# Numerical Calculation of the Output Power of a MHD Generator

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***Abstract**: Using Lazăr Dragoş's analytic solution for the electric potential we perform some numerical calculations in order to find the characteristics of a Faraday magnetohydrodymamics (MHD) power generator (total power, useful power and Joule dissipation power).*

***Key Words**: MHD generator, electric potential, conformal mapping, Voltera-Signorini problem, Gauss quadrature formulas, power.*

## 1. INTRODUCTION

The magnetohydrodynamic (MHD) generators transform the thermal and mechanical (kinetic) energy into electricity. Unlike the traditional electric generators, they use hot electroconductive plasmas (ionized gases, liquid metals) and have no moving parts. References concerning the mathematical theory of the MHD power generators may be found in L. Dragoş's book [1], Chapter 4. In the last years, new applications of MHD generators to hypersonic aircrafts have been considered (see for example the papers of Petit and Geffray [2] and those of Sheikin and Kuranov [3]).

The generated electricity can be used to power electromagnetic devices on board or to the so-called MHD bypass (i.e. MHD acceleration of the engine exhaust flow). The basic elements of a simple MHD generator (the so-called continuous electrode Faraday generator) are shown in Fig. 1.

In the domain bounded by the electrodes, a magnetic field of induction $\vec{B}_0$ is applied transversely to the motion of an electrically conducting fluid flowing with velocity $\vec{V}_0$ in an insulated duct. The motion of electrically conducting fluids in a duct was investigated in many papers (se for example the articles of Carabineanu et al. [4], [5], Tezer-Sezgin [6], Tezer-Sezgin and Bozkaya [7], [8], Çelik [9], Tezer-Sezgin and Han Aydin [10]). Electrically charged particles flowing with the fluid determine an induced electric field $\vec{V}_0 \times \vec{B}_0$ which drives an electric current in a direction orthogonal to $\vec{V}_0$ and $\vec{B}_0$. This current is collected by the electrodes and flows in an external load circuit. We denote by $2L_0$ the distance between the electrodes. The electric current flowing across the electroconductive plasma between the electrodes is the *Faraday current*. It provides the main electrical output of the MHD generator. The Faraday current reacts with the applied magnetic field creating a *Hall effect* current perpendicular to the Faraday current.

In this paper we present a simplified version of the MHD generators theory. Besides the simple geometry of the generator, we neglect the Hall effect and the effect of the Loretz force against the fluid flow, whence it follows the uniform flow of the plasma. Thermal effects, compressibility and viscosity are also neglected and the electromagnetic field is considered stationary. In order to calculate the MHD generator characteristics we use Lazăr Dragoş's analytic solution for the electric potential and perform some numerical calculations.

