Attitude Control Synthesis for Small Satellites Using Gradient Method
Part II Linear Equations, Synthesis

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Abstract: *In order to continue paper [5] which presented the nonlinear equations of the movement for small satellite, this paper presents some aspects regarding the synthesis of the attitude control. After the movement equation linearization, the stability and command matrixes will be established and by using the gradient methods controller we will obtain them. Two attitude control cases will be analysed: the reaction wheels and the micro thrusters. The results will be used in the project European Space Moon Orbit - ESMO, founded by the European Space Agency in which the POLITEHNICA University of Bucharest is involved.*

Key Words: *linear equations, small satellites, gradient methods, mathematical model*

NOMENCLATURE

ξ - Rotation angle around body X_B axis

η - Rotation angle around body Y_B axis

ζ - Rotation angle around body Z_B axis

ψ - Attitude angle around z axis

θ - Attitude angle around y axis

ϕ - Attitude angle around x axis

$\boldsymbol{\omega}_{BI}$ -Angular velocity of the body frame relative to the inertial frame expressed in body frame;

$\boldsymbol{\omega}_{RI}$ -Angular velocity of the reference frame relative to the inertial frame expressed in relative frame;

$\boldsymbol{\omega}_{RIB}$ -Angular velocity of the reference frame relative to the inertial frame expressed in body frame;

$\boldsymbol{\omega}_{BR}$ -Angular velocity of the body frame relative to the reference frame expressed in body frame;

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A, B, C, E - Satellite inertia moments;
m - Satellite Mass;
a - Major semi axis of elliptical satellite orbit;
e - Eccentricity of elliptical satellite orbit;
t - Time;
$\mathbf{r}$ - Position vector of the satellite relative to the origin of the inertial frame – centre of the Earth;
T -Orbital period;
$\mathbf{v}$ - Velocity.

1. BALANCE MOUVEMENT

The study of flight stability will be made accordingly to Lyapunov theory, considering that the system of movement equations, established in paper [5], is perturbed around the balanced movement. This involves a disturbance shortly applied on the balance movement, which will produce deviation of the state variables. If we do a series development of the perturbed movement equations in relation to status variables and take into account the first order terms of the detention, we will get a system of linear equations which can be used to analyse the stability in the first approximation, as we proceed in most dynamic non linear problems. To determine the basic movement parameters in these equations we consider that the vehicle is stabilized in the position where the body frame overlaps the reference frame [4], [5]. This means that the rotation angles [5], [9] are nulls:

$$\mathbf{a}_R = [0 \quad 0 \quad 0]^T \tag{1}$$

and also that the angular velocity of the body frame [5] related to the reference frame is null.

$$\boldsymbol{\omega}_{BR} = [0 \quad 0 \quad 0]^T \tag{2}$$

In this case the link between angular velocity, for the balance movement equation becomes:

$$\boldsymbol{\omega}_{BI} = \boldsymbol{\omega}_{RIB} \tag{3}$$

Moreover, because the attitude or the rotation angles are nulls and the rotation matrix is a unitary matrix, the previously relation becomes:

$$\boldsymbol{\omega}_{BI} = \boldsymbol{\omega}_{RI} \tag{4}$$

If we write it in scalar form we obtain:

$$\omega_x = 0; \omega_y = \omega_j;\ \omega_z = 0 \tag{5}$$

In order to have a stationary movement, we must admit that the orbit is circular. This hypothesis leads to a constant orbit range, $r = a$ which allows us to have a constant value for the orbital angular velocity:

$$\omega_j = -\frac{v}{r} \tag{6}$$

If we suppose that the angular velocity ω_j was obtained using only the reaction wheels by spinning them around the Y_B axis, then the momentum exchange device for balance can be put in the form:

$$\mathbf{h}_w = [0 \quad h_{wy} \quad 0]^T, \tag{7}$$

where:

$$h_{wy} = -\omega_j B\,, \tag{8}$$

If the angular velocity ω_j was obtained using the rotation micro thrusters around the Y_B axis, then the momentum exchange device for balance can be considerate null:

$$\mathbf{h}_w = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \tag{9}$$

This situation will be further analysed.

2. LINEAR FORM OF THE GENERAL EQUATIONS

If we consider the base general equation in accordance with Kepler's model we can obtain the linear form of these equations.

From dynamic Euler equation we can obtain the following two linear forms:

- for the thrusters control case

$$\Delta\dot{\boldsymbol{\omega}}_{BI} = \mathbf{M}_\omega \Delta\boldsymbol{\omega}_{BI} + \mathbf{M}_R \Delta\mathbf{a}_R + \mathbf{J}^{-1}\Delta\mathbf{M}_C \tag{10}$$

- for the reaction wheels control case:

$$\Delta\dot{\boldsymbol{\omega}}_{BI} = \mathbf{M}_\omega \Delta\boldsymbol{\omega}_{BI} + \mathbf{M}_R \Delta\mathbf{a}_R + \mathbf{M}_h \Delta\mathbf{h}_w + \mathbf{J}^{-1}\Delta\mathbf{y}_w \tag{11}$$

The gyroscopic terms are:

$$\mathbf{M}_\omega = \omega_j \mathbf{J}^{-1} \begin{bmatrix} E & 2D & B-C \\ -D & 0 & F \\ A-B & -2F & -E \end{bmatrix} \tag{12}$$

The reaction wheel term is:

$$\mathbf{M}_h = \omega_j \mathbf{J}^{-1} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{13}$$

Starting from the gravity gradient moment components, we can obtain the following relations for the small rotation angles:

$$\Delta L_g = 3\omega_j^2 (B-C)\Delta\xi\,;\ \Delta M_g = 3\omega_j^2 (A-C)\Delta\eta\,;\ \Delta N_g = 0 \tag{14}$$

The matrix form in both cases is:

$$\mathbf{M}_R = 3\omega_j^2 \mathbf{J}^{-1} \begin{bmatrix} B-C & 0 & 0 \\ 0 & A-C & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

