Aerodynamic Interference between Oscillating Lifting Surfaces and Fuselage
Part 5: A Panel Method for Non-Lifting Bodies

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Abstract: In the fifth article of our series we will deal with the calculation of the unsteady aerodynamic forces on non-lifting bodies. We present here a contribution to the problem of the flow about non-lifting bodies. It is a panel method available for subsonic unsteady flow. The method will be used further to the unsteady body-body and wing-body interference problems.

Key Words: Flow about bodies, lifting surface, integral equation, harmonic oscillations, generalised airforces, flutter, oscillating bodies.

1. INTRODUCTION

In the previous articles of this series we dealt with the flow about the oscillating lifting surfaces, [1]-[4]. In the next articles we will continue to present some other problems regarding the unsteady flow about wings. For example, we haven’t yet presented a method for calculation of the suction force in unsteady flow.

However, the goal of our study is the unsteady wing-body combination. Therefore, we make a break in presenting the issues on wing. It is now the time to attack the problem of the non-lifting body. This is the second stage after the wing case approach.

The work presents resume the results obtained by the author long time ago [5], [6], [7]. For lack of time, we have never extended the analysis, except for the case of the body-body interference, [8].

The problem of estimating the unsteady aerodynamics of elongated bodies that perform harmonic oscillations of small amplitude, has been the subject of several studies appeared over time. We must mention that, in 1923 Munk developed the concept of “flow in the transverse plane”, [9] available for steady flow case.

This concept applies to very elongated bodies. It postulates that in the transverse plane, the flow is predominantly two-dimensional. Stewartson's method which applies to the unsteady flow about the elongated bodies [10], implicitly recognizes the same postulate of the transverse plane flow.

Finally, we mention that important contributions to the theory of unsteady flow about the revolution bodies had Wu, Garcia - Fogeda and Liu, [11], [12].
These latter works exceed the narrow theory of the elongated bodies. So they respond to the demands imposed by modern configurations of missiles.

However, their method cannot be used to wing-body combination which is analysed in [13]. This paper is devoted to development of an oscillating source panel method by which we can study the distribution of pressure on non-lifting bodies of any aerodynamic shape. It will be used for applications to the combinations of wings and bodies that perform harmonic oscillations.

As we will see, this case raises specific problems that are distinct from those found in the case of lifting surfaces.

2. HARMONIC OSCILLATING SOURCE PANEL METHOD

2.1 Formulation of the problem

Consider an elongated body (for example a rocket, a tank, an isolated fuselage). This body is immersed in a stream. The stream is uniform and subsonic at infinite.

One the following situations are available: either the body perform harmonic oscillations of small amplitude, or it is located in the neighborhood of an oscillator.

The flow about this body can be represented by a simple layer (or source) potential of intensity \( q(\xi, \eta, \zeta; M, \kappa) \):

\[
\phi(x, y, z; M, \omega) = -\frac{i}{4\pi} \int_{(\Sigma_0)} e^{i(M\bar{R} - R)} q(\xi, \eta, \zeta; M, \kappa)dS
\]  

(1)

In the above formula, we introduced the following symbols and notations:

\( \Sigma_0 \) represents the surface of the body, \((x, y, z)\) is a point in space, \((\xi, \eta, \zeta)\) is a point on \( \Sigma_0 \), \( U_\infty \) is the wind velocity at infinity, \( M \) is the Mach number of the wind \((\beta^2 = 1 - M^2)\), and \( \omega \) is the angular frequency of oscillations.

Moreover, \( i = \sqrt{-1} \), and:

\[
x_0 = x - \xi \; ; \; y_0 = y - \eta \; ; \; z_0 = z - \zeta \; ; \; (\xi, \eta, \zeta) \in (\Sigma_0)
\]

\[
R = \sqrt{x_0^2 + \beta^2 r^2} \; ; \; r^2 = y_0^2 + z_0^2 \; ; \; \kappa = \frac{\omega}{U_\infty \beta^2}
\]  

(2)

Let us find the potential \( \phi \) and the velocity field generated by it when the source strength \( q(\xi, \eta, \zeta) \) is known.

To solve the above problem, we will use a numerical method called the panel method. Consider the equation of the body surface \( (\Sigma_0) \) written as:

\[
P = P_0(U, V)
\]  

(3)

In the above equation \( U \) and \( V \) are parameters that define a certain point \( P \) on \( (\Sigma_0) \). One can imagine a certain discretization into \( N \) curvilinear panels, as that represented in fig. 1.

