frac{\partial \vec{P}_1}{\partial u} e^{i\omega t}\right) \times \left(\frac{\partial \vec{P}_0}{\partial v} + \delta \frac{\partial \vec{P}_1}{\partial v} e^{i\omega t}\right) = \\ \left(\frac{\partial \vec{P}_0}{\partial u} \times \frac{\partial \vec{P}_0}{\partial v}\right) + \left(\frac{\partial \vec{P}_0}{\partial u} \times \frac{\partial \vec{P}_1}{\partial v} - \frac{\partial \vec{P}_0}{\partial v} \times \frac{\partial \vec{P}_1}{\partial u}\right)\delta \cdot e^{i\omega t} + \frac{\partial \vec{P}_1}{\partial u} \times \frac{\partial \vec{P}_1}{\partial v}\delta^2 e^{i\cdot 2\omega t}\end{gathered} \tag{A5}$$

Let us divide (A5) by $\left|\frac{\partial \vec{P}_0}{\partial u} \times \frac{\partial \vec{P}_0}{\partial v}\right|$ and we get:

$$\vec{n} = \vec{n}_0(U,V) + \delta \cdot \vec{n}_1(U,V)e^{i\omega t} + \delta^2 \cdot \vec{n}_2(U,V)e^{i2\omega t} \tag{A6}$$

This is the Fourier expansion of the ”quasi-unit” normal vector $\vec{n}$. One can see that the last term can be neglected. In the above equation $\vec{n}_0$ is the unit normal vector of (Σ_0). The term “quasi-unit” is justified by the fact that its module is slightly different by 1:

$$\vec{n} \approx \vec{n}_0(U,V) + \delta \cdot \vec{n}_1(U,V)e^{i\omega t}, \quad |\vec{n}| = 1 + O(\delta) \text{ as } \delta \to 0 \tag{A7}$$

This last expression for $\vec{n}$ will be used in the main text of the article.

APPENDIX B
VELOCITY POTENTIALS AND PRESSURE COEFFICIENTS

The velocity potential and pressure distribution are assumed to be of the form:

$$\begin{cases}\varphi(x,y,z;t) = U_\infty x + \varphi_0(x,y,z) + \delta \cdot \overline{\varphi}(x,y,z)e^{i\omega t} \\ p(x,y,z;t) = p_\infty + \Delta p(x,y,z) + \delta \cdot \overline{p}(x,y,z)e^{i\omega t}\end{cases} \tag{B1}$$

On the other hand, for steady flow φ_0 and Δp (see for example [1]) satisfy the following equations:

$$\begin{gathered}\left[(1-M^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\varphi_0 = 0 \\ \frac{\Delta p}{\rho_\infty} = -U_\infty \frac{\partial \varphi}{\partial x}\end{gathered} \tag{B2}$$

For the oscillatory flow, the following relationships for $\overline{\varphi}$ and $\overline{p}$ hold true:

$$\begin{gathered}\left[(1-M^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - 2i\omega\frac{M}{a_\infty} + \frac{\omega^2}{a_\infty^2}\right]\overline{\varphi} = 0 \\ \frac{\overline{p}}{\rho_\infty} + U_\infty \frac{\partial \overline{\varphi}}{\partial x} + i\omega\overline{\varphi} = 0\end{gathered} \tag{B3}$$

We remark that if ω=0, the equations (B3) transfom into (B2).
Then we make a change of function,

$$\overline{\varphi}(x,y,z)=\exp\left(i\frac{\omega}{U_\infty}\frac{M^2}{1-M^2}x\right)\overline{\overline{\varphi}}(x,y,z) \tag{B4}$$

This change of function introduced in the first equation in (B3), brings it to the simpler form:

$$\left[\left(1-M^2\right)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{U_\infty^2}\frac{M^2}{\beta^2}\right]\overline{\overline{\varphi}}=0 \tag{B5}$$