In this case, for equation (10) the command moment is given by rotation thrusters:

$$\Delta\mathbf{M}_C = \begin{bmatrix} \Delta L_C & \Delta M_C & \Delta N_C \end{bmatrix}^T, \tag{16}$$

and for equation (11) the command is given by reaction wheels:

$$\Delta\mathbf{y}_w = \begin{bmatrix} \Delta y_{wx} & \Delta y_{wy} & \Delta y_{wz} \end{bmatrix}^T, \tag{17}$$

where:

$$\begin{bmatrix}\Delta y_{wx} & \Delta y_{wy} & \Delta y_{wz}\end{bmatrix}^T = \begin{bmatrix}\Delta \dot{h}_{wx} & \Delta \dot{h}_{wy} & \Delta \dot{h}_{wz}\end{bmatrix}^T \tag{18}$$

From kinematic equation, if we use rotation angles, the kinematic equation from paper [5] in its linear form becomes:

$$\Delta\dot{\mathbf{a}}_R = \mathbf{W}_R\Delta\boldsymbol{\omega}_{BI} + \mathbf{A}_R\Delta\mathbf{a}_R \tag{19}$$

where:

$$\mathbf{A}_R = \mathbf{W}_{\omega R} - \mathbf{W}_R\mathbf{A}_{e\omega R} \tag{20}$$

whereabouts:

$$\mathbf{W}_{\omega R} = \begin{bmatrix}\dfrac{\partial \mathbf{W}_A}{\partial \xi} & \dfrac{\partial \mathbf{W}_A}{\partial \eta} & \dfrac{\partial \mathbf{W}_A}{\partial \zeta}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_{BR} & 0 & 0\\ 0 & \boldsymbol{\omega}_{BR} & 0\\ 0 & 0 & \boldsymbol{\omega}_{BR}\end{bmatrix} \quad \mathbf{A}_{e\omega R} = \begin{bmatrix}\dfrac{\partial \mathbf{A}_e}{\partial \xi} & \dfrac{\partial \mathbf{A}_e}{\partial \eta} & \dfrac{\partial \mathbf{A}_e}{\partial \zeta}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_{RI} & 0 & 0\\ 0 & \boldsymbol{\omega}_{RI} & 0\\ 0 & 0 & \boldsymbol{\omega}_{RI}\end{bmatrix} \tag{21}$$

Finally, using relations (10), (19), with rotation angles, the stability and the command matrices for thrusters control case are:

Table 1.The stability matrix for the thrusters control case

	$\boldsymbol{\omega}_{BI}$	$\mathbf{a}_R$
$\boldsymbol{\omega}_{BI}$	$\mathbf{M}_\omega$	$\mathbf{M}_R$
$\mathbf{a}_R$	$\mathbf{W}_R$	$\mathbf{A}_R$

Table 2. The command matrix for the thrusters control case

	$\mathbf{M}_C$
$\boldsymbol{\omega}_{BI}$	$\mathbf{J}^{-1}$
$\mathbf{a}_R$	

or, if used the reaction wheels to control the attitude, from relations (11) and (19) we'll obtain:

Table 3. Stability matrix for the reaction wheels case

	$\boldsymbol{\omega}_{BI}$	$\mathbf{a}_R$	$\mathbf{h}_w$
$\boldsymbol{\omega}_{BI}$	$\mathbf{M}_\omega$	$\mathbf{M}_R$	$\mathbf{M}_h$
$\mathbf{a}_R$	$\mathbf{W}_R$	$\mathbf{A}_R$	
$\mathbf{h}_w$			

In the case of reaction wheels command, the last column remains, because this parameter is being controlled by command.

Table 4.The command matrix for the reaction wheels case

	$\mathbf{y}_w$
$\boldsymbol{\omega}_{BI}$	$\mathbf{J}^{-1}$
$\mathbf{a}_R$	
$\mathbf{h}_w$	$-\mathbf{I}_3$

In any of these cases the system can be put in standard form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{22}$$

where:

$$\mathbf{x} = \begin{bmatrix}\boldsymbol{\omega}_{BI} & \mathbf{a}_A\end{bmatrix}^T;\ \mathbf{u} = \begin{bmatrix}L_C & M_C & N_C\end{bmatrix}^T \tag{23}$$

or:

$$\mathbf{x} = \begin{bmatrix}\boldsymbol{\omega}_{BI} & \mathbf{a}_A & \mathbf{h}_w\end{bmatrix}^T \mathbf{u} = \begin{bmatrix}y_{wx} & y_{wy} & y_{wz}\end{bmatrix}^T \tag{24}$$

Observation. For the balance movement described above, where the body frame coincides with the reference frame, the stability and command matrix are identical for attitude angles and rotation angles.

Next, we will try to find an analytical solution of these equations. For this purpose we consider that the vehicle has a symmetric plane OX_BZ_B with two inertial products being nulls: $D=F=0$. As we will show in this paper [5], for our model the inertial products $D;F$ are small comparatively with the inertial product E. From the kinematic equation we obtain:

$$\Delta\dot{\xi} = \Delta\omega_x - \omega_j\Delta\zeta\,;$$
$$\Delta\dot{\eta} = \Delta\omega_y \tag{25}$$
$$\Delta\dot{\zeta} = \Delta\omega_z + \omega_j\Delta\xi$$

From the dynamic equations we can write:

$$\Delta\dot{\omega}_x = -a_x^x\omega_j\Delta\omega_x + a_x^z\omega_j\Delta\omega_z + 3a_x^{\phi}\omega_j^2\Delta\phi + b_x^l\Delta L_C + b_x^n\Delta N_C + \omega_j b_x^n\Delta h_{wx} - $$
$$- \omega_j b_x^l\Delta h_{wz} + b_x^l\Delta y_{wx} + b_x^n\Delta y_{wz}\,;$$
$$\Delta\dot{\omega}_y = 3a_y^{\theta}\omega_j^2\Delta\theta + b_y^m\Delta M_C + b_y^m\Delta y_{wy} \tag{26}$$
$$\Delta\dot{\omega}_z = -a_z^x\omega_j\Delta\omega_x - a_z^z\omega_j\Delta\omega_z + 3a_z^{\phi}\omega_j^2\Delta\phi + b_z^l\Delta L_C + b_z^n\Delta N_C + \omega_j b_z^n\Delta h_{wx} -$$
$$- \omega_j b_z^l\Delta h_{wz} + b_z^l\Delta y_{wx} + b_z^n\Delta y_{wz}$$