However, we will not use curvilinear panels; we will rather prefer planar ones. They were denoted here by \((\Pi)_k\), \( k=1...N \). The way we get the planar panel to replace the curvilinear one is described by Hess, Smith and Woodward.

\footnote{The proof in the next paper}
We will write (1) as:

$$\varphi(x, y, z; M; \omega) \approx \sum_{k=1}^{N} q_k \varphi_k(x, y, z; M; \omega),$$

$$\varphi_k = -\frac{1}{4\pi} \int_{(\Pi)_k} \xi_k \varphi_k(\xi) dS.$$  (4)

In the above equations, $q_k$ is the mean value of $q(\xi, \eta, \zeta)$ in the curvilinear domain that was approximated by the planar quadrilateral $(\Pi)_k$.

So the previous formulae (4) introduced two approximations, namely:

1) $q_k$ is considered constant, whereas it depends on $(x, y, z)$, $M$, $\kappa$;
2) the curvilinear panel is considered as a plane domain, $(\Pi)_k$, whereas it is curvilinear.

The problem is thus reduced to the calculation of the double integrals in (4) and their gradients, when $(\Pi)_k$ are planar quadrilaterals and $q = \text{const}$.

### 2.2 The expressions of the induced velocity potential and the induced velocity field of an oscillating source planar quadrilateral domain, $q=\text{const}$.

Consider a generic quadrilateral domain $(\Pi)$. We have written the velocity potential in (4), while the velocity field is easily obtained applying the gradient operator:

$$\mathbf{v}(Q; M; \omega) = -\frac{1}{4\pi} \int_{(\Pi)} \mathbf{W}(x_0, y_0, z_0; M; \omega) dS$$

$$\mathbf{W}(x_0, y_0, z_0; M; \omega) = \left\{ -\frac{\mathbf{R}}{R^3} + i\kappa \left[ M\mathbf{i} - \frac{\mathbf{R}}{R} \right] \frac{1}{R} \right\} e^{i(k(M_0 - R))}.$$  (5)

In the above,

$$\mathbf{R} = x_0 \cdot \mathbf{i} + \beta \cdot \mathbf{r}, \quad \mathbf{r} = y_0 \cdot \mathbf{j} + z_0 \cdot \mathbf{k}.$$  (6)

From now on $Q$ will stand for the point of coordinates $(x, y, z)$, where the potential $\varphi$ and the velocity $\mathbf{v}$ are calculated. We will use the scalar expressions

$$\varphi(Q; M; \omega) = -\frac{1}{4\pi} q I_{01}$$

$$v_x(Q; M; \omega) = \frac{1}{4\pi} q (I_{x3} + i\kappa \cdot I_{x2} - i\kappa \cdot M I_{01})$$

$$v_y(Q; M; \omega) = \frac{\beta^2}{4\pi} q (I_{y3} + i\kappa \cdot I_{y2})$$

$$v_z(Q; M; \omega) = \frac{\beta^2}{4\pi} q (I_{z3} + i\kappa \cdot I_{z2}).$$  (7)

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In the above equations:

\[
I_{01} = \iint_{(\Pi)} \frac{1}{R} \cdot e^{i\omega(M_{01} - R)} \, dS; \\
I_{ym} = \iint_{(\Pi)} \frac{\gamma_0}{R^m} \cdot e^{i\omega(M_{ym} - R)} \, dS;
\]

\[
I_{zm} = \iint_{(\Pi)} \frac{z_0}{R^m} \cdot e^{i\omega(M_{zm} - R)} \, dS;
\]

\[
m = 2, 3
\]

The above integrals cannot be expressed by elementar functions, except when the frequency is zero (\(\omega = 0\)) or the steady case. Therefore we will seek approximate expressions for the integrals in (8), when the planar domain is bordered by a convex quadrilateral.