In addition to the foregoing formula, a change of variables

$$x=x';\qquad y=\frac{y'}{\beta}\qquad z=\frac{z'}{\beta} \tag{B6}$$

will simplify (B5) to the next form:

$$\left[\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}+\frac{\omega^2}{U_\infty^2}\frac{M^2}{\beta^4}\right]\overline{\overline{\varphi}}'=0 \tag{B7}$$

which is known as the Helmholtz equation. Here,

$$\overline{\overline{\varphi}}(x,y,z)=\overline{\overline{\varphi}}\left(x',\frac{y'}{\beta},\frac{z'}{\beta}\right)=\overline{\overline{\varphi}}'(x',y',z') \tag{B8}$$

Let (Σ) be the closed surface of our body. The integral form of the solution of (B7) is:

$$\begin{aligned}
&\overline{\overline{\varphi}}'(x',y',z')=\overline{\overline{\varphi}}_s'(x',y',z')+\overline{\overline{\varphi}}_d'(x',y',z')\\
&\overline{\overline{\varphi}}_s'(x',y',z')=\frac{1}{4\pi}\int_{(\Sigma')}\frac{\exp\left(-i\frac{\omega}{U_\infty}\frac{M}{\beta^2}R'\right)}{R'}\frac{\partial\overline{\overline{\varphi}}'}{\partial n'}d\Sigma';\\
&\overline{\overline{\varphi}}_d'(x',y',z')=-\frac{1}{4\pi}\int_{(\Sigma')}\frac{\partial}{\partial n'}\left(\frac{\exp\left(-i\frac{\omega}{U_\infty}\frac{M}{\beta^2}R'\right)}{R'}\right)\cdot\overline{\overline{\varphi}}'\,d\Sigma'\\
&(x',y',z')\in R^3-(\Sigma')\,;\ (\xi',\eta',\zeta')\in(\Sigma')
\end{aligned} \tag{B9}$$

In the above equations,

$$R(x,y,z)=R\left(x',\frac{y'}{\beta},\frac{z'}{\beta}\right)=R'(x',y',z'). \tag{B10}$$

On the other hand, the body surface can be written in its parametric form as

$$(\Sigma):\begin{cases} X = X(u,v) = X'(u,v) \\ Y = Y(u,v) = \dfrac{1}{\beta}Y'(u,v) \\ Z = Z(u,v) = \dfrac{1}{\beta}Z'(u,v) \end{cases} \qquad (\Sigma'):\begin{cases} X' = X(u,v) \\ Y' = \beta\, Y(u,v) \\ Z' = \beta\, Z(u,v) \end{cases} \tag{B11}$$

The proof of the previous solution (B9) can be found in the works dealing with the equations of mathematical physics (see, for example [17]). In the above equations, the function $\overline{\overline{\varphi}}_s$ is called the simple layer potential, while $\overline{\overline{\varphi}}_d$ is called the double layer potential; $\dfrac{\partial(\bullet)}{\partial n'} = \vec{n}' \cdot \nabla'(\bullet)$ where $\vec{n}'$ represents the outward unit nomal vector to (Σ').
The double layer potential is used for the lifting surface theory, which is not the object of our study. So in what follows, we will consider only the simple layer potential, i.e.

$$\overline{\overline{\varphi}}'(\xi',\eta',\zeta') = 0 \Rightarrow \overline{\varphi}(\xi,\eta,\zeta) = 0, \quad (\xi',\eta',\zeta') \in (\Sigma')\,\text{and}\,(\xi,\eta,\zeta) \in (\Sigma). \tag{B12}$$

Returning to the initial function $\overline{\varphi}(x,y,z)$, we can write:

$$\begin{gathered} \overline{\varphi}(x,y,z) = \exp\left(i\frac{\omega}{U_\infty}\frac{M^2}{\beta^2}x\right)\overline{\overline{\varphi}}'_s(x',y',z') = \\ \frac{1}{4\pi}\exp\left(i\frac{\omega}{U_\infty}\frac{M^2}{\beta^2}x\right)\int_{(\Sigma')}\frac{\exp\left(-i\dfrac{\omega}{U_\infty}\dfrac{M}{\beta^2}R'\right)}{R'}\frac{\partial\,\overline{\overline{\varphi}}'}{\partial\,n'}\,d\Sigma' \end{gathered} \tag{B13}$$