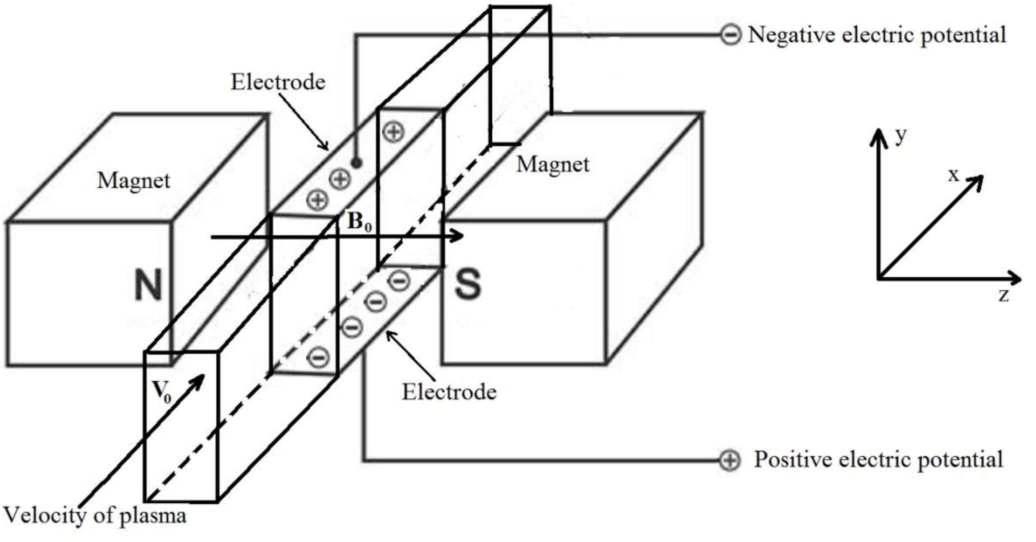


Fig. 1 – MHD generator

## 2. MATHEMATICAL FORMULATION OF THE PROBLEM

We shall use non-dimensional variables, by referring the electromagnetic field variables to $L_0, V_0$ and $B_0$. Denoting by $a$ the non-dimensional length of the electrodes and by $\vec{i}, \vec{j}, \vec{k}$ the unit vectors of the Cartesian axes, we deduce that the non-dimensional velocity and magnetic induction are:

$$\vec{V} = \vec{i}, \vec{B} = \begin{cases} \vec{k}, |x| \le a \\ \vec{0}, |x| > a \end{cases}. \tag{1}$$

Denoting by $\vec{J}$ the non-dimensional current density and by $\vec{E}$ the non-dimensional intensity of the electric field, we use Ohm's law

$$\vec{J} = Rm\left(\vec{E} + \vec{V} \times \vec{B}\right), \tag{2}$$

where

$$Rm = \sigma \mu L_0 V_0, \tag{3}$$

is the magnetic Reynolds' number, $\sigma$ is the electrical conductivity and $\mu$ is the magnetic permeability.

From Faraday's law

$$curl\,\vec{E} = 0, \tag{4}$$

we deduce that there exists a function $\varphi$ (the electric potential) such that

$$\vec{E} = -grad\ \varphi. \tag{5}$$

From the boundary conditions imposed on the insulating parts of the walls of the duct

$$\vec{J}\cdot\vec{n} = 0, \left[\vec{B}\right]\cdot\vec{n} = 0, \vec{V}\cdot\vec{n} = 0, \tag{6}$$

we deduce that the flow is plane-parallel and the functions we are dealing with, do not depend on the $z$ variable.

In the sequel we shall calculate the potential of the electric field. From the continuity equation

$$div\,\vec{J} = 0 \tag{7}$$

from Ohm's law and from (5) it follows that

$$\Delta\varphi(x,y) = 0, -\infty < x < \infty, -1 < y < 1. \tag{8}$$

The value of the electric potential is imposed on the electrodes:

$$\varphi(x,1) = -\varphi_w,\ \varphi(x,-1) = \varphi_w,\ x \in (-a,a). \tag{9}$$

The following relationships

$$\frac{\partial\varphi}{\partial y}(1,x) = 0, \frac{\partial\varphi}{\partial y}(-1,x) = 0,\ x \in (-\infty,-a)\cup(a,\infty). \tag{10}$$

can be deduced on the insulating parts of the walls of the duct, from Ohm's law, from (5) and from the boundary conditions (6).

The condition

$$\lim_{|x|\to\infty} grad\ \varphi = 0. \tag{11}$$

is imposed at infinity.

## 3. LAZĂR DRAGOŞ'S ANALYTICAL SOLUTION

Since $\varphi(x,y)$ is a harmonic function, there exists its harmonic conjugate $\chi(x,y)$, related by $\varphi(x,y)$ through the Cauchy-Riemann equations

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\chi}{\partial y}, \frac{\partial\varphi}{\partial y} = -\frac{\partial\chi}{\partial x}. \tag{12}$$

We shall also introduce the complex holomorphic function

$$f(z)=\varphi(x,y)+i\chi(x,y),\ z=x+iy. \tag{13}$$

From the boundary conditions (10), one deduces the boundary conditions

$$\chi(x,\pm 1)=b,\ x\in(-\infty,-a), \tag{14}$$

$$\chi(x,\pm 1)=0,\ x\in(a,\infty). \tag{15}$$

$\chi$ is determined up to an additive constant and $b$ has to be calculated.