2.3 A rapid numerical method for calculating the integrals in (8)

Let \(l\) be a characteristic length of the panel, for example \(\frac{1}{2}\) of its longest side or diagonal. Then we will define the panel’s reduced frequency as

\[
k = \frac{\omega \cdot l}{U_\infty}
\]

We can write the \((\Pi)\) plane equation as

\[
\vec{P} = \vec{a} + \vec{b} \cdot \lambda + \vec{c} \cdot \mu + \vec{d} \cdot \lambda \mu
\]

In the above equation,

\[
\vec{a} = \frac{1}{4} \left( \vec{N}_1 + \vec{N}_2 + \vec{N}_3 + \vec{N}_4 \right); \\
\vec{b} = \frac{1}{4} \left( -\vec{N}_1 + \vec{N}_2 + \vec{N}_3 - \vec{N}_4 \right)
\]

\[
\vec{c} = \frac{1}{4} \left( -\vec{N}_1 - \vec{N}_2 + \vec{N}_3 + \vec{N}_4 \right); \\
\vec{d} = \frac{1}{4} \left( \vec{N}_1 - \vec{N}_2 + \vec{N}_3 - \vec{N}_4 \right)
\]

\(\vec{N}_i\), \(i = 1, 2, 3, 4\) represent the vector positions of the panel corners.

We can see that, for \(\lambda = -1, \mu = -1 \Rightarrow \vec{P} \equiv \vec{N}_1; \lambda = 1, \mu = -1 \Rightarrow \vec{P} \equiv \vec{N}_2\), and so on. Then a point on the panel is characterized by the standard domain: \((\lambda, \mu) \in (-1,1) \times (-1,1)\).

We can now write the potential and the velocity field as:

\[
\varphi(Q; M; \omega) = -\frac{1}{4\pi} \int_{-1}^{1} \int_{-1}^{1} I_{01}[Q - P(\lambda, \mu)] \sqrt{g_{11}g_{22} - g_{12}^2} \, d\lambda \, d\mu
\]

\[
v(Q; M; \omega) = -\frac{1}{4\pi} \int_{-1}^{1} \int_{-1}^{1} W[Q - P(\lambda, \mu)] \sqrt{g_{11}g_{22} - g_{12}^2} \, d\lambda \, d\mu
\]

In (12) \(g_{ij}\) represent the metric coefficients of \((\Pi)\). If the point \(Q\) is far enough from the panel, the integrands in (12) can be well approximated by polynomials, so that we can use Gauss quadrature formulae:

\[
\varphi(Q; M; \omega) = -\frac{1}{4\pi} \sum_{m=1}^{N} \sum_{n=1}^{N} A_m A_n (I_{01} \cdot \sqrt{g_{11}g_{22} - g_{12}^2})_{\lambda=\lambda_m, \mu=\mu_n}
\]

\[
v(Q; M; \omega) = -\frac{1}{4\pi} \sum_{m=1}^{N} \sum_{n=1}^{N} A_m A_n (\tilde{W} \sqrt{g_{11}g_{22} - g_{12}^2})_{\lambda=\lambda_m, \mu=\mu_n}
\]
We used \( N \) points for the quadratures, \( A_m \) and \( A_n \) are the Gauss coefficients and \( \lambda_m, \mu_n \) are the Gauss integration points.

Obviously, if \( Q \) is too close to the panel, we have to use another method.

### 2.4 Coordinate systems and change of variables

The equations (7) and (8) are written using the \( oxyz \) system, with \( o \) along the free stream. Let \( \alpha \) be the angle between the \( Ox \) and \( (\Pi) \) plane (fig. 2).

We can write a parametric equation of \( (\Pi) \),

\[
\begin{align*}
(\Pi) : & \quad \xi = \xi' \cos \alpha ; \quad \eta = \eta' ; \quad \zeta = \zeta' \sin \alpha
\end{align*}
\]  

We will define a new system of coordinates, namely:

\[
(oy'z') \quad (o' \equiv o, o'y' \equiv oy, o'x' \in (\Pi))
\]

Between the \( oxyz \) and \( o'x'y'z' \) coordinate systems there is the next transformation:

\[
\begin{align*}
x = x' \cos \alpha - z' \sin \alpha \\
y = y' \\
z = x' \sin \alpha + z' \cos \alpha
\end{align*}
\]  

It is available both for \( (x, y, z) \rightarrow (x', y', z') \) and for \( (\xi, \eta, \zeta) \rightarrow (\xi', \eta', 0) \). The principal difficulty of the integrals is given by the polar singularity introduced by \( R^2 \):

\[
R^2 = a_{11} \xi'^2 + a_{22} \eta'^2 - 2a_{10} \xi' \eta' - 2a_{02} \eta' + a_{00}
\]  

where:

\[
\begin{align*}
a_{11} &= 1 - M^2 \sin^2 \alpha ; \quad a_{22} = \beta^2 ; \quad a_{10} = x \cos \alpha + \beta^2 z \sin \alpha \\
a_{02} &= \beta^2 y ; \quad a_{00} = x^2 + \beta^2 (y^2 + z^2)
\end{align*}
\]  

