Similitude Criteria for Aeroelastic Models

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Abstract: *This article presents principles of similarity modeling, dimensional analysis, and the Buckingham* (Π) *theorem. A number of examples from different fields are given. Based on these principles, the similar phenomena concept was developed. The main goal of these forays is to help the work in the domain of experimental aeroelasticity.*

Key Words: *scale, similitude, Pi theorem, dimensional analysis, aeroelasticity, effect of structural flexibility on wing loading, flutter.*

1. INTRODUCTION

Many technical problems lead to mathematical equations that are difficult to solve. To do it, some of the equations could be brought by approximations to forms which have known solutions. Others prefer especially numerical solutions. It is always a good practice to seek an experimental verification. But now another problem arises: how to perform this experimental verification? The first idea is to verify the **or**iginal object(s), denoted here *Or* in real conditions. However, it is not always possible to do it; rather it is always impossible. Therefore we have to build the object(s) at a different scale and then to test them in certain conditions which show some similarity for the real ones. So we follow the idea that there is a certain similitude between the real and the experimental phenomena [1].

We will call ***mo****del* (*Mo*) the experimental object, and simply ***or****iginal* (*Or*) the initial body we are interested to test. The idea is that between *Mo* and *Or* must be a certain correspondence. Then we must find some principles of invariance, so that both *Mo* and *Or* are governed by similar laws.

A model is usually defined [2] as "*An abstract or material system which, being put in correspondence with another system previously given, can serve to indirect study of the properties of this complex system (the original) and with which the model shows a certain analogy".*

Some models coincide perfectly with the original, for instance two spheres made of the same material but with different radii. So there is a one-to-one correspondence between the *Or* and *Mo*. Another example: two beams made of different elastic materials and built at different scales. For a certain bending load of a beam (*Or*), there is a similar bending load of the second beam (*Mo*) so that their deflections are similar. In this case, there is only a partial correspondence between *Or* and *Mo*. For example, they can behave differently when they are subjected to fatigue.

In this article we treat the case of *similarity modeling*. In passing we mention that the term "modeling" has a broader meaning:

- there is a *modeling by analogy*, for example: the elastic membrane analogy, slide rule analogy, clocks of any type, planetaria, electromecanic analogy devices and analog computers, etc.
- the *more complex theoretical analogies*, such as the one made by Rutherford between the atom and the solar system: he inferred from the well known attributes of the solar system some probable attributes of the atom.

To get a similarity model one must obtain at least one of the next conditions:

- geometric similarity;
- static similarity which coincides with the geometric similarity for rigid models but includes elastic similarity in case of elastic models;
- kinematic similarity;
- dynamic similarity.

The geometric similarity is intuitively understood and known from the theoretical geometry. It is always used for wind tunnel tests. For example:

$$\frac{l_{\mathrm{mod}}}{l} = k_L \quad \frac{S_{\mathrm{mod}}}{S} = k_L^{\ 2} \tag{1}$$

In the above equation, l is a reference length of the *Or*, S is an area of the *Or*, the index 'mod' signifies the model (reference length, surface, etc.). In (1) k_L stands for the length [L] scale. Obviously, the angles of the *Or* and *Mo* are equal.

The static similarity means both the geometric similarity, but also the similarity of the external forces F and the weights G, the similarity of the positions of the gravity centres (x_{CG}, y_{CG}, z_{CG}).

$$\frac{F_{\mathrm{mod}}}{F} = \frac{G_{\mathrm{mod}}}{G} = k_F; \qquad \frac{x_{CG\,\mathrm{mod}}}{x_{CG}} = \frac{y_{CG\,\mathrm{mod}}}{y_{CG}} = \frac{z_{CG\,\mathrm{mod}}}{z_{CG}} = k_L \tag{2}$$

These are available for the rigid models. If the model is considered elastic, we must have in addition:

The elastic similarity which means that the force scale k_F has been chosen such that the homologous forces produce similar linear displacements (δ). On the other hand, homologous moments produce equal angular displacements (ε). So, we have to build the model at a linear scale k_L, than to calibrate the force scale (k_F) so that the linear/angular displacements respect the geometric similarity condition:

$$\frac{\delta_{\mathrm{mod}}}{\delta} = k_L; \quad \varepsilon_{\mathrm{mod}} = \varepsilon \tag{3}$$

The kinematic similarity means first of all that *Mo* and *Or* have both the same degrees of freedom. For example, the wing- flap mechanism model. Then we need a time scale, a speed scale, an acceleration scale:

$$\frac{t_{\mathrm{mod}}}{t} = k_T; \qquad \frac{v_{\mathrm{mod}}}{v} = k_V; \qquad \frac{a_{\mathrm{mod}}}{a} = k_A. \tag{4}$$

The dynamic similarity is achieved when all the forces and moment acting on the parts of the model and original are similar. To get a dynamic similarity we must obtain simultaneously geometric (1), static (2) and kinematic similarity (3). Moreover, we need a similarity of the masses and moments of inertia of the moving parts. As in the case of the elastic similarity, *there are correlations between these scale numbers*.