Using (B4) again, we find that

$$\overline{\overline{\varphi}}(\xi,\eta,\zeta) = \exp\left(-i\frac{\omega}{U_\infty}\frac{M^2}{1-M^2}\xi\right)\overline{\varphi}(\xi,\eta,\zeta) = \exp\left(-i\frac{\omega}{U_\infty}\frac{M^2}{1-M^2}\xi'\right)\overline{\varphi}(\xi',\eta',\zeta') \tag{B14}$$

Then, taking into account (A13), we find that

$$\begin{gathered} \left[\frac{\partial}{\partial n'}\overline{\overline{\varphi}}(\xi',\eta',\zeta')\right]_{(\Sigma')} = \left\{\frac{\partial}{\partial n'}\left[\exp\left(-i\frac{\omega}{U_\infty}\frac{M^2}{1-M^2}\xi'\right)\overline{\varphi}'(\xi',\eta',\zeta')\right]\right\}_{(\Sigma')} = \\ \left\{\frac{\partial}{\partial n'}\left[\exp(\ldots)\right]\underbrace{\overline{\varphi}'}_{=0}\right\}_{(\Sigma')} + \left\{\exp(\ldots)\frac{\partial\overline{\varphi}'}{\partial n'}\right\}_{(\Sigma')} = \left\{\exp\left(-i\frac{\omega}{U_\infty}\frac{M^2}{1-M^2}\xi'\right)\frac{\partial\overline{\varphi}'}{\partial n'}\right\}_{(\Sigma')} \end{gathered} \tag{B15}$$

It is now time to return to the old variables (*x,y,z*). Let us consider from (B13) the integral

$$I = \int_{(\Sigma')}\frac{\exp\left(i\dfrac{\omega}{U_\infty}\cdot\dfrac{-M^2\xi'-MR'}{\beta^2}\right)}{R'}\frac{\partial\,\overline{\overline{\varphi}}'}{\partial\,n'}\,d\Sigma' = \int_{(\Sigma')}\frac{\exp(\ldots)}{R'}\frac{\vec{N}'}{\left|\vec{N}'\right|}\nabla'\overline{\overline{\varphi}}'\cdot\left|\vec{N}'\right|dudv \tag{B16}$$

In the above equations, $\nabla' = \underbrace{grad}_{x',y',z'}(\bullet)$. The normal vectors to (Σ) and (Σ') are:

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ X_u & Y_u & Z_u \\ X_v & Y_v & Z_v \end{vmatrix}; \quad \vec{N}' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ X'_u & Y'_u & Z'_u \\ X'_v & Y'_v & Z'_v \end{vmatrix} = \beta\left[\vec{N} - (1-\beta)N_x\vec{i}\,\right] \tag{B17}$$

Similarly,

$$\underbrace{grad}_{x',y',z'}(\bullet) = \frac{\beta-1}{\beta}\frac{\partial}{\partial x}(\bullet)\vec{i} + \frac{1}{\beta}\underbrace{grad}_{x,y,z}(\bullet) = \frac{1}{\beta}\left[\underbrace{grad}_{x,y,z}(\bullet) - (1-\beta)\frac{\partial}{\partial x}(\bullet)\vec{i}\,\right] \tag{B18}$$

Then, giving up the variables (*x',y',z'*) we find

$$I = \int_{(\Sigma')} \frac{\exp\left(i\dfrac{\omega}{U_\infty}\cdot\dfrac{-M^2\xi - MR}{\beta^2}\right)}{R}\left(\vec{N}\cdot grad\,\varphi - M^2 N_x\frac{\partial\varphi}{\partial x}\right)\frac{|\vec{N}|}{|\vec{N}|}dudv \tag{B19}$$