where we denoted:

$$a_x^x = \frac{E(B-A-C)}{AC-E^2};\; a_x^z = \frac{CB-C^2-E^2}{AC-E^2};\; a_x^{\phi} = \frac{CB-C^2}{AC-E^2};\; a_y^{\theta} = \frac{A-C}{B};$$
$$a_z^x = \frac{AB-A^2-E^2}{AC-E^2};$$
$$a_z^z = \frac{E(A+C-B)}{AC-E^2};\, a_z^{\phi} = \frac{E(B-C)}{AC-E^2};\; b_x^l = \frac{C}{AC-E^2};\; b_x^n = \frac{E}{AC-E^2}; \tag{27}$$
$$b_z^n = \frac{A}{AC-E^2};$$
$$b_z^l = \frac{E}{AC-E^2};\; b_y^m = \frac{1}{B}$$

Deriving equations (25) and substituting them in equations (26) we obtain:

$$\Delta\ddot{\xi} + a_x^x\omega_j\Delta\dot{\xi} + (a_x^z - 3a_x^{\xi})\omega_j^2\Delta\xi = (a_x^z - 1)\omega_j\Delta\dot{\zeta}$$
$$- a_x^x\omega_j^2\Delta\zeta + b_x^l\Delta L_C + b_x^n\Delta N_C + \omega_j b_x^n\Delta h_{wx} - \omega_j b_x^l\Delta h_{wz} + b_x^l\Delta y_{wx} + b_x^n\Delta y_{wz}$$
$$\Delta\ddot{\eta} - 3a_y^{\eta}\omega_j^2\Delta\eta = b_y^m\Delta M_C + b_y^m\Delta y_{wy} \tag{28}$$
$$\Delta\ddot{\zeta} + a_z^z\omega_j\Delta\dot{\zeta} + a_z^x\omega_j^2\Delta\zeta = (1-a_z^x)\omega_j\Delta\dot{\xi} +$$
$$(a_z^z + 3a_z^{\xi})\omega_j^2\Delta\xi + b_z^l\Delta L_C + b_z^n\Delta N_C + \omega_j b_z^n\Delta h_{wx} - \omega_j b_z^l\Delta h_{wz} + b_z^l\Delta y_{wx} + b_z^n\Delta y_{wz}$$

First, we can notice that the second equation can be analysed separately. For the first and the third equations, if we consider the value of ω_j constant, we can apply the Laplace transformation, and then put these relations in the matrix form:

$$\mathbf{A}(s)\mathbf{x} = \mathbf{bu} \tag{29}$$

where:

$$\mathbf{A}(s)=\begin{bmatrix} s^2+a_x^x\omega_j s+(a_x^z-3a_x^\xi)\omega_j^2 & -(a_x^z-1)\omega_j s+a_x^x\omega_j^2 \\ (a_z^x-1)\omega_j s-(a_z^z+3a_z^\xi)\omega_j^2 & s^2+a_z^z\omega_j s+a_z^x\omega_j^2 \end{bmatrix} \tag{30}$$

$$\mathbf{b}=\begin{bmatrix} b_x^l & b_x^n \\ b_z^l & b_z^n \end{bmatrix}\ \mathbf{x}=[\xi \quad \zeta]^T\ \mathbf{;}\ \mathbf{u}=[L_C \quad N_C]^T \tag{31}$$

or, if we use the reaction wheels:

$$\mathbf{b}=\begin{bmatrix} b_x^l-s\omega_j b_x^n & b_x^n+s\omega_j b_x^l \\ b_z^l-s\omega_j b_z^n & b_z^n+s\omega_j b_z^l \end{bmatrix}\ \mathbf{u}=[y_{wx} \quad y_{wz}]^T \tag{32}$$

Easily we can obtain the inverse of the $\mathbf{A}(s)$ matrix:

$$\mathbf{A}^{-1}=\frac{1}{P}\begin{bmatrix} s^2+a_z^z\omega_j s+a_z^x\omega_j^2 & (a_x^z-1)\omega_j s-a_x^x\omega_j^2 \\ -(a_z^x-1)\omega_j s+(a_z^z+3a_z^\xi)\omega_j^2 & s^2+a_x^x\omega_j s+(a_x^z-3a_x^\xi)\omega_j^2 \end{bmatrix} \tag{33}$$

where the characteristic polynomial is:

$$\begin{aligned} P(s)=s^4+\left(a_x^x+a_z^z\right)\omega_j s^3+(a_x^x a_z^z+a_z^x a_x^z+1-3a_x^\xi)\omega_j^2 s^2+ \\ +(a_x^x+a_z^z-3a_x^\xi a_z^z-3a_z^\xi a_x^z+3a_z^\xi)\omega_j^3 s+ \\ +(a_x^x a_z^z+a_z^x a_x^z-3a_x^\xi a_z^x+3a_z^\xi a_x^x)\omega_j^4 \end{aligned} \tag{34}$$

Using these results we can put the previous relations in the form

$$\mathbf{x}=\mathbf{A}^{-1}(s)\mathbf{bu} \tag{35}$$

This form represents the analytical solution of the commanded linear equations.
Observation. *For balance movement described above, the stability matrix* $\mathbf{A}$, *defined by relation (30) is identical for the attitude angles and rotation angles.*

3. EXTENDED STABILITY AND CONTROL MATRICES

Besides the general motion equations in their linear form described above, the S/C needs other equations to be added. Among them, the actuator equations and the auxiliary guidance equations cannot be neglected. For the autonomous flight, as is the case of S/C's, the auxiliary guidance equations are necessary in order to introduce integrated terms specific to PID-type controllers.