With the conformal mapping

$$\zeta=i\exp\frac{\pi}{2}(z+a),\zeta=\xi+i\eta, \tag{16}$$

the strip $-1\le y\le 1$ is mapped onto the upper half-plane $\eta\ge 0$ with the following point-to-point correspondence (Fig. 2):

$$A(-\infty,\pm 1)\to A'(0,0),\ B(-a,-1)\to B'(1,0),\ C(a,-1)\to C'(\exp\pi a,0),$$
$$D(\infty,\pm 1)\to D'(\pm\infty,0),\ E(a,1)\to E'(-\exp\pi a,0),\ F(-a,1)\to F'(-1,0).$$

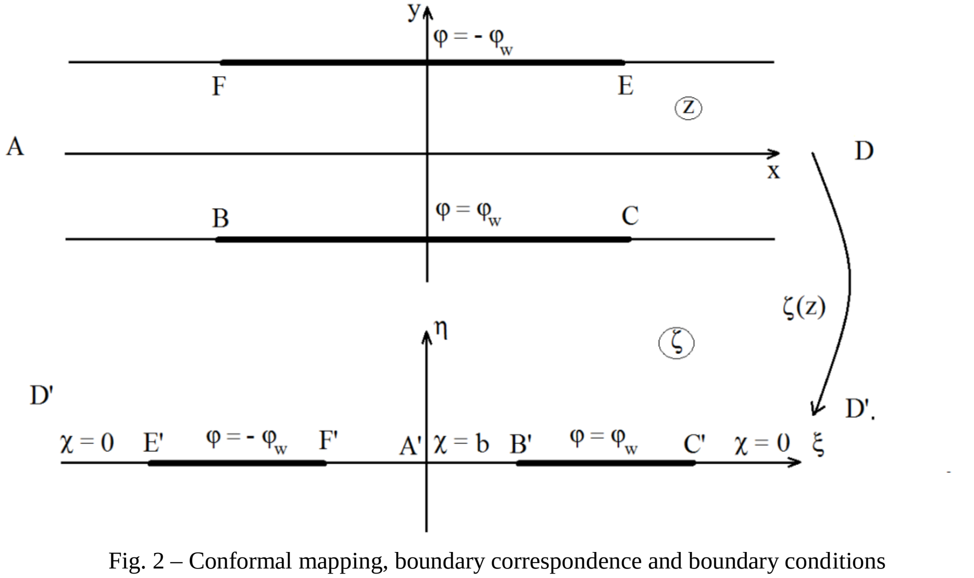


Fig. 2 – Conformal mapping, boundary correspondence and boundary conditions

The boundary value problem (8) – (11) was reduced to the following Volterra-Signorini problem: find a holomorphic function $f(\zeta)=\varphi(\xi,\eta)+i\chi(\xi,\eta)$ in the upper half-plane $\eta>0$, with the following boundary conditions:

$$\chi(\xi,0)=0,\ \xi\in(-\infty,-\exp\pi a)\cup(\exp\pi a,\infty),\ \chi(\xi,0)=b,\ \xi\in(-1,1), \tag{17}$$

$$\varphi(\xi,0)=-\varphi_w,\ \xi\in(-\exp\pi a,-1),\ \varphi(\xi,0)=\varphi_w,\ \xi\in(1,\exp\pi a). \tag{18}$$