The next change of variables,

\[
\begin{align*}
\xi &= \frac{\xi''}{\beta} + \frac{x \cos \alpha + \beta^2 z \sin \alpha}{\beta^2} \\
\eta' &= \frac{\eta''}{\beta} + y
\end{align*}
\]  

brings (16) to the simpler form:

\[
\text{Fig. 2 The panel plane and axes of coordinates} 
\]
\[
R = \sqrt{\xi^2 + \eta^2 + z''^2}, \quad z'' = \frac{\beta}{\beta_\alpha} z' = \frac{\beta}{\beta_\alpha} (-x \sin \alpha + z \cos \alpha)
\] (19)

In the above equations,
\[
\beta_\alpha = \sqrt{1 - M^2 \sin^2 \alpha} = \sqrt{\cos^2 \alpha + \beta^2 \sin^2 \alpha}
\] (20)

Now we can write (8) as:
\[
I_{0l} = \frac{1}{\beta \beta_\alpha} \exp(-i\kappa M \frac{\beta}{\beta_\alpha} z'' \sin \alpha) J_{0l}
\]
\[
I_{lm} = \frac{1}{\beta \beta_\alpha} \exp(-i\kappa M \frac{\beta}{\beta_\alpha} z'' \sin \alpha) \cdot (J_{\zeta m} \cos \alpha + \beta z'' J_{0m} \sin \alpha)
\]
\[
I_{\zeta m} = \frac{1}{\beta \beta_\alpha} \exp(-i\kappa M \frac{\beta}{\beta_\alpha} z'' \sin \alpha) J_{\zeta m}
\]
\[
I_{\eta m} = \frac{1}{\beta \beta_\alpha} \exp(-i\kappa M \frac{\beta}{\beta_\alpha} z'' \sin \alpha) \cdot (J_{\zeta m} \sin \alpha - \frac{z''}{\beta} J_{0m} \cos \alpha)
\]
\[
m = 2,3
\]

where:
\[
J_{0n} = \iint_{(11)} \exp\left\{i\kappa \left(\mu \xi'' - R\right)\right\} \frac{d\xi''}{R^n} d\eta'', \quad n = 1,2,3
\]
\[
J_{\zeta m} = \iint_{(11)} \xi'' \exp\left\{i\kappa \left(\mu \xi'' - R\right)\right\} d\xi'' d\eta''
\]
\[
J_{\eta m} = \iint_{(11)} \eta'' \exp\left\{i\kappa \left(\mu \xi'' - R\right)\right\} d\xi'' d\eta'', \quad m = 2,3, \mu = -\frac{M}{\beta_\alpha} \cos \alpha
\] (22)

It now is convenient to use the polar coordinates:
\[
\xi'' = \rho \cos \theta; \quad \eta'' = \rho \sin \theta
\] (23)

Now we can write:
\[
R = \sqrt{\rho^2 + z''^2}
\] (24)

Fig. 3 The panel divided into four triangular domains
We will use the centre o" as the pole of the coordinate system and perform the integrals over the four triangular domains (fig. 3).

Let Δ be such a triangular domain.

The exterior side of this domain has the equation:

\[ r(\theta) = \frac{r_0}{\cos(\theta - \theta_0)} , \quad \theta_i \leq \theta \leq \theta_2 \]  

(25)

Here \( r_0 \) is distance from o" to the linear side of the panel; \( \theta_1 \) and \( \theta_2 \) are the position angles of the two ends of the side, measured from the o"ξ".

So the integrals in (22) can be written as a sum of four integrals of the type:

\[
\int_0^{\theta_2} \int_0^{\theta_1} \int_0^{\theta_2} \int_0^{\theta_1} \]

\[ J_{0n}^\Delta = \int_0^{\theta_2} \left[ J_{r0}^{(1)}(r(\theta)) \right] d\theta ; \quad n = 1, 2, 3 \]

\[ J_{\xi m}^\Delta = \int_0^{\theta_2} \left[ J_{r\xi}^{(2)}(r(\theta)) \cos \theta d\theta ; \quad m = 2, 3 \]

\[ J_{\eta m}^\Delta = \int_0^{\theta_2} \left[ J_{r\eta}^{(2)}(r(\theta)) \sin \theta d\theta ; \quad m = 2, 3 \]

In the above equations,

\[
J_{r0}^{(1)}(r) = \int_0^r \frac{\rho}{(\rho^2 + z''^2)^{m/2}} \exp\left\{ i\kappa (\mu \rho \cos \theta - \sqrt{\rho^2 + z''^2} \right\} d\rho , \quad n = 1, 2, 3 \]

(27)

\[
J_{r\xi}^{(2)}(r) = \int_0^r \frac{\rho^2}{(\rho^2 + z''^2)^{m/2}} \exp\left\{ i\kappa (\mu \rho \cos \theta - \sqrt{\rho^2 + z''^2} \right\} d\rho , \quad m = 2, 3 \]

Some of the above integrals are singular if \( z'' \to 0 \).