In order to give commands using the micro thrusters, we introduce a controller, called the Trigger Schmidt element. For the linearization of this element, we applied the method given to us by paper [7], using the Fourier transformation. Thus, taking into account only the first harmonic approximation, we obtain a linear transfer function of the form:

$$N(s)\cong k_M^u\frac{s+\Omega_M}{s} \tag{36}$$

where we denoted:

$$k_M^u=\frac{a_1}{x_0};\ \Omega_M=\frac{-b_1\omega}{a_1} \tag{37}$$

where:

$$a_1=\frac{2M}{\pi}\left(\sqrt{1-\frac{(a_M+b_M)^2}{x_0^2}}+\sqrt{1-\frac{{a_M}^2}{x_0^2}}\right);\ b_1=-\frac{2M}{\pi}\frac{b_M}{x_0} \tag{38}$$

where a_M, b_M, M define the non linear function from figure 7, paper [5]. Sizes x_0 and ω represent the amplitude, and the pulsation of the input signal.

In this case, considering the integrator element and feedback loop, the linear transfer function of the command system for a channel can be written as:

$$H_0(s) = \frac{k_M^u (s + \Omega_M)}{\tau_M s^2 + \left(k_M^u + 1\right)s + k_M^u \Omega_M} \tag{39}$$

or, if we neglect the pulse term Ω_M, we obtain the following simplified linear equation:

$$H_0(s) = \frac{k_M^u}{\tau_M s + k_M^u + 1} \tag{40}$$

Starting from the previous relation, the linear form of the command equation becomes:

$$\left[\Delta \dot{L}_C \quad \Delta \dot{M}_C \quad \Delta \dot{N}_C\right]^T = \mathbf{D}_M \left[\Delta L_C \quad \Delta M_C \quad \Delta N_C\right]^T + \mathbf{D}_u \Delta \mathbf{u} \tag{41}$$

where:

$$\mathbf{D}_M = \begin{bmatrix} -\left(1+k_M^u\right)/\tau_M & 0 & 0 \\ 0 & -\left(1+k_M^u\right)/\tau_M & 0 \\ 0 & 0 & -\left(1+k_M^u\right)/\tau_{Mr} \end{bmatrix}; \mathbf{D}_u = \begin{bmatrix} k_M^u/\tau_M & 0 & 0 \\ 0 & k_M^u/\tau_M & 0 \\ 0 & 0 & k_M^u/\tau_M \end{bmatrix}, \tag{42}$$

Similarly, if we use the reaction wheels, the command equation proposed in paper [8] becomes:

$$\Delta \dot{\mathbf{y}}_w = \mathbf{D}_{yw} \Delta \mathbf{y}_w + \mathbf{D}_{hw} \Delta \mathbf{h}_w + \mathbf{D}_{uw} \Delta \mathbf{u}_w ; \quad \Delta \dot{\mathbf{h}}_w = -\mathbf{I}_3 \Delta \mathbf{y}_w , \tag{43}$$

where:

$$\mathbf{D}_{yw} = \begin{bmatrix} -a_w & 0 & 0 \\ 0 & -a_w & 0 \\ 0 & 0 & -a_w \end{bmatrix}; \ \mathbf{D}_{hw} = \begin{bmatrix} b_w & 0 & 0 \\ 0 & b_w & 0 \\ 0 & 0 & b_w \end{bmatrix} \ \mathbf{D}_{uw} = \begin{bmatrix} k_w^u & 0 & 0 \\ 0 & k_w^u & 0 \\ 0 & 0 & k_w^u \end{bmatrix} \tag{44}$$

For integral terms, linear form of auxiliary equation described in paper [5] becomes:

$$\left[\Delta \dot{I}_\xi \quad \Delta \dot{I}_\eta \quad \Delta \dot{I}_\zeta\right]^T = \left[\Delta \xi \quad \Delta \eta \quad \Delta \zeta\right]^T \tag{45}$$

For the thrusters command case, by using linear relation (41) and (45) we can build extended stability and control matrixes.

Table 5. Extended stability matrix $\mathbf{A}$ for the thrusters control case

	$\boldsymbol{\omega}_{BI}$	$\mathbf{a}_R$	$\mathbf{I}$	$\mathbf{M}_C$
$\boldsymbol{\omega}_{BI}$	$\mathbf{M}_\omega$	$\mathbf{M}_R$		$\mathbf{J}^{-1}$
$\mathbf{a}_R$	$\mathbf{W}_R$	$\mathbf{A}_R$		
$\mathbf{I}$		$\mathbf{I}_3$		
$\mathbf{M}_C$				$\mathbf{D}_M$

Table 6. Transposed extended command matrix $\mathbf{B}^T$ for the thrusters control case

	$\boldsymbol{\omega}_{BI}$	$\mathbf{a}_R$	$\mathbf{I}$	$\mathbf{M}_C$
$\mathbf{u}$				$\mathbf{D}_u$

Similarly, for the reaction wheels case, from relation (43), the extended matrix and command matrix become:

Table 7. Extended stability matrix $\mathbf{A}$ for the reaction wheels control case

	$\boldsymbol{\omega}_{BI}$	$\mathbf{a}_R$	$\mathbf{h}_w$	$\mathbf{y}_w$
$\boldsymbol{\omega}_{BI}$	$\mathbf{M}_\omega$	$\mathbf{M}_R$		$\mathbf{J}^{-1}$
$\mathbf{a}_R$	$\mathbf{W}_R$	$\mathbf{A}_R$		
$\mathbf{h}_w$				$-\mathbf{I}_3$
$\mathbf{y}_w$			$\mathbf{D}_{hw}$	$\mathbf{D}_{yw}$

Table 8. Transposed extended command matrix $\mathbf{B}^T$ for the reaction wheels control case

	$\boldsymbol{\omega}_{BI}$	$\mathbf{a}_R$	$\mathbf{h}_w$	$\mathbf{y}_w$
$\mathbf{u}$				$\mathbf{D}_{uw}$

In this case the system can be put in the standard form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\,;\ \mathbf{u} = -\mathbf{K}\mathbf{x} \tag{46}$$

where, for the thrusters control case: $\mathbf{x} = \begin{bmatrix}\boldsymbol{\omega}_{\mathbf{BI}} & \mathbf{a}_{\mathbf{R}} & \mathbf{I} & \mathbf{M}_{\mathbf{C}}\end{bmatrix}^T$; $\mathbf{u} = \begin{bmatrix}u_l & u_m & u_n\end{bmatrix}^T$, and for the reaction wheels control case: $\mathbf{x} = \begin{bmatrix}\boldsymbol{\omega}_{\mathbf{BI}} & \mathbf{a}_{\mathbf{R}} & \mathbf{h}_w & \mathbf{y}_w\end{bmatrix}^T$; $\mathbf{u} = \begin{bmatrix}u_l & u_m & u_n\end{bmatrix}^T$. For both cases the control matrix $\mathbf{K}$ will be defined further.