2. DIMENSIONAL ANALYSIS AND SIMILITUDE

We call “fundamental dimensions” or “primary quantities” each of the seven quantities: length (L), time (T), mass (M), temperature (Θ), current intensity (I), luminous intensity (J) and amount of substance (Q).

Then any physical quantity A has the following *dimension*:

$$[A] = L^{\alpha}T^{\beta}M^{\gamma}\Theta^{\tau}I^{\lambda}J^{\mu}Q^{\nu} \tag{5}$$

In (5) $\alpha, \beta, \gamma, \ldots \nu$ are *rational numbers*. If $\alpha = \beta = \gamma = \ldots = \nu = 0$, A is called *dimensionless*.

In what follows we will deal only with qantities B that involve the following dimensions: L, T, M:

$$[B] = L^{\alpha}T^{\beta}M^{\gamma} \tag{6}$$

In the above formulae L, T, M are regarded as *primary quantities*. However, we could use other basis, for example L, T, $[F]$. So we can replace M with $[F]$, because $F = ma$, and it contains M. This was the case of MKS (meter, kilogram, second) system. Here, $[F]$=1kilogram (force unit).

We emphasize that:

- The first condition that must be met by the equations used in geometry and physics is the *dimensional homogeneity*.
- The second condition that must be met by a physical equation refers to the fact that we cannot have, for example L^{π}, T^{e}, $M^{\sqrt{2}}$ since the exponents are irrational, etc., but L^{0}. T^{-1}, M^{2}, $L^{3/2}$, $L^{-2/3}$ are all available.
- The third condition a physical equation must fulfill is that it is never possible to have a dimensional quantity as argument of a function. So it is wrong to write sin$(3t)$, but it is correct sin(ωt), with $\omega=3s^{-1}$and $[t]$=s.

Some Examples of Dimensional Analysis

1. Consider a rectangular triangle a, b, c and $<A=\pi/2$. Then:

$$\begin{aligned} &a^2 = b^2 + c^2; \\ &[a] = [b] = [c] = L; \quad \text{with} \\ &\Pi_1 = \frac{b^2}{a^2} = \cos^2 B;\ \Pi_2 = \frac{c^2}{a^2} = \cos^2 C \quad \text{we get :} \\ &\Pi_1 + \Pi_2 = 1 \end{aligned} \tag{7}$$

2. Consider the example of the example of the Bernoulli’s equation (see, for example equation (2.69), pg. 72, [3]), in compressible isentropic flow:

$$\frac{V^2}{2} + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0}\ ; \quad \frac{p}{\rho^{\gamma}} = \frac{p_0}{\rho_0^{\ \gamma}} \tag{8}$$

We have used the classical notations: V is the fluid speed, p is the pressure and ρ is the fluid density.

The index 0 signifies the stagnation values, and γ is the adiabatic constant. One can see that the formula is homogenious:

$$\left[V^2\right] = L^2T^{-2}; \qquad \left[\frac{p}{\rho}\right] = \left[\frac{p_0}{\rho_0}\right] = \frac{MLT^{-2}L^{-2}}{ML^{-3}} = L^2T^{-2}$$

Equation (8) may be written as

$$\frac{\frac{1}{2}\rho_0 V^2}{p_0} = \frac{\gamma}{\gamma-1}\left[1-\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right] \text{ or, } \Pi_1 = \Pi_2 \quad \text{where}$$

$$\Pi_1 = \frac{\frac{1}{2}\rho_0 V^2}{p_0}; \ \Pi_2 = \frac{\gamma}{\gamma-1}\left[1-\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right] \tag{9}$$

One can easily see that *both* Π_1 and Π_2 *are dimensionless*; they are *the same for all unit systems*.

3. Consider now another formula containing a differential equation ([3], pg. 121, eq. (5.4)):

$$\frac{d\sigma}{\sigma} + \frac{dV}{V}\left(1-\frac{V^2}{a^2}\right) = 0 \tag{10}$$

This is the *Hugoniot formula* of the flows through pipes. Here σ signifies the pipe section area and a is the local sound speed. The equation contains terms with no dimensions. It is not correct to write (10) with $\frac{d\sigma}{\sigma} = \frac{d}{d\sigma}(\ln\sigma)$ because $[\sigma] = L^2$. Instead we can write $\frac{d\sigma}{\sigma} = \frac{d}{d\sigma}\left(\ln\frac{\sigma}{\sigma_0}\right)$, where σ_0 is a reference section area. This transformation is available for the speed as well: $\frac{dV}{V} = \frac{d}{dV}\left(\ln\frac{V}{V_0}\right)$. On the other hand, we can write (10) as:

$$\Pi_1\Pi_2 + 1 = 0, \text{ where } \Pi_1 = \frac{\sigma}{V}\frac{dV}{d\sigma}, \ \Pi_2 = 1-\frac{V^2}{a^2} \tag{11}$$

Again Π_1 and Π_2 *are dimensionless* and are *the same whatever unit system we use*.