The solution of the Volterra – Signorini problem, is given by a formula which may be found for example in [11], and it is ([1], Chapter 4):

$$f(\zeta)=\frac{\sqrt{\left(\zeta^2-\exp 2\pi a\right)\left(\zeta^2-1\right)}}{\pi}\int_{-\exp\pi a}^{\exp\pi a}\frac{\nu(\xi)}{\sqrt{\left|\left(\xi^2-\exp 2\pi a\right)\left(\xi^2-1\right)\right|}}\frac{d\xi}{\zeta-\xi}, \tag{19}$$

where

$$\nu(\xi)=\begin{cases}\varphi_w,\ \xi\in\left(-\exp\pi a,-1\right)\cup\left(1,\exp\pi a\right),\\ b,\ \xi\in\left(-1,1\right)\end{cases}. \tag{20}$$

From condition (11) we deduce that $\lim\limits_{\zeta\to\infty}\dfrac{df}{d\zeta}=0$, whence, taking into account (19) and (20) it follows that the constant $b$ must satisfy the equation

$$b\int_{-1}^{1}\frac{d\xi}{\sqrt{\left(\exp 2\pi a-\xi^2\right)\left(1-\xi^2\right)}}=-2\varphi_w\int_{1}^{\exp\pi a}\frac{d\xi}{\sqrt{\left(\exp 2\pi a-\xi^2\right)\left(\xi^2-1\right)}}. \tag{21}$$

## 4. NUMERICAL RESULTS

We use the Gauss quadrature formulas for continuous functions ([12], Appendix F):

$$\int_{-1}^{1}\frac{F(x)}{\sqrt{1-x^2}}dx\cong\frac{\pi}{n}\sum_{\alpha=1}^{n}F(x_\alpha),\ x_\alpha=\cos\frac{(2\alpha-1)\pi}{2n},\ \alpha=1,\dots,n. \tag{22}$$

Hence

$$b=-\varphi_w b'(a), b'(a)=2\frac{I_1(a)}{I_2(a)} \tag{23}$$

with

$$I_2(a)=\frac{\pi}{n}\sum_{\alpha=1}^{n}\frac{1}{\sqrt{\exp 2\pi a-x_\alpha^2}}. \tag{24}$$

We shall also use the Gauss quadrature formulas in order to calculate the integral from the right hand part of (21).

To this aim we consider the change of variable

$$\xi=\frac{\exp\pi a-1}{2}\theta+\frac{\exp\pi a+1}{2} \tag{25}$$

and obtain

$$\int_{-1}^{\exp\pi a}\frac{d\xi}{\sqrt{\left(\exp 2\pi a-\xi^2\right)\left(\xi^2-1\right)}}=$$

$$2\int_{-1}^{1}\frac{d\theta}{\sqrt{1-\theta^2}}\frac{1}{\sqrt{\left[\left(\exp\pi a-1\right)\theta+3\exp\pi a+1\right]\left[\left(\exp\pi a-1\right)\theta+\exp\pi a+3\right]}}\cong \tag{26}$$

$$\frac{2\pi}{n}\sum_{\alpha=1}^{n}\frac{1}{\sqrt{\left[\left(\exp\pi a-1\right)x_\alpha+3\exp\pi a+1\right]\left[\left(\exp\pi a-1\right)x_\alpha+\exp\pi a+3\right]}}=I_1(a).$$

In Fig. 3 we use a continuous line to give the graphical representation of *b' (a)*. We also use stars '*' to represent *2a* versus *a* and notice that