4. GUIDANCE COMMAND SYNTHESIS

4.1 Optimal control using coupled states

If we assume that we can access the extend state vector $\mathbf{x}$ of the system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{47}$$

then we could obtain directly the controller $\mathbf{K}$ for optimal command:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \tag{48}$$

In order to satisfy the linear quadratic performance index (cost function):

$$\min J = \int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,\mathrm{d}t \tag{49}$$

where the extended pair $(\mathbf{A},\mathbf{B})$ is controllable and the state weighting matrix $\mathbf{Q}$ is symmetric and quasi positive:

$$\mathbf{Q} \geq 0;\ \mathbf{Q} = \mathbf{Q}^T \tag{50}$$

the control weighting matrix $\mathbf{R}$ must be symmetric and positive:

$$\mathbf{R} > 0;\ \mathbf{R} = \mathbf{R}^T; \tag{51}$$

In this case, the following relation gives the optimal controller

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \tag{52}$$

where the matrix $\mathbf{P}$ is the solution of the algebraic Riccati equation:

$$\mathbf{A}^{\mathbf{T}}\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{53}$$

4.2 Optimal control using Kalman filter

The usage of the optimal controller designed above requires access to all system's states, which is very difficult from the perspective of the limited number of sensors. In this case, for a complete description of the system we use a linear state estimator constructed as a Kalman filter. For this purpose we start from the regular relations:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} + \mathbf{Gw} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du} + \mathbf{v}\end{aligned} \tag{54}$$

where **w** is the external noise and **v** is the internal noise introduced by the sensors.

The idea of an estimator operation works like this: if the deliver the system $\Sigma_1 : (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ with the state **x**, then, by using the system $\Sigma_2 : (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ that requires state **z**, which is accessible in this case to the controller, we can predict the state x. In order that the system Σ_2 follows the system Σ_1 we calculate a regulator $\mathbf{L}$ which brings the difference between the actual read states $\mathbf{y_1}$ and the estimated states $\mathbf{y_2}$ as a correction into the system Σ_2. In this case we can write:

$$\Sigma_1 : \begin{cases} \dot{x} = \boldsymbol{Ax} + \boldsymbol{Bu} + x_0\delta + \boldsymbol{Gw} \\ y_1 = \boldsymbol{Cx} + \boldsymbol{Du} + \boldsymbol{v} \end{cases} \tag{55}$$

$$\Sigma_2 : \begin{cases} \dot{\mathbf{z}} = \mathbf{Az} + \mathbf{Bu} + \mathbf{z}_0\delta + \mathbf{L}(\mathbf{y}_1 - \mathbf{y}_2) \\ \mathbf{y}_2 = \mathbf{Cz} + \mathbf{Du} \end{cases} \tag{56}$$

where initial conditions are introduced by $\mathbf{x}_0$ and $\mathbf{z}_0$, respectively. The possibility of tracking error, including into the initial conditions is given by:

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{z}; \quad \tilde{\mathbf{x}}_0 = \mathbf{x}_0 - \mathbf{z}_0 \tag{57}$$

If we decrease Σ_2 from Σ_1 and neglect the noise we obtained:

$$\tilde{\mathbf{x}} = he^{(\mathbf{A}-\mathbf{LC})t}\tilde{\mathbf{x}}_\mathbf{0} \tag{58}$$

Hence if **L** is dimensioned such that **A-LC** has eigenvalues with a negative real part, then the estimation error tends to zero. Since **z** is provided by the estimator, we have access to all the states in order to make the control of the form:

$$\mathbf{u} = -\mathbf{Kz} \tag{59}$$

In this case the system Σ_1 is described by the equation:

$$\dot{\mathbf{x}} = \mathbf{Ax} - \mathbf{BKz} + \mathbf{x_o}\delta = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BK}\tilde{\mathbf{x}} + \mathbf{x_o}\delta \tag{60}$$

which has the solution:

$$\mathbf{x} = he^{(\mathbf{A}-\mathbf{BK})t}(\mathbf{x_0}\delta + h\mathbf{BK}e^{(\mathbf{A}-\mathbf{LC})t}\tilde{\mathbf{x}}_\mathbf{0}) \tag{61}$$

The process of calculating the estimator is similar to that described above for the optimal regulator. This is based on the dual system:

$$\dot{\bar{\mathbf{x}}} = \mathbf{A}^T\bar{\mathbf{x}} + \mathbf{C}^T\bar{\mathbf{u}} \tag{62}$$

for which we consider the performance index:

$$\min J = \int_0^\infty [\bar{\mathbf{x}}'(\mathbf{G}\bar{\mathbf{Q}}\mathbf{G}^T)\bar{\mathbf{x}} + \bar{\mathbf{u}}^T\bar{\mathbf{P}}\bar{\mathbf{u}}]\mathrm{d}t \tag{63}$$

By solving the matrix Riccati equation:

$$\mathbf{AR} + \mathbf{RA}^T - \mathbf{RC}^T\bar{\mathbf{P}}^{-1}\mathbf{CR} + \mathbf{G}\bar{\mathbf{Q}}\mathbf{G}^T = \mathbf{0} \tag{64}$$

the matrix estimator is obtained:

$$\mathbf{L} = \mathbf{RC}^T\bar{\mathbf{P}}^{-1} \tag{65}$$

where R is the solution of the Riccati equation.