4. Now consider the case of a harmonic oscillator with elastic coefficient k, viscous damping coefficient n and an external force $F(t)$. It is governed by the equation:

$$m\ddot{x} + n\dot{x} + kx = F(t) \tag{12}$$

One can see that the dimension of each term is $LT^{-2}M$, i.e a dimension of a force. Consider a reference length l_{ref} and a reference time, T_{ref}. We make a change of variable and function:

$$t = T_{ref}\tau; \ x(t) = l_{ref}\xi(\tau); \ \dot{x}(t) = \frac{l_{ref}}{T_{ref}}\dot{\xi}(\tau); \ \ddot{x}(t) = \frac{l_{ref}}{T_{ref}^{\ 2}}\ddot{\xi}(\tau); \ F(t) = m\frac{l_{ref}}{T_{ref}^{\ 2}}\Phi(t) \tag{13}$$

Then, the motion equation becomes:

$$\ddot{\xi} + \Pi_1 \dot{\xi} + \Pi_2 \xi = \Pi_3 \Phi(t) \tag{14}$$

In the above equation, we put:

$$\Pi = \begin{pmatrix} \frac{n}{m} T_{ref} & \frac{k}{m} T_{ref}{}^2 & 1 \end{pmatrix}^T \tag{15}$$

We observe again that the vector matrix elements are dimensionless and $\xi(\tau)$ and its derivatives are all dimensionless as well. Our equation of motion can now be interpreted simply as differential equation, ***the same one*** *regardless the unit system we use.*

We can now give the fundamental theorem of the dimensional analysis:

The Buckingham **(Π)** ***Theorem***

Consider the equation that is relevant in a given problem,

$$\Phi(q_1, q_2, ..., q_n) = 0 \tag{A}$$

where q_1, q_2,...,q_n *are physical quantities. Then, equation* (A) *can be written as:*

$$\varphi(\Pi_1, \Pi_2, ..., \Pi_m) = 0 \tag{B}$$

In the above equation, $m = n - p$*, where p is the number of primary quantities involved in* (A).

In the first case we have a simple metric equation between 3 lengths:a, b, c. We have only 1 primary quantity, L. So, in the first case there are 3-1=2 dimensionless numbers: Π_1 and Π_2.

We can see that, in the second case, we have 5 physical quantities: V, p, ρ, p_0, and ρ_0. The primary quantities involved in these quantities are L, T, and M, that is 3 primary quantities. So the pysical phenomenon is described by 5 – 3 = 2 dimensionless quantities, Π_1 and Π_2.

In the third case, we have 4 physical quantities: V, σ, $\frac{dV}{d\sigma}$, a and 2 primary quantities: L and T. Then we have only 2 dimensionless quantities Π_1 and Π_2.

For the fourth case, we will prefer to write the *dimension matrix*:

$$\begin{array}{c c} & \begin{array}{cccccc} x & t & m & n & k & F \end{array} \\ \begin{array}{c} L \\ T \\ M \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \tag{16}$$

So, we have 6 – 3 = 3 values for Π.

The Π vector is *not unique*, but the vector dimension is the same ($n - p = m$). For example, in the fourth case, we obtained the Π values so that the coefficient of $\ddot{\xi}$ in (14) is 1. It is also poosible to impose a value (usually 1) to the coefficient of $\dot{\xi}$ (or ξ).

We will present now a different problem solved using the Buckingham theorem:

Find the drag force D of a sphere of radius R in a stream. What are the physical quantities that can influence the drag? We identify all the interest data: $(D, R, \mu, U_\infty, p_\infty, \rho_\infty)$, that is 6 numbers. We choose the 3 basic dimensions (L, T, F). We can determine 6 – 3 = 3 "Pi groups". For example:

$$\Pi_1 = \frac{D}{\rho U_\infty{}^2 R^2};\ \Pi_2 = \frac{\rho U_\infty R}{\mu};\ \Pi_3 = \frac{p_\infty}{\rho U_\infty{}^2}$$

Then the sought after equation will be of the form $F(\Pi_1, \Pi_2, \Pi_3) = 0$, or $\Pi_1 = f(\Pi_2, \Pi_3)$ which can be written as:

$$\frac{D}{\frac{1}{2}\rho U_\infty{}^2 \cdot \pi R^2} = \varphi\left(\frac{\rho U_\infty R}{\mu}, \frac{p_\infty}{\frac{1}{2}\rho U_\infty{}^2}\right)$$

The first term slighty modified, is the *drag coefficient*, Π_2 is the *Reynols number*, while Π_3 slighty modified too, is the *pressure coefficient at infinity*.

In conclusion, if we have metric relationship (involving only lengths *L*) or, more generally, a physical formula (involving two or more fundamental dimensions) represented by equation (A), we can express it as an another equation (B) that contains *m = n - p* “Pi groups” $\Pi_1, \Pi_2, ..., \Pi_m$, which are dimensionless.