2.5 Polar singularities

Let us introduce the following notations:

\[ E(\rho, \theta; z''; \kappa; \mu) = \exp\left\{ i\kappa (\mu \rho \cos \theta - \sqrt{\rho^2 + z''^2} \right\} \]

\[ E(0, 0; z''; \kappa; \mu) = \exp\left\{- i\kappa |z''| \right\} = E_0 \]

(28)

The following integrands in (27) do not generate singular behaviours:

\[ j_{p1}^{(1)} = \frac{\rho}{\sqrt{\rho^2 + z''^2}} E ; \quad j_{p2}^{(2)} = \frac{\rho}{\sqrt{\rho^2 + z''^2}} (E - E_0) \]

\[ j_{p3}^{(1)} = \frac{\rho}{\sqrt{\rho^2 + z''^2}} [E - (1 - \kappa \mu \rho \cos \theta) E_0] \]

(29)

\[ j_{p2}^{(2)} = \frac{\rho^2}{\rho^2 + z''^2} E ; \quad j_{p3}^{(2)} = \frac{\rho^2}{(\rho^2 + z''^2)^{3/2}} (E - E_0) \]

We can now write the five integrals in (27) as:
\[ J_{p1}^{(1)} = \int_0^r j_{p1}^{(1)}(\rho, \theta; z^*; \kappa; \mu) d\rho \]
\[ J_{p2}^{(1)} = \int_0^r j_{p2}^{(1)}(\rho, \theta; z^*; \kappa; \mu) d\rho + E_0 \cdot \left( \ln \sqrt{r^2(\theta) + z^{*2}} - \ln\eta \right) \]
\[ J_{p3}^{(1)} = \int_0^r j_{p3}^{(1)}(\rho, \theta; z^*; \kappa; \mu) d\rho + \]
\[ E_0 \left( -\frac{1}{\sqrt{r^2(\theta) + z^{*2}}} + i\kappa \cos \theta \right) + \ln \left( \frac{r(\theta)}{\sqrt{r^2(\theta) + z^{*2}}} \right) \]
\[ \left( \frac{1}{|z^*|} - i\kappa \mu \cos \theta \ln |z^*| \right) \]
\[ J_{p2}^{(2)} = \int_0^r j_{p2}^{(2)}(\rho, \theta; z^*; \kappa; \mu) d\rho \]
\[ J_{p3}^{(2)} = \int_0^r j_{p3}^{(2)}(\rho, \theta; z^*; \kappa; \mu) d\rho + E_0 \left( -\frac{r(\theta)}{\sqrt{r^2(\theta) + z^{*2}}} + \right. \]
\[ \left. \ln \left( r(\theta) + \sqrt{r^2(\theta) + z^{*2}} \right) - \ln |z^*| \right] \]