4.3 Optimal control using uncoupled states (type PID after each of the satellite axes)

In order to have a simpler control matrix and also a more robust one, by resuming papers [4],[5][6], the guidance commands for uncoupled state vector has the following form:

$$\mathbf{u} = \mathbf{U}_R \begin{bmatrix} u_\xi & u_\eta & u_\zeta \end{bmatrix}^T \tag{66}$$

where the main control signals are PID structured:

$$u_\xi = -(k_u^\xi \tilde{\xi} + k_u^{\dot{\xi}} \dot{\tilde{\xi}} + k_u^{I\xi} \tilde{I}_\xi)\,;\; u_\eta = -(k_u^\eta \tilde{\eta} + k_u^{\dot{\eta}} \dot{\tilde{\eta}} + k_u^{I\eta} \tilde{I}_\eta)\,;$$
$$u_\zeta = -(k_u^\zeta \tilde{\psi} + k_u^{\dot{\zeta}} \dot{\tilde{\zeta}} + k_u^{I\zeta} \tilde{I}_\zeta)\;, \tag{67}$$

The matrix $\mathbf{U}_R$, was previously presented. The parameters relative $\tilde{\xi};\tilde{\eta};\tilde{\zeta}$ are given by:

$$\tilde{\xi} = \xi - \xi_d;\; \tilde{\eta} = \eta - \eta_d;\; \tilde{\zeta} = \zeta - \zeta_d; \tag{68}$$

where ξ_d, η_d, ζ_d are the input reference values, and the integrals terms are defined hereby:

$$\begin{bmatrix} \dot{\tilde{I}}_\xi & \dot{\tilde{I}}_\eta & \dot{\tilde{I}}_\zeta \end{bmatrix} = \begin{bmatrix} \tilde{\xi} & \tilde{\eta} & \tilde{\zeta} \end{bmatrix}. \tag{69}$$

For balance movement we can write relation (160) in their linear form:

$$\Delta\mathbf{u} = -(\mathbf{K}_\Omega \Delta\boldsymbol{\omega}_{BI} + \mathbf{K}_P \Delta\mathbf{a}_R + \mathbf{K}_I \Delta\mathbf{I}) + \Delta\mathbf{f} \tag{70}$$

where:

$$\Delta\mathbf{f} = \mathbf{K}_\Omega \Delta\boldsymbol{\omega}_{BId} + \mathbf{K}_P \Delta\mathbf{a}_{Rd} + \mathbf{K}_I \Delta\mathbf{I}_d \tag{71}$$

$$\mathbf{K}_\Omega = \begin{bmatrix} k_u^{\dot{\xi}} & 0 & 0 \\ 0 & k_u^{\dot{\eta}} & 0 \\ 0 & 0 & k_u^{\dot{\zeta}} \end{bmatrix};\; \mathbf{K}_P = \begin{bmatrix} k_u^{\xi} & 0 & 0 \\ 0 & k_u^{\eta} & 0 \\ 0 & 0 & k_u^{\zeta} \end{bmatrix};\; \mathbf{K}_I = \begin{bmatrix} k_u^{I\xi} & 0 & 0 \\ 0 & k_u^{I\eta} & 0 \\ 0 & 0 & k_u^{I\zeta} \end{bmatrix} \tag{72}$$

In this case we can obtain the control matrix for uncoupled case:

Table 9. Control matrix $\mathbf{K}$ for thruster's case

	$\boldsymbol{\omega}_{BI}$	$\mathbf{a}_R$	$\mathbf{I}$	$\mathbf{M}_C$
$\mathbf{u}$	$\mathbf{K}_\Omega$	$\mathbf{K}_P$	$\mathbf{K}_I$	

Heaving the system in its regular form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{73}$$

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \tag{74}$$

we defined the performance index

$$J = \int_0^\infty (\mathbf{x}^T \mathbf{Q}\mathbf{x} + \mathbf{u}^T \mathbf{R}\mathbf{u})\,\mathrm{d}t \tag{75}$$

where the extended pair $(\mathbf{A},\mathbf{B})$ is controllable and the state weighting matrix $\mathbf{Q}$ is symmetric and quasi positive:

$$\mathbf{Q} \geq 0;\; \mathbf{Q} = \mathbf{Q}^T. \tag{76}$$

The control weighting matrix $\mathbf{R}$ is symmetric and positive:

$$\mathbf{R} > 0;\; \mathbf{R} = \mathbf{R}^T; \tag{77}$$

In this case we can obtain the controller $\mathbf{K}$ terms by using the gradient method for minimizing the performance index (75). In order to improve this method we will use the adjoint theory as it is described in paper [12]. First, we will try to obtain a simplified solution for the guidance command defined previously in PID form. For this purpose we will start from the scalar equations established for the commanded linear equations (28).

Moreover we will neglect cross influence introduced by the angular velocity ω_j and also we will consider the inertial product moment null:

$$E = 0 \tag{78}$$

In this case all angular equations (28) have a similar form:

$$\Delta\ddot{\xi} = \frac{\Delta L}{A}; \Delta\ddot{\eta} = \frac{\Delta M}{B}; \Delta\ddot{\zeta} = \frac{\Delta N}{C} \tag{79}$$

If we neglect the actuator delay time: $\tau_M = 0$, from the guidance command form established previously, we can write the following linear forms:

$$\begin{aligned}
\Delta L &= -(k_u^{\dot{\xi}}\Delta\dot{\tilde{\xi}} + k_u^{\xi}\Delta\tilde{\xi} + k_u^{I\xi}\Delta\tilde{I}_{\xi})k_M^u / \left(k_M^u + 1\right);\\
\Delta M &= -(k_u^{\dot{\eta}}\Delta\dot{\tilde{\eta}} + k_u^{\eta}\Delta\tilde{\eta} + k_u^{I_\eta}\Delta\tilde{I}_{\eta})k_M^u / \left(k_M^u + 1\right);\\
\Delta N &= -(k_u^{\dot{\zeta}}\Delta\dot{\tilde{\zeta}} + k_u^{\zeta}\Delta\tilde{\zeta} + k_u^{I_\zeta}\Delta\tilde{I}_{\zeta})k_M^u / \left(k_M^u + 1\right)
\end{aligned} \tag{80}$$