In case we do not know the (A) equation, identify the important *independent* variables $q_1, q_2, ..., q_n$. Express the independent variabiles in terms of the fundamental dimensions. Determine the number of fundamental dimensions involved, say *p*. For example use the dimension matrix (16). Then find the *m = n - p* “Pi groups” $\Pi_1, \Pi_2, ..., \Pi_m$. Then express $\Pi_1 = f(\Pi_2, \Pi_3, ... \Pi_m)$, where Π_1 is quantity of the most interest. So, even if we did not find the functional dependence of the *independent* variables $q_1, q_2, ..., q_n$, we found at least the the ”Pi groups” $\Pi_1, \Pi_2, ..., \Pi_m$ that play a role in the phenomenon. It is then easier to find this function experimentally, or otherwise, the “Pi groups” could help us try a theoretical approach.

3. A CLASS OF SIMILAR PHENOMENA

Let us examine again the four cases presented before. In our first example, the rectangular triangles that have equal Π_1 and Π_2, ($\Pi_1 > 0$, $\Pi_2 > 0$) are similar rectangular triangles. In the (Π_1, Π_2) coordinate system all the similar rectangle triangles are located on a segment on the line $\Pi_1 + \Pi_2 = 1$. Consider a point (Π_1, Π_2) = (3/4, 1/4); it represents all the rectangular triangles with < B = 60^0 and < C = 30^0.

In the second example, the Bernoulli’s equation (fig. 2), all the flow cases are located on the bisector $\Pi_1 - \Pi_2 = 0$.

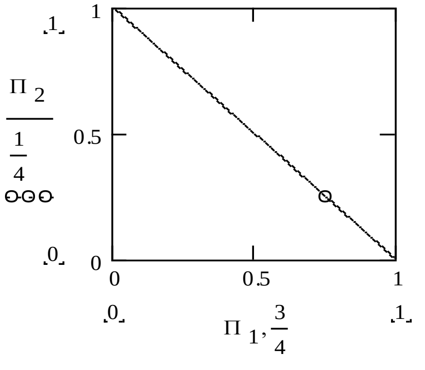

Fig. 1 Rectangular triangles, with the example when <B=60^0 and <C=30^0

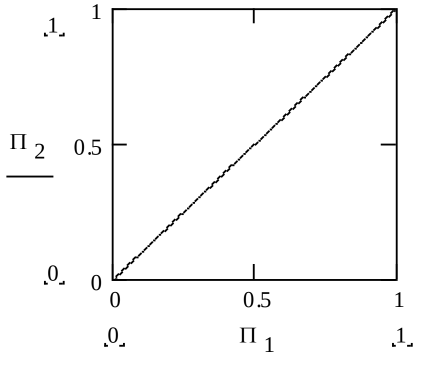

Fig. 2 Bernoulli’s equation as a function of the Pi group Π_1, Π_2

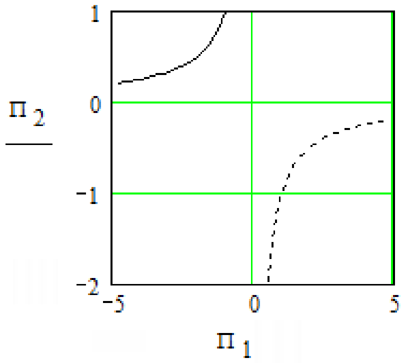

Fig. 3 Tube flow formula; the left branch (M < 1);
the right branch (M > 1); $\Pi_2=1-M^2$

The third case is represented in fig. 3. The flow in a tube is expressed by the rectangular hyperbola (11). The left branch of the curve (equilateral hyperbola) represents the subsonic flows, while the right branch represents the supersonic flows.

So a certain point on these diagrams correspond to a certain *state* of a ***physical system***.

The fourth case is a different one: here the time plays an important role. Actually equation (12) describes a ***phenomenon*** (a physical event=something taking place in time). The Pi coefficients determine the characteristic equation and so the nature of the solution of the differential equation.

Suppose we have a mechanical system or mechanical phenomenon described by an equation written in the (A) form, i.e. as a function of independent variables $q_1, q_2,..., q_n$. Consider that, using a certain method as those presented before, one obtaines the (B) form say $f(\Pi_1, \Pi_2, \Pi_3,...\Pi_m)$. If we consider $\Pi_1, \Pi_2,..., \Pi_m$ as a class of *constants*, then the formula $f(\Pi_1, \Pi_2, \Pi_3,...\Pi_m)$ describes a ***class of similar mechanical systems or phenomena***.

We will present some applications of the above theoretical considerations in aeroelasticity.

4. APPLICATIONS IN STATIC AEROELASTICITY

The first example of applications in the field of aeroelasticity is the aeroelastic redistribution of airload on a wing. So we study an elastic mechanical system in equilibrium state.

Consider a classical straight wing with the following characteristics: the dihedral angle is 0, its aspect ratio is big enough to let us use the following theories of aerodynamics and elasticity:

- The lift can be considered distributed along the *¼ chord line*.
- We assume the existence of an *elastic axis*; it will be used for the calculation of wing structure deformations.

The wing twist effect on aerodynamic load redistribution is the most important; the wing bending effect is negligible.

We will start with the aerodynamic forces. Let us denote by α the local angle of attack measured between the wind direction and the untwisted wing section zero-lift line. The local twist is $\theta(y)$.