2.6 Expressions for \( J_{0n}^\Delta, J_{\xi m}^\Delta, J_{\eta n}^\Delta \)

Using the above equations and the panel side equation given in (25) which is a line segment, we get:
\[ J_{01}^\Delta = \int_0^{\theta_1} \int_0^{r(\theta)} j_{p1}^{(1)} d\rho d\theta \]
\[ J_{02}^\Delta = \int_0^{\theta_2} \int_0^{r(\theta)} j_{p2}^{(1)} d\rho d\theta + E_0 \left[ \lambda_{02} \lambda_{10} - (\theta_2 - \theta_1) E_0 \right] \ln |z^*| \]
\[ J_{03}^\Delta = \int_0^{\theta_3} \int_0^{r(\theta)} j_{p3}^{(1)} d\rho d\theta + E_0 \left[ -\lambda_{03} \lambda_{10} + i\kappa \mu \left( -\lambda'_{\xi 3} + \lambda''_{\xi 3} \right) \lambda_{01} + \frac{\theta_2 - \theta_1}{|z^*|} - i\kappa \mu (\sin \theta_2 - \sin \theta_1) \ln |z^*| \right] \]
\[ J_{\eta 2}^\Delta = \int_0^{\theta_2} \int_0^{r(\theta)} j_{p2}^{(2)} \sin \theta d\rho d\theta \]
\[ J_{\eta 3}^\Delta = \int_0^{\theta_3} \int_0^{r(\theta)} j_{p3}^{(2)} \sin \theta d\rho d\theta + E_0 \left[ -\lambda'_{\eta 3} + \lambda''_{\eta 3} \lambda_{01} + E_0 (\cos \theta_2 - \cos \theta_1) \ln |z^*| \right] \]
\[ J_{\xi 2}^\Delta = \int_0^{\theta_2} \int_0^{r(\theta)} j_{p2}^{(2)} \cos \theta d\rho d\theta \]
\[ J_{\xi 3}^\Delta = \int_0^{\theta_3} \int_0^{r(\theta)} j_{p3}^{(2)} \cos \theta d\rho d\theta + E_0 \left[ -\lambda'_{\xi 3} + \lambda''_{\xi 3} \lambda_{01} - E_0 (\sin \theta_2 - \sin \theta_1) \ln |z^*| \right] \]

The functions \( \lambda'_{03}, \lambda'_{\xi 3}, \lambda''_{\xi 3}, \lambda'_{\eta 3}, \lambda''_{\eta 3} \) are given in the next paragraph. The integrals which appear in (31) are calculated numerically, for example using the method given in 2.8.
On the other hand the function $\lambda_{02}$ can also be calculated numerically. Before that, we need to separate the logarithmic singularity:

$$\lambda_{02} = \int_{\theta_1}^{\theta_2} \ln \sqrt{r^2(\theta) + z''^2} d\theta = \int_{\theta_1}^{\theta_2} F(\theta, r, \theta_0, z'') d\theta +$$

$$+ \left[ (\frac{z'}{z} + \theta_0 - \theta) \ln(\frac{z'}{z} + \theta_0 - \theta) \right]_{\theta_1}^{\theta_2} - \left[ (\frac{z'}{z} - \theta_0 + \theta) \ln(\frac{z'}{z} - \theta_0 + \theta) \right]_{\theta_1}^{\theta_2}$$

$$+ (\theta_2 - \theta_1) (\ln \sqrt{r^2(\theta) + z''^2} + 2), \text{ where :}$$

$$F(\theta; r, \theta_0, z'') = \int \left( 1 - \frac{z''^2 \sin^2(\theta - \theta_0)}{r^2 + z''^2} \cos(\theta - \theta_0) \right) d\theta$$

One can see that the numerical method proposed here is always available when $o''$ is inside the panel. If $o''$ is outside the panel, it should not be too far. Otherwise, there is a parasitic domain outside the panel which introduces new unnecessary approximations.

### 2.7 Expressions for $\lambda'_{03}, \lambda'_{\xi3}, \lambda''_{\xi3}, \lambda'_{\eta3}, \lambda''_{\eta3}$

The next integrals can be expressed using elementary functions:

$$\lambda_{03}(\theta) = \int \frac{d\theta}{\sqrt{r^2(\theta) + z''^2}} ; \lambda'_{\xi3}(\theta) = \int \frac{r(\theta)}{\sqrt{r^2(\theta) + z''^2}} \cos \theta d\theta$$

$$\lambda''_{\xi3}(\theta) = \int \ln \left[ r(\theta) + \sqrt{r^2(\theta) + z''^2} \right] \cos \theta d\theta$$

$$\lambda'_{\eta3}(\theta) = \int \frac{r(\theta)}{\sqrt{r^2(\theta) + z''^2}} \sin \theta d\theta ; \lambda''_{\eta3}(\theta) = \int \ln \left[ r(\theta) + \sqrt{r^2(\theta) + z''^2} \right] \sin \theta d\theta$$

In the above formulae, $r(\theta)$ is given by the straight line equation (25).