$$b'(a)>2a. \tag{27}$$

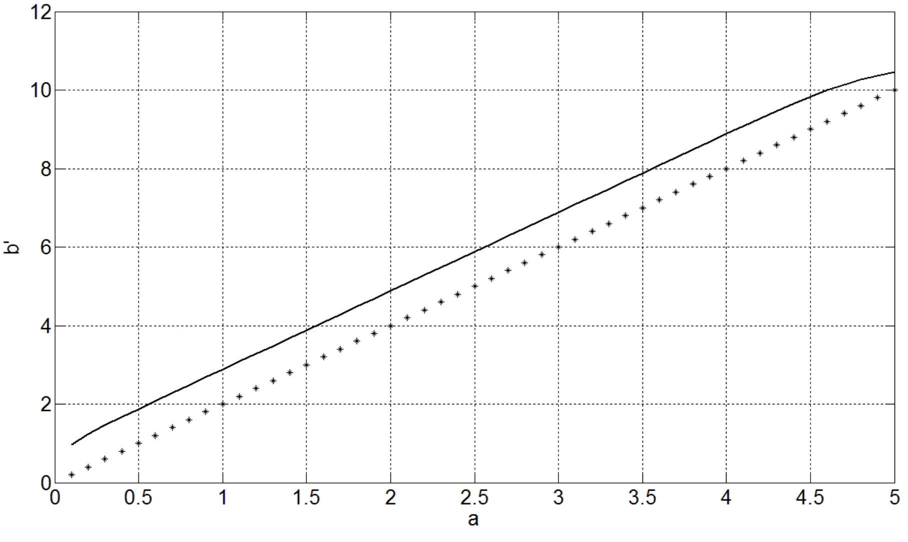


Fig. 3 – *b'(a)* and *2a* versus *a*

In the points $\left(x_p, y_s\right)$ of a certain grid we calculate

$$\frac{\partial\varphi\left(x_p,y_s\right)}{\partial x}-i\frac{\partial\varphi\left(x_p,y_s\right)}{\partial y}=\frac{\pi i}{2}\frac{df\left(\zeta_{ps}\right)}{d\zeta}\cdot\exp\frac{\pi}{2}\left(z_{ps}+a\right),\ z_{ps}=x_p+iy_s,\ \zeta_{ps}=\zeta\left(z_{ps}\right) \tag{28}$$

We take into account that

$$\begin{aligned}\frac{df\left(\zeta_{ps}\right)}{d\zeta}&=\frac{\zeta_{ps}\left(2\zeta_{ps}{}^2-1-\exp 2\pi a\right)}{\sqrt{\left(\zeta_{ps}{}^2-\exp 2\pi a\right)\left(\zeta_{ps}{}^2-1\right)}}\int_{-\exp\pi a}^{\exp\pi a}\frac{\nu(\xi)}{\sqrt{\left|\left(\xi^2-\exp 2\pi a\right)\left(\xi^2-1\right)\right|}}\frac{d\xi}{\zeta_{ps}-\xi}\\&-\frac{\sqrt{\left(\zeta_{ps}{}^2-\exp 2\pi a\right)\left(\zeta_{ps}{}^2-1\right)}}{\pi}\int_{-\exp\pi a}^{\exp\pi a}\frac{\nu(\xi)}{\sqrt{\left|\left(\xi^2-\exp 2\pi a\right)\left(\xi^2-1\right)\right|}}\frac{d\xi}{\left(\zeta_{ps}-\xi\right)^2}.\end{aligned} \tag{29}$$

For calculating $\dfrac{df\left(\zeta_{ps}\right)}{d\zeta}$ we have to numerically compute the integrals

$$\int_{-1}^{1}\frac{b}{\sqrt{\left(1-\xi^2\right)\left(\exp 2\pi a-\xi^2\right)}}\frac{d\xi}{\zeta_{ps}-\xi},\int_{-1}^{1}\frac{b}{\sqrt{\left(1-\xi^2\right)\left(\exp 2\pi a-\xi^2\right)}}\frac{d\xi}{\left(\zeta_{ps}-\xi\right)^2},\tag{30}$$

$$\int_{1}^{\exp\pi a}\frac{\varphi_w}{\sqrt{\left(\xi^2-1\right)\left(\exp 2\pi a-\xi^2\right)}}\frac{d\xi}{\zeta_{ps}-\xi},\int_{1}^{\exp\pi a}\frac{\varphi_w}{\sqrt{\left(\xi^2-1\right)\left(\exp 2\pi a-\xi^2\right)}}\frac{d\xi}{\left(\zeta_{ps}-\xi\right)^2},\tag{31}$$

$$\int_{1}^{\exp\pi a}\frac{\varphi_w}{\sqrt{\left(\xi^2-1\right)\left(\exp 2\pi a-\xi^2\right)}}\frac{d\xi}{\zeta_{ps}+\xi},\int_{1}^{\exp\pi a}\frac{\varphi_w}{\sqrt{\left(\xi^2-1\right)\left(\exp 2\pi a-\xi^2\right)}}\frac{d\xi}{\left(\zeta_{ps}+\xi\right)^2}.\tag{32}$$