Separating angular inputs and applying Laplace transformation, from the previous relations we will obtain:

$$\begin{aligned}
&\left(A\frac{\left(k_M^u+1\right)}{k_M^u}s^2 + \frac{k_u^{\dot{\xi}}s^2 + k_u^{\xi}s + k_u^{I\xi}}{s}\right)\xi = \frac{k_u^{\dot{\xi}}s^2 + k_u^{\xi}s + k_u^{I\xi}}{s}\xi_D\\
&\left(B\frac{\left(k_M^u+1\right)}{k_M^u}s^2 + \frac{k_u^{\dot{\eta}}s^2 + k_u^{\eta}s + k_u^{I\eta}}{s}\right)\eta = \frac{k_u^{\dot{\eta}}s^2 + k_u^{\eta}s + k_u^{I\eta}}{s}\eta_D\\
&\left(C\frac{\left(k_M^u+1\right)}{k_M^u}s^2 + \frac{k_u^{\dot{\zeta}}s^2 + k_u^{\zeta}s + k_u^{I\zeta}}{s}\right)\zeta = \frac{k_u^{\dot{\zeta}}s^2 + k_u^{\zeta}s + k_u^{I\zeta}}{s}\zeta_D
\end{aligned} \tag{81}$$

Admitting proportionality between the coefficients and the inertial moment we can write:

$$\begin{aligned}
&k_1 = \frac{k_u^{\dot{\xi}}k_M^u}{A\left(k_M^u+1\right)} = \frac{k_u^{\dot{\eta}}k_M^u}{B\left(k_M^u+1\right)} = \frac{k_u^{\dot{\zeta}}k_M^u}{C\left(k_M^u+1\right)};\; k_2 = \frac{k_u^{\xi}k_M^u}{A\left(k_M^u+1\right)} = \frac{k_u^{\eta}k_M^u}{B\left(k_M^u+1\right)} = \frac{k_u^{\zeta}k_M^u}{C\left(k_M^u+1\right)}\\
&k_3 = \frac{k_u^{I\xi}k_M^u}{A(k_M^u+1)} = \frac{k_u^{I\eta}k_M^u}{B(k_M^u+1)} = \frac{k_u^{I\zeta}k_M^u}{C(k_M^u+1)}
\end{aligned} \tag{82}$$

Using these new coefficients, the transfer function for angular size has the form:

$$H_0(s) = \frac{k_1 s^2 + k_2 s + k_3}{s^3 + k_1 s^2 + k_2 s + k_3} \tag{83}$$

Next we will use the pole-zero allocation method [4]. For this purpose we use an optimal function quite similar with the one previously obtained:

$$H_0(s) = \frac{6.7\Omega_0 s^2 + 6.7\Omega_0^2 s + \Omega_0^3}{s^3 + 6.7\Omega_0 s^2 + 6.7\Omega_0^2 s + \Omega_0^3} \tag{84}$$

with $\Omega_0 = \tau_r / t_r$, where $\tau_r = 1.5$, and the response time is chosen.

Identifying between functions coefficients, we can obtain the following useful relations:

$$H_0(s) = \frac{6.7\Omega_0 s^2 + 6.7\Omega_0^2 s + \Omega_0^3}{s^3 + 6.7\Omega_0 s^2 + 6.7\Omega_0^2 s + \Omega_0^3} \tag{84'}$$

Finally, choosing the response time $t_r = 5\,s$ we obtain: $k_1 = 2.01$; $k_2 = 0.603$; $k_3 = 0.027$. The obtained values k_1, k_2 and k_3 can be used as initial values for the gradient method 0. The gradient of the performance index with respect to the parameters k_1, k_2, k_3 are then the partial derivatives

$$\frac{\partial J}{\partial k_s},\ s = 1,2,3. \tag{85}$$

This gradient can be calculated given a base case pair $[\mathbf{x}, \mathbf{K}(k_1, k_2, k_3)]$ and then solving the so-called forward problem:

$$\dot{\mathbf{x}}_s = \tilde{\mathbf{A}}\mathbf{x}_s - \mathbf{B}\mathbf{K}_s\mathbf{x}, \qquad s = 1,2,3, \tag{86}$$

with the initial conditions:

$$\mathbf{x}_s(0) = \mathbf{x}_{s,0} \tag{87}$$

where:

$$\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K},\ \ \mathbf{K}_s = \partial\mathbf{K}/\partial k_s,\ \ \mathbf{x}_s = \frac{\partial \mathbf{x}}{\partial k_s}\ . \tag{88}$$

Once these nine problems are solved, the gradient of the performance index is calculated with

$$\frac{\partial J}{\partial k_s} = 2\int_0^\infty \left(\boldsymbol{x}^T\tilde{\boldsymbol{Q}}\mathbf{x}_s + \boldsymbol{x}^T\boldsymbol{K}^T\boldsymbol{R}\boldsymbol{K}_s\boldsymbol{x}\right)dt; \qquad s = 1,2,3 \tag{89}$$

where:

$$\tilde{\mathbf{Q}} = \mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}\ . \tag{90}$$

The first term in the integral requires the repeated solution of the forward problem. To circumvent this disadvantage one can use the adjoint function method. Let's consider $\boldsymbol{\lambda}$ to be the adjoint of $\mathbf{x}$. The solution of the adjoint problem

$$\dot{\boldsymbol{\lambda}} = -\tilde{\mathbf{A}}^T\boldsymbol{\lambda} - \tilde{\mathbf{Q}}^T\mathbf{x}, \tag{91}$$

with a homogeneous final condition:

$$\boldsymbol{\lambda}(t_f) = 0, \tag{92}$$

allows us the direct calculation of the gradient the performance index. To do this, we multiply the equation (80) with the adjoint variable λ and then by integration we obtain:

$$\int_0^\infty \boldsymbol{\lambda}^{\mathbf{T}}\dot{\mathbf{x}}_{\mathbf{s}}dt = \int_0^\infty \boldsymbol{\lambda}^{\mathbf{T}}\tilde{\mathbf{A}}\mathbf{x}_{\mathbf{s}}dt + \int_0^\infty \boldsymbol{\lambda}^{\mathbf{T}}\tilde{\mathbf{A}}_{\mathbf{s}}\mathbf{x}dt\,. \tag{93}$$