Then the spanwise distribution of the aerodynamic moment about the local elastic centre is given by [4], (7.10), pg. 491:

$$m(y,\alpha) = qC_{L\alpha}c(y)e(y)\cdot[\alpha + \theta(y)] + qc^2C_{m0} \tag{17}$$

where q is the dynamic pressure, $C_{L\alpha}$ is the mean lift coefficient of the wing, C_{m0} is the pitcing moment coefficient, *c(y)* represents the local chord and

$$e(y) = x_e(y) - x_{1/4}(y) \tag{18}$$

In (17) *we neglected the effect of the inertial forces*.

We will consider the wing as a cantilevered beam, (fig.4).

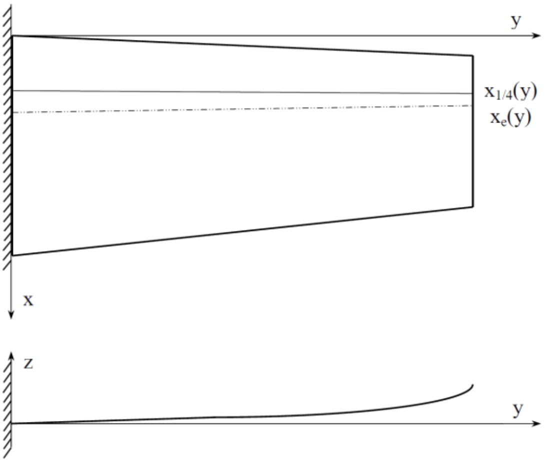

Fig. 4 The right part of a wing approximated as a cantilevered beam; in this figure, only wing bending is represented; however wing torsion is the most important cause airload redistribution

The torsion equation is (see [4], equation (3.33)):

$$\frac{d}{dy}\left(GJ(y)\frac{d\theta}{dy}\right) = -m(y,\alpha) \tag{19}$$

In the above formula, *GJ(y)* represents the twist stiffness.

Consider a reference length $l_{ref} = b/2$ and also the reference stiffness GI_{ref}, which can be a mean stiffness, or better the maximum stiffness at *y = 0*. We introduce the dynamic pressure

$$q = \frac{\rho}{2}U^2$$

and

$$\begin{aligned} &y = \frac{b}{2}\eta;\ \frac{d}{dy} = \frac{2}{b}\frac{d}{d\eta};\ GJ = (GJ)_{ref}\cdot\overline{G}\overline{J}(\eta); \\ &m = \frac{\rho}{2}U^2C_{L\alpha}\frac{b^2}{4}\,\overline{c}(\eta)[\overline{e}(\eta)(\alpha + \theta(\eta)) + \overline{c}(\eta)C_{m0}] \end{aligned} \tag{20}$$

In the above equation, the bars above the letters mean that the quantities are dimensionless.

Now we can write (19) as

$$\frac{d}{d\eta}\left(\overline{G}\overline{J}(\eta)\frac{d\theta}{d\eta}\right)=-\frac{\rho U^2 b^4}{16(GJ)_{ref}}C_{L\alpha}\overline{c}(\eta)\left[\overline{e}(\eta)(\alpha+\theta(\eta))+\overline{c}(\eta)C_{m0}\right]$$

Let us put

$$\Pi=\frac{\rho U^2 b^4}{16(GJ)_{ref}}C_{L\alpha} \tag{21}$$

So we have:

$$\frac{d}{d\eta}\left(\overline{G}\overline{J}(\eta)\frac{d\theta}{d\eta}\right)+\Pi\cdot\overline{c}(\eta)\overline{e}(\eta)\cdot\theta(\eta)=-\Pi\overline{c}(\eta)\left[\overline{e}(\eta)\alpha+\overline{c}(\eta)C_{m0}\right] \tag{22}$$

For two wings to be similar, they shall be governed by the same differential equation (22), so that the following conditions must be met:

- The functions $\overline{G}\overline{J}(\eta)$, $\overline{c}(\eta)$, $\overline{e}(\eta)$ must be the same;
- The numbers Π and C_{m0} must be the same for the two wings.

Consider an original wing and a model. We have

$$\overline{G}\overline{J}(\eta)\equiv\overline{G}\overline{J}_{\mathrm{mod}}(\eta);\ \overline{c}(\eta)=\overline{c}_{\mathrm{mod}}(\eta);\ \overline{e}(\eta)=\overline{e}_{\mathrm{mod}}(\eta);$$

where index mod stands for "model".

They also have approximately the same pitching moment coefficient $C_{m0}\approx C_{m0m}$ (their value difference is due to the Reynolds number effect).