For calculating the integrals from (30) we use the Gauss quadrature formulas (22) and for calculating the integrals from (31) and (32) we perform the change of variable (25) and then use the Gauss quadrature formulas (22).

We use (2), (5), (28) and (29) to calculate the current density in the grid points. In Fig. 4 we present the current density field $J/Rm$ for $a=1/2$ and $\varphi_w=1$.

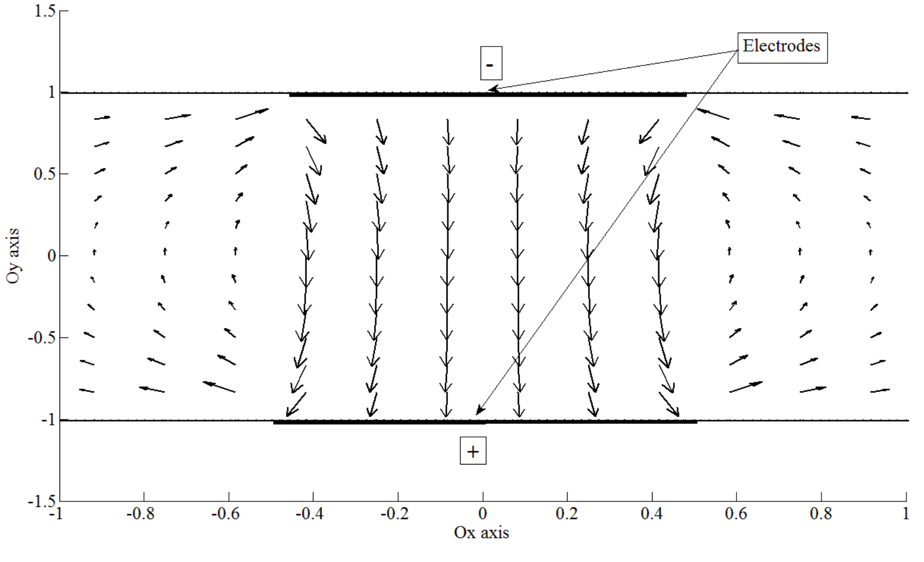


Fig. 4 – Current density field

## 5. THE CHARACTERISTICS OF THE MHD GENERATOR

The non-dimensional power on the unit of length developed by plasma in the motion against the electromagnetic field is

$$A=-\iint\limits_{(-\infty,\infty)\times[-1,1]}\vec{V}\cdot\left(\vec{J}\times\vec{B}\right)dxdy=\iint\limits_{(-\infty,\infty)\times[-1,1]}\vec{J}\cdot\left(\vec{V}\times\vec{B}\right)dxdy\tag{33}$$

and it represents in fact the power dissipated by the Lorentz force with changed sign.

Taking into account the relations (1), (5), (9), (10) and Ohm's law (2) we get

$$A = Rm \iint_{[-a,a]\times[-1,1]} \left(\frac{\partial\varphi}{\partial y}+1\right)dxdy = Rm\int_{-a}^{a}\left[\varphi(x,1)-\varphi(x,-1)-2\right]dx = 4a\ Rm(1-\varphi_w) \quad (34)$$

and also

$$A = \iint_{(-\infty,\infty)\times[-1,1]} \vec{J}\cdot\left(\frac{\vec{J}}{Rm}-\vec{E}\right)dxdy = Q+W \quad (35)$$

where

$$Q = \iint_{(-\infty,\infty)\times[-1,1]} \frac{\vec{J}^2}{Rm}dxdy > 0 \quad (36)$$

stands for the Joule dissipation power and

$$W = -\iint_{(-\infty,\infty)\times[-1,1]} \vec{E}\cdot\vec{J}dxdy \quad (37)$$

is the useful output power of the generator.