Let us introduce the simplifying notation:

$$R_0^2 = \sqrt{r_0^2 + z''^2}$$

Integrating the indefinite integrals (33), we get:

$$\lambda_{03} = \frac{1}{z''} \arcsin \left[ \frac{z''}{R_0} \sin(\theta - \theta_0) \right], \text{ if } z'' \neq 0$$

$$\lambda_{03} = \frac{1}{r_0} \sin(\theta - \theta_0), \text{ if } z'' = 0$$

$$\lambda'_{\xi2} = \frac{r_0}{|z''|} \left[ I_y(\theta - \theta_0) \cos \theta_0 + I_c(\theta - \theta_0) \sin \theta_0 \right], \text{ if } z'' \neq 0$$

$$\lambda'_{\xi2} = \sin \theta, \text{ if } z'' = 0$$

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\[
\lambda'_{n3} = \frac{r_0}{|z'|} \left[ - I_c(\theta - \theta_0) \cos \theta_0 + I_c(\theta - \theta_0) \sin \theta_0 \right], \text{ if } z'' \neq 0
\]
\[
\lambda''_{n3} = -\cos \theta, \text{ if } z'' = 0
\]

In the above two formulae,
\[
I_s(\tau) = \arcsin \left( \frac{|z'|}{R_0} \sin \tau \right); \quad I_c(\tau) = \ln \left( \cos \tau + \frac{1}{|z'|} \sqrt{r_0 + z''^2 \cos^2 \tau} \right)
\]

Then,
\[
\lambda''_{\xi3} = (\lambda''_{\xi3})_1 - (\lambda''_{\xi3})_2; \quad \lambda''_{n3} = (\lambda''_{n3})_1 - (\lambda''_{n3})_2
\]
\[
(\lambda''_{\xi3})_1 = J_s(\sin(\theta - \theta_0)) \cos \theta_0 + J_c(\cos(\theta - \theta_0)) \sin \theta_0, \text{ if } z'' \neq 0
\]
\[
(\lambda''_{n3})_1 = J_s(\sin(\theta - \theta_0)) \sin \theta_0 - J_c(\cos(\theta - \theta_0)) \cos \theta_0, \text{ if } z'' \neq 0
\]
\[
(\lambda''_{\xi3})_2 = \ln(2r_0) \sin \theta; \quad (\lambda''_{n3})_2 = -\ln(2r_0) \cos \theta, \text{ if } z'' = 0
\]

\[
J_s(u) = u \ln(r_0 + \sqrt{R_0^2 - z''^2 u^2}) +
\]
\[
\frac{1}{|z'|} \ln \left| \frac{z''u(R_0 - r_0)}{|z'|(R_0 + \sqrt{R_0^2 - z''^2 u^2})} \right| + \frac{1}{|z'|} \left[ \arcsin \left( \frac{z''}{R_0} \right) u - z'' u \right]
\]

\[
J_c(u) = v \left[ \ln(r_0 + \sqrt{r_0^2 + z''^2 v^2}) - 1 \right] - \frac{r_0}{z''} \ln \left( \frac{r_0}{z'' \sqrt{r_0^2 + z''^2 v^2}} \right)
\]

\[
K_c(t) = \sin t \cdot \left[ \ln(\cos t) - 1 \right] + \ln \left( \frac{\cos t}{1 - \sin t} \right)
\]

### 2.8 Approximate calculation of the integrals in (31)

There are a lot of integrals in (31) which cannot be expressed by elementary functions. They will be estimated numerically. They are of the form:

\[
\Omega(\theta_1, \theta_2; r_0; \theta_0) = \int_{\theta_1}^{\theta_1 + r(\theta)} \int_0^{\theta_2} F(\rho, \theta)d\rho d\theta
\]

where \( F(\rho, \theta) \) has no singularities in the integration domain. Making the change of variables,

\[
\theta = \frac{1}{2} [ (\theta_2 - \theta_1) \cdot \bar{\theta} + \theta_1 + \theta_2 ]
\]
\[
\rho = \frac{1}{2} r(\theta) \cdot (\bar{\rho} + 1)
\]
where \( r(\theta) \) is given by (25), we obtain:

\[
\Omega(\theta_1, \theta_2; \eta_0; \theta_0) = \frac{1}{4} (\theta_2 - \theta_1) \int_{-1}^{1} \int_{-1}^{1} r[\theta(\bar{\theta})] \cdot F[\rho(\bar{\theta}, \bar{\phi}), \theta(\bar{\theta})] \rho d\bar{\theta} = \\
= \frac{1}{4} (\theta_2 - \theta_1) \sum_{i=1}^{N_\eta} \sum_{j=1}^{N_\rho} A_i A_j \rho_\eta \rho \theta(\bar{\theta}_j), \theta(\bar{\theta}_j)]
\]

(41)

Here \((A_i, A_j, (\rho_j, \theta_j))\) are the Gauss coefficients and respectively the positions of the integration points in the domain of integration.