The next condition is $\Pi=\Pi_{\mathrm{mod}}$ or,

$$\frac{\rho U^2 b^4}{16(GJ)_{ref}}C_{L\alpha}=\frac{\rho U_{\mathrm{mod}}^2 b_{\mathrm{mod}}^4}{16(GJ)_{ref\,\mathrm{mod}}}C_{L\alpha\,\mathrm{mod}}$$

From the above equation, we find that

$$k_{stif}=k_L^4 k_V^2, \tag{23}$$

because $C_{L\alpha}\approx C_{L\alpha\,\mathrm{mod}}$. In the previous equation,

$$k_{stif}=\frac{(GJ)_{ref\,\mathrm{mod}}}{(GJ)_{ref}};\quad k_L=\frac{b_{\mathrm{mod}}}{b};\quad k_V=\frac{U_{\mathrm{mod}}}{U} \tag{24}$$

We usually know all about the "original wing" geometry and structure. We also know the "model size", the wind speeds both for the "original wing" and for the "model wing", so that we can calculate k_L and k_V.

Then we find k_{stif} and then

$$\left(GJ\right)_{ref\,\mathrm{mod}} = k_{stif}\left(GJ\right)_{ref}, \tag{25}$$

This is the reference stifness of the model we have to build for experimental purposes.

5. APPLICATIONS IN DYNAMIC AEROELASTICITY

We will use the flutter equations of a cantilevered wing, as given by Fung in [6]. The figure below is from the same book.

We draw attention to the fact that Fung used a different frame from the one given in fig. 4, as can be easily seen below.

$$\begin{aligned} &\frac{\partial^2}{\partial y^2}\left(EI\frac{\partial^2 h}{\partial y^2}\right) + m\frac{\partial^2 h}{\partial t^2} + mx_\alpha\frac{\partial^2 \alpha}{\partial t^2} + L = 0 \\ &\frac{\partial}{\partial y}\left(GJ\frac{\partial \alpha}{\partial y}\right) - I_\alpha\frac{\partial^2 \alpha}{\partial t^2} - mx_\alpha\frac{\partial^2 h}{\partial t^2} + M = 0 \end{aligned} \tag{26}$$

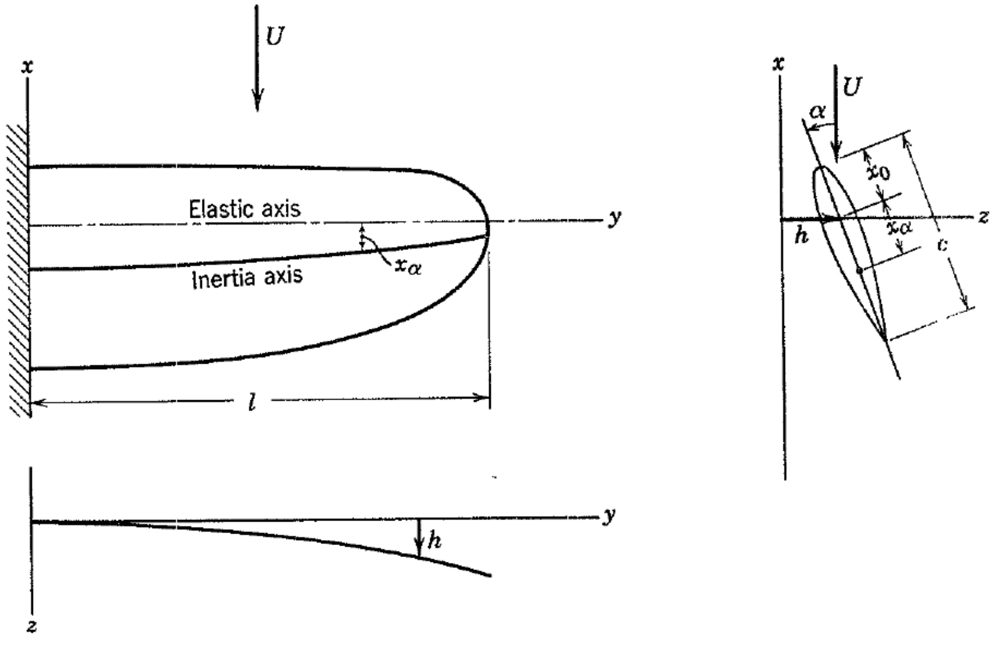

Fig. 5 Notations for a cantilever wing, after [6], pag. 194, sec. 6.3

Some quantities used in (26) are given in fig. 5: the deflection *h*, the elastic axis position x_α, etc. Others were defined in the previous chapter.

Newly introduced are: *EI* which represents the bending stiffness, *m* which is the unit length mass at the wing section *y*, $[m]=ML^{-1}$, I_α is unit length momet of inertia with respect to the elastic axis at the same wing section *y*, $[I_\alpha]=ML^2/L=ML$.