From (5), from the continuity equation (7) and from the boundary conditions imposed on the insulating walls, one deduces that

$$W = \iint_{(-\infty,\infty)\times[-1,1]} div\left(\varphi\vec{J}\right)dxdy = \oint\varphi\vec{J}\cdot\vec{n}\ ds = -\varphi_w\int_{-a}^{a}\left[\vec{J}(x,-1)+\vec{J}(x,-1)\right]\cdot\vec{j}dx =$$

$$\varphi_w Rm\int_{-a}^{a}\left(\frac{\partial\varphi}{\partial y}(x,-1)+\frac{\partial\varphi}{\partial y}(x,1)+2\right)dx = \varphi_w Rm\int_{-a}^{a}\left(-\frac{\partial\chi}{\partial x}(x,-1)-\frac{\partial\chi}{\partial x}(x,1)+2\right)dx = \quad (38)$$

$$\varphi_w Rm\left[\chi(-a,-1)-\chi(a,-1)+\chi(-a,1)-\chi(a,1)+2a\right] = 2\varphi_w Rm(2a+b).$$

We notice that, from (34) and (38) it follows

$$Q = A-W = 2Rm(-b\varphi_w-4a\varphi_w+2a) = 2Rm\left(b'\varphi_w^2-4a\varphi_w+2a\right). \quad (39)$$

Since $b'>0$ and the discriminant $\Delta = 16a^2-8ab' = 8a(2a-b')<0,$ we verify again that $Q>0.$

In Fig. 5 we represent the useful output power, the total power, the Joule dissipation power (divided by $Rm$) and the useful power/ Joule dissipation power ratio against the electrode length $a$ and electric potential $\varphi_w$.

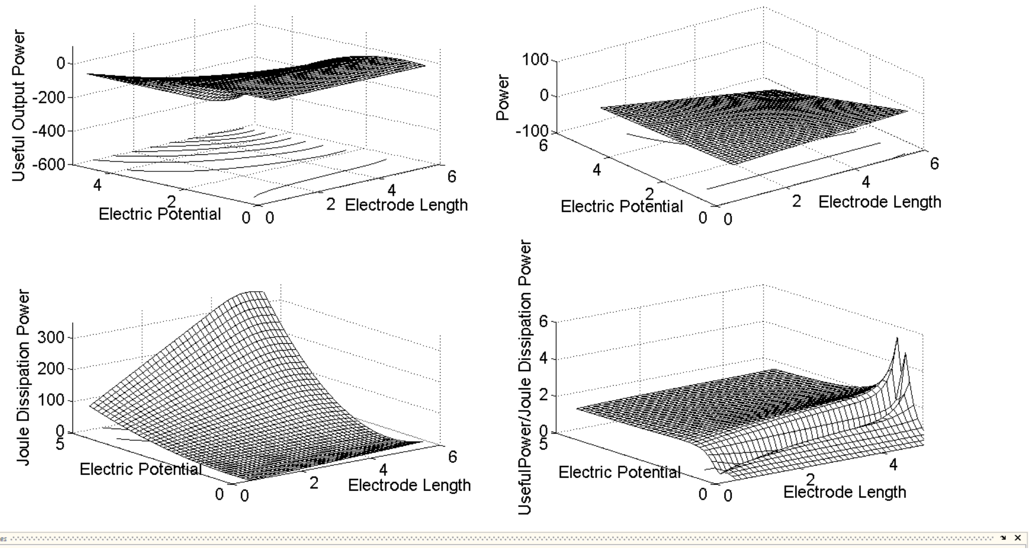


Fig. 5 – MHD generator power

These graphical representations may be helpful in designing a MHD generator and optimizing its characteristics.

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