So, (B13) becomes:

$$\overline{\varphi}(x,y,z) = \frac{1}{4\pi}\int_{(\Sigma)} \frac{\exp\left(i\dfrac{\omega}{U_\infty}\cdot\dfrac{M^2(x-\xi) - MR}{\beta^2}\right)}{R}\left(\vec{n}\cdot grad\,\varphi - M^2 n_x\frac{\partial\varphi}{\partial x}\right)d\Sigma \tag{B20}$$

Since

$$\begin{gathered}\vec{n}\cdot grad\,\varphi - M^2 n_x\frac{\partial\varphi}{\partial\xi} = n_x\beta^2\frac{\partial\overline{\varphi}}{\partial\xi} + n_y\frac{\partial\overline{\varphi}}{\partial\eta} + n_z\frac{\partial\overline{\varphi}}{\partial\zeta} = \\ \vec{n}\cdot\nabla\varphi(\beta^2\xi,\eta,\zeta) = \frac{\partial}{\partial n}\varphi(\beta^2\xi,\eta,\zeta)\end{gathered} \tag{B21}$$

we can write the solution as

$$\overline{\varphi}(x,y,z) = -\frac{1}{4\pi}\int_{(\Sigma_0)} \frac{\exp\left(i\dfrac{\omega}{U_\infty}\cdot\dfrac{M^2(x-\xi)-MR}{\beta^2}\right)}{R}q(\xi,\eta,\zeta)d\Sigma_0 \tag{B22}$$

where

$$q(\xi,\eta,\zeta) = -\frac{\partial}{\partial n_0}\varphi(\beta^2\xi,\eta,\zeta), \quad (\xi,\eta,\zeta)\in(\Sigma_0) \tag{B23}$$

is the strength of the simple layer or source sheet.

One can see that the integral is calculated over the steady surface (Σ_0) rather than (Σ) since the differences between them are very small (see Appendix A). This approximation is also used in the unsteady lifting surface theory.

If we put ω=0 in (B19), we get the steady flow case:

$$\varphi_0(x,y,z)=\frac{-1}{4\pi}\int_{(\Sigma_0)}\frac{1}{R}q_0(\xi,\eta,\zeta)d\Sigma_0 \tag{B24}$$

In the above equation,

$$q_0(\xi,\eta,\zeta)=-\frac{\partial}{\partial n_0}\varphi_0(\beta^2\xi,\eta,\zeta),\quad (\xi,\eta,\zeta)\in(\Sigma_0) \tag{B25}$$

Therefore it is a good practice to use (B22) for both oscillatory and steady cases.
Let us focus our attention on the *pressure coefficient*:

$$c_p(x,y,z;t)=\frac{p(x,y,z;t)-p_\infty}{\frac{1}{2}\rho_\infty U_\infty{}^2}=\frac{\Delta p(x,y,z)}{\frac{1}{2}\rho_\infty U_\infty{}^2}+\delta\,\frac{\overline{p}(x,y,z)}{\frac{1}{2}\rho_\infty U_\infty{}^2}e^{i\omega t} \tag{B26}$$

From (B2-II) we find that

$$c_{p0}=\frac{\Delta p}{\frac{1}{2}\rho_\infty U_\infty{}^2}=-2\frac{1}{U_\infty}\frac{\partial\varphi_0}{\partial x} \tag{B27}$$

Similarly, from (B3 II)

$$c_{p1}=\frac{\overline{p}}{\frac{1}{2}\rho_\infty U_\infty{}^2}=-2\frac{1}{U_\infty}\frac{\partial\varphi_1}{\partial x}-2i\frac{\omega}{U_\infty}\frac{\varphi_1}{U_\infty} \tag{B28}$$

Finally, the total pressure coefficient becomes:

$$c_p=c_{p0}+\delta\cdot c_{p1}\cdot e^{i\omega t} \tag{B29}$$

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