2.9 Final formulae

After the calculation of all \( J_{01}^\Delta, J_{02}^\Delta, \ldots J_{n3}^\Delta \), we will get:

\[
J_{01} = \sum_{i=1}^{4} J_{01}^\Delta; \quad J_{02} = \sum_{i=1}^{4} J_{01}^\Delta; \quad \ldots J_{n3} = \sum_{i=1}^{4} J_{n3}^\Delta
\]

(42)

So we got all the terms of (22). We now will introduce these values in (20) and so obtain \( \Omega_{11}, I_{1m}, I_{jm}, I_{1m}, (m = 2, 3) \). So we can calculate the potential and the velocity field with (7).

It is still important to note that we can write (7) as follows:

\[
\phi = -\frac{qE}{4\pi} J_{01}
\]

\[
\nu_x = \frac{qE}{4\pi \beta_\alpha} (-J_{\xi32} \cos \alpha - \frac{\beta^2 z'}{\beta_\alpha} J_{032} \sin \alpha - i\kappa M \beta_\alpha J_{01})
\]

\[
\nu_y = -\frac{qE \beta}{4\pi} J_{\eta32}
\]

\[
\nu_z = \frac{qE}{4\pi \beta_\alpha} (-\beta^2 J_{\xi32} \sin \alpha + \frac{\beta^2 z'}{\beta_\alpha} J_{032} \cos \alpha)
\]

where,

\[
J_{032} = J_{03} + i\kappa J_{02} \\
J_{\xi32} = J_{\xi3} + i\kappa J_{\xi2} \\
J_{\eta32} = J_{\eta3} + i\kappa J_{\eta2} \\
E = \frac{1}{\beta \beta_\alpha} \exp(-i\kappa M \frac{\beta}{\beta_\alpha} z'' \sin \alpha)
\]

(44)

Written in the panel system of coordinates \( o' x' y' z' \), the induced velocity becomes:

\[
\nu_x' = -\frac{qE}{4\pi} (\beta_\alpha J_{\xi32} + i\kappa M J_{01} \cos \alpha)
\]

\[
\nu_y' = -\frac{qE}{4\pi} \beta J_{\eta32}
\]

\[
\nu_z' = \frac{qE}{4\pi} \left( \frac{M^2}{\beta^2_\alpha} J_{\xi32} \sin \alpha \cos \alpha + \frac{\beta^2}{\beta^2_\alpha} z' J_{032} + i\kappa M J_{01} \sin \alpha \right)
\]

(45)
We observe that in the steady flow conditions $\omega = 0$, and then $\kappa = 0$. So in the above relations only $J_{03}, J_{\xi 3}, J_{\eta 3}$ do not cancel (steady state effect). Remarkably, all these terms are expressed by closed formulae. The other functions are expressed both by analytic functions and by numerical quadratures. Only these functions contribute to the proper oscillatory effect.

We will also remark that the velocity field is defined throughout the space except the panel domain. The normal component of the velocity has a discontinuity across the panel,

$$
\lim_{\zeta \to 0^+} v_\zeta' - \lim_{\zeta \to 0^-} v_\zeta' = \Delta v_\zeta = -\frac{q}{\beta_0^2}
$$

(46)

This jump is produced only by the terms that do not depend on $\omega$, i.e. it is produced by the steady effect.

Finally also note that, in neighbourhood of the panel sides, the velocity field has a singular behavior. This behavior is given both by the steady and by the oscillatory terms.

3. CONCLUSIONS

Two numerical methods for calculating the potential and velocity fields about a planar quadrilateral panel have been presented. The first one is a simple numerical method that can be applied for points located not too close to the panel, but at a certain distance from it. This minimum distance will be find numerically and it will be presented in the next paper. The second method applies to points located however close to the panel. The velocity field in the neighbourhood of the oscillating source panel has some remarkable properties:

1. The steady effect (i.e. the terms of the velocity field that are independent of $\omega$) can be decoupled for $\omega = 0$ ($k = 0$); in this case (steady flow), the method is exact.

2. When $z' \to 0$ and $\omega'$ is inside the panel, the velocity field has polar singularities. The polar singularities that occur are due to only the steady effect. When crossing the panel, there is a jump of the normal component of the velocity. This jump is influenced by $\alpha$ and $M$.

3. There is a singular behaviour of the velocity in the proximity of the panel borders. It is generated by both steady and unsteady terms.

REFERENCES


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