There remain *L* and *M* which mean the unsteady lift and pitching moment calculated with respect to the elastic axis (not to be confused with the dimensions of length and mass!). So we can write, after [6], formula (2), sec. 6.3, pag 194:

$$L=\frac{\rho U^2}{2}cC_L;\quad M=\frac{\rho U^2}{2}c^2C_M=\frac{\rho U^2}{2}c^2\left[\left(C_M\right)_{l.e.}+\frac{x_0}{c}C_L\right] \tag{27}$$

We proceed as in the previous case. Firstly, we choose some new non-dimensional quantities:

$$\begin{gathered}y=\frac{b}{2}\eta;\ EI=\left(EI\right)_{ref}\overline{EI};\ GJ=\left(GJ\right)_{ref}\overline{GJ};\ h=\frac{b}{2}\chi;\ m=m_{ref}\frac{2}{b}\mu;\ x_\alpha=\frac{b}{2}\xi_\alpha;\\ t=\frac{b}{2U}\tau;I_\alpha=I_m\frac{2}{b}I_{\alpha\alpha}\ L=\frac{\rho U^2}{2}\frac{b}{2}\left(\frac{2c}{b}C_L\right);\ M=\frac{\rho U^2}{2}\frac{b^2}{4}\left(\frac{4c^2}{b^2}C_M\right)\end{gathered} \tag{28}$$

With these changes of functions and variables, system (26) becomes:

$$\begin{gathered}\frac{8}{b^3}\left(EI\right)_{ref}\frac{\partial^2}{\partial\eta^2}\left(\overline{EI}\frac{\partial^2\chi}{\partial y^2}\right)+m_{ref}\frac{2}{b}\mu\frac{b}{2}\frac{4U^2}{b^2}\frac{\partial^2\chi}{\partial\tau^2}+m_{ref}\frac{2}{b}\mu\frac{b}{2}\xi_a\frac{4U^2}{b^2}\frac{\partial^2\alpha}{\partial\tau^2}+\\ \frac{\rho U^2}{2}\frac{b}{2}\left(\frac{2c}{b}C_L\right)=0\\ \frac{4}{b^2}\left(GJ\right)_{ref}\frac{\partial}{\partial\eta}\left(\overline{GJ}\frac{\partial\alpha}{\partial\eta}\right)-I_m\frac{2}{b}\frac{4U^2}{b^2}I_{\alpha\alpha}\frac{\partial^2\alpha}{\partial\tau^2}-m_{ref}\frac{2}{b}\mu\frac{b}{2}\xi_\alpha\frac{4U^2}{b^2}\frac{\frac{b}{2}\partial^2\chi}{\partial\tau^2}+\\ \frac{\rho U^2}{2}\frac{b}{4}^2\left(\frac{4c^2}{b^2}C_M\right)=0\end{gathered} \tag{29}$$

We will introduce the following notations for the non-dimensional quantities

$$\begin{gathered}\Pi_1=\frac{1}{2}\frac{m_{ref}U^2b}{\left(EI\right)_{ref}};\quad \Pi_2=\frac{1}{32}\frac{\rho U^2b^4}{\left(EI\right)_{ref}};\quad \Pi_3=\frac{2I_mU^2}{b\left(GJ\right)_{ref}}\\ \Pi_4=\frac{1}{2}\frac{m_{ref}U^2b}{\left(GJ\right)_{ref}};\quad \Pi_5=\frac{1}{32}\frac{\rho U^2b^4}{\left(GJ\right)_{ref}}\end{gathered} \tag{30}$$

So, the wing motion equations (29) become:

$$\begin{gathered}\frac{\partial^2}{\partial\eta^2}\left(\overline{EI}(\eta)\frac{\partial^2\chi(\eta,\tau)}{\partial y^2}\right)+\Pi_1\mu(\eta)\frac{\partial^2\chi(\eta,\tau)}{\partial\tau^2}+\Pi_1\mu(\eta)\xi_a(\eta)\frac{\partial^2\alpha(\eta,\tau)}{\partial\tau^2}+\\ \Pi_2\left(\frac{2c}{b}C_L\right)=0\\ \frac{\partial}{\partial\eta}\left(\overline{GJ}(\eta)\frac{\partial\alpha(\eta,\tau)}{\partial\eta}\right)-\Pi_3I_{\alpha\alpha}(\eta)\frac{\partial^2\alpha(\eta,\tau)}{\partial\tau^2}-\Pi_3\mu(\eta)\xi_\alpha(\eta)\frac{\partial^2\chi(\eta,\tau)}{\partial\tau^2}+\\ \Pi_5\left(\frac{4c^2}{b^2}C_M\right)=0\end{gathered} \tag{31}$$

Consider now the “original” wing and its “model” wing- index “mod”. The similarity conditions for the “original” and “model” mean that the non dimensional functions

$\overline{EI}(\eta)$, $\overline{GJ}(\eta)$, $\mu(\eta)$, $\xi_a(\eta)$, and $I_{\alpha\alpha}(\eta)$ must be identical, and the Pi coefficients must be equal as well:

$$\Pi_1 = \frac{1}{2}\frac{m_{ref}U^2b}{(EI)_{ref}} = \frac{1}{2}\frac{(m_{ref})_{mo}U_{\mathrm{mod}}{}^2 b_{\mathrm{mod}}}{\left[(EI)_{ref}\right]_{\mathrm{mod}}};\ \Pi_2 = \frac{1}{32}\frac{\rho U^2 b^4}{(EI)_{ref}} = \frac{1}{32}\frac{\rho U_{\mathrm{mod}}{}^2 b_{\mathrm{mod}}{}^4}{\left[(EI)_{ref}\right]_{\mathrm{mod}}}$$
$$\Pi_3 = \frac{2I_mU^2}{b(GJ)_{ref}} = \frac{2(I_m)_{\mathrm{mod}}U_{\mathrm{mod}}{}^2}{b_{mo}\left[(GJ)_{ref}\right]_{\mathrm{mod}}};\ \Pi_4 = \frac{1}{2}\frac{m_{ref}U^2b}{GJ_{ref}} = \frac{1}{2}\frac{(m_{ref})_{\mathrm{mod}}U_{\mathrm{mod}}{}^2 b_{\mathrm{mod}}}{\left[(GJ)_{ref}\right]_{\mathrm{mod}}}; \qquad (32)$$
$$\Pi_5 = \frac{1}{32}\frac{\rho U^2 b^4}{(GJ)_{ref}} = \frac{1}{32}\frac{\rho U_{\mathrm{mod}}{}^2 b_{\mathrm{mod}}{}^4}{\left[(GJ)_{ref}\right]_{\mathrm{mod}}}$$

From the five equations in (32) we conclude that:

$$\begin{aligned} k_m &= k_L^3 \\ k_I &= k_L^5 \\ k_{stif\ b} &= k_v^2 k_L^4 \\ k_{stif\ \tau} &= k_m k_v^2 k_L = k_L^4 k_v^2 \\ k_{stif\ \tau} &= k_L^4 k_v^2 \end{aligned} \qquad (33)$$

In the above equations, *k* numbers represent the *scales*:

$$\begin{aligned} k_L &= \frac{l_{\mathrm{mod}}}{l} && \text{length scale} \\ k_m &= \frac{(m_{ref})_{\mathrm{mod}}}{m_{ref}} && \text{reference mass scale} \\ k_I &= \frac{(I_m)_{\mathrm{mod}}}{I_m} && \text{moment of inertia scale} \\ k_{stif\ b} &= \frac{\left[(EI)_{ref}\right]_{\mathrm{mod}}}{(EI)_{ref}} && \text{bending stiffness scale} \\ k_{stif\ \tau} &= \frac{\left[(GJ)_{ref}\right]_{\mathrm{mod}}}{(GJ)_{ref}} && \text{torsion stiffness scale} \\ k_V &= \frac{U_{\mathrm{mod}}}{U} && \text{wind speed scale} \end{aligned} \qquad (34)$$

Looking at (33), we find that the last two equations are identical, so that there remain only three equations:

$$\begin{aligned} k_m &= k_L^3 \\ k_I &= k_L^5 \\ k_{stif\ b} &= k_L^4 k_v^2 \\ k_{stif\ \tau} &= k_L^4 k_v^2 \end{aligned} \qquad (35)$$

Returning to the aeroelastic problem: to build an aeroelastic model of a wing in an *incompressible flow*, the next procedure seems to be reasonable:

- Suppose that the "original wing" is completely known geometrically and structurally;
- Suppose that the maximum wing speed $U_{\max}$ is known and the maximum admissible wind tunnel speed $(U_{\max})_{\text{mod}}$ is also known; we find $k_V = \dfrac{(U_{\max})_{\text{mod}}}{U_{\max}}$;
- Choose a model length scale k_L to match the size of the tunnel;
- Now calculate the unknown scales in (33) and build the model; the "model" will withstand the static loads to which it is subjected in terms of similarity, because the "original wing" itself is supposed to withstand the "original condition loads";
- Test the model wing in the tunnel, starting from a small wind tunnel speed and increasing the speed up to its maximum value, $(U_{\max})_{\text{mod}}$. If the flutter develops at $(U_f)_{\text{mod}} < (U_{\max})_{\text{mod}}$, then we can calculate the "original wing" flutter speed $U_f = \dfrac{(U_f)_{\text{mod}}}{k_V}$. If the flutter does not develop in the speed range $(U_f)_m < (U_{max})_{\text{mod}}$, then it is no danger for the original wing to meet the flutter conditions in real flight.

6. CONCLUSIONS

In this work are summarized the principles of similarity modeling, dimensional analysis, and the Buckingham (Π) theorem is reminded. The theoretical considerations are illustrated with some examples from geometry, fluid mechanics and mechanics.

From these examples, the concept of similar mechanical problems/phenomena was developed: two states/phenomena are similar if they are described by equations (of any type algebraic, differential, integral, etc.) that can be brought to a form involving identical non-dimensional functions and equal Pi grup coefficients ($\Pi_1, \Pi_2, ..., \Pi_m$).

The main applications of this general theory are in the field of aeroelasticity. The paper presents how, starting from the general equations governing the aerodynamic load redistibution (static aeroelasticity) and flutter (dynamic aeroelasticity), one can get equations involving Pi groups and identical non-dimensional functions.

Equating the Pi terms for the "original wing and flight conditions" with the Pi terms written for the "model wing and wind-tunnel conditions", one gets the algebraic relations between the scale factors.

These relations are always monomials (in a broad sense, i.e negative powers are acceptable). A general procedure for the experimental study of the aeroelastic phenomenon of flutter is described.

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