Integrating by parts the left term, we obtain:

$$\boldsymbol{\lambda}^{\mathbf{T}}\mathbf{x}_{\mathbf{s}}\Big|_0^\infty - \int_0^\infty \mathbf{x}_{\mathbf{s}}^{\mathbf{T}}\dot{\boldsymbol{\lambda}}dt = \int_0^\infty \boldsymbol{\lambda}^{\mathbf{T}}\tilde{\mathbf{A}}\mathbf{x}_{\mathbf{s}}dt + \int_0^\infty \boldsymbol{\lambda}^{\mathbf{T}}\tilde{\mathbf{A}}_{\mathbf{s}}\mathbf{x}dt\,. \tag{94}$$

By grouping the two terms result we get:

$$\boldsymbol{\lambda}^{\mathbf{T}}\mathbf{x}_{\mathbf{s}}\Big|_0^\infty - \int_0^\infty \mathbf{x}_{\mathbf{s}}^{\mathbf{T}}\left(\dot{\boldsymbol{\lambda}} + \tilde{\mathbf{A}}^{\mathbf{T}}\boldsymbol{\lambda}\right)dt = \int_0^\infty \boldsymbol{\lambda}^{\mathbf{T}}\tilde{\mathbf{A}}_{\mathbf{s}}\mathbf{x}dt\,. \tag{95}$$

On the other hand, if we consider the adjoint equation (91) with the final conditions (92), we get:

$$\int_0^\infty \mathbf{x}^T \tilde{\mathbf{Q}} \mathbf{x}_\mathbf{s} dt = \boldsymbol{\lambda}_0^T \mathbf{x}_{\mathbf{s}0} - \int_0^\infty \boldsymbol{\lambda}^\mathbf{T} \mathbf{B} \mathbf{K}_\mathbf{s} \mathbf{x} dt . \tag{96}$$

In this case the relation (89) becomes:

$$\frac{\partial J}{\partial k_s} = 2\boldsymbol{\lambda}_0^T \mathbf{x}_{s,0} + 2\int_0^\infty \left(\mathbf{x}^T \mathbf{K}^T \mathbf{R} \mathbf{K}_s \mathbf{x} - \boldsymbol{\lambda}^T \mathbf{B} \mathbf{K}_s \mathbf{x}\right) dt , \tag{97}$$

for which, it is necessary to determine the direct and the adjoint states, which is an obvious advantage in terms of required calculations from the previous algorithm.

Moreover, if we admit that the linear relation between the direct and adjoint variable is the one proposed by [3], we have:

$$\boldsymbol{\lambda} = \mathbf{P}\mathbf{x} \tag{98}$$

or after deriving:

$$\dot{\boldsymbol{\lambda}} = \mathbf{P}\dot{\mathbf{x}} , \tag{99}$$

By introducing the state equation (73) and the adjoint equation (91) we get:

$$\mathbf{P}\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T \mathbf{P} + \tilde{\mathbf{Q}} = \mathbf{0} , \tag{100}$$

which is the Riccati algebraic equation in variable $\mathbf{P}$. In this case the relation (97) becomes:

$$J_s = 2\boldsymbol{\lambda}_0^T \mathbf{x}_{\mathbf{s}0} + 2\int_0^\infty \mathbf{x}^\mathbf{T} (\mathbf{K}^\mathbf{T}\mathbf{R} - \mathbf{P}^\mathbf{T}\mathbf{B}) \mathbf{K}_\mathbf{s} \mathbf{x} dt \tag{101}$$

which supposes to evaluate the direct states $\mathbf{x}$ by solving the state equation (73) and then by solving the Riccati equation (100). Similarly, for the reaction wheels control case we have:

$$\Delta\ddot{\xi} = \frac{\Delta y_{wx}}{A} ; \Delta\ddot{\eta} = \frac{\Delta y_{wy}}{B} ; \qquad \Delta\ddot{\zeta} = \frac{\Delta y_{wz}}{C} \tag{102}$$

Taking into account that:

$$\begin{gathered} \Delta\dot{y}_{wx} = -a_w \Delta y_{wx} + b_w \Delta h_{wx} + k_w^u \Delta u_\xi ;\ \Delta\dot{y}_{wy} = -a_w \Delta y_{wy} + b_w \Delta h_{wy} + k_w^u \Delta u_\eta ; \\ \Delta\dot{y}_{wz} = -a_w \Delta y_{wz} + b_w \Delta h_{wz} + k_w^u \Delta u_\zeta , \end{gathered} \tag{103}$$

and:

$$\Delta u_\xi = -(k_u^{\dot{\xi}} \Delta\dot{\tilde{\xi}} + k_u^{\xi} \Delta\tilde{\xi}) ; \Delta u_\eta = -(k_u^{\dot{\eta}} \Delta\dot{\tilde{\eta}} + k_u^{\eta} \Delta\tilde{\eta}) ; \Delta u_\zeta = -(k_u^{\dot{\zeta}} \Delta\dot{\tilde{\zeta}} + k_u^{\zeta} \Delta\tilde{\zeta}) \tag{104}$$

For the balance movement considered we can write the command relation in linear form:

$$\Delta\mathbf{u} = -(\mathbf{K}_\Omega \Delta\boldsymbol{\omega}_{BI} + \mathbf{K}_P \Delta\mathbf{a}_R) + \Delta\mathbf{f} \tag{105}$$

where:

$$\Delta\mathbf{f} = \mathbf{K}_\Omega \Delta\boldsymbol{\omega}_{BId} + \mathbf{K}_P \Delta\mathbf{a}_{Rd} \tag{106}$$

In this case we can obtain the control matrix for uncoupled case:

Table 10. Control matrix $\mathbf{K}$ for the reaction wheels case

	$\boldsymbol{\omega}_{BI}$	$\mathbf{a}_R$	$\mathbf{h}_w$	$\mathbf{y}_w$
$\mathbf{u}$	$\mathbf{K}_\Omega$	$\mathbf{K}_P$		

Similarly, we can obtain the controller $\mathbf{K}$ terms by using the gradient method.

We separate the angular inputs and then by applying Laplace transformation, from the previous relations we will obtain:

$$\left(\frac{A}{k_w^u}s+\frac{k_u^{\dot{\xi}}s+k_u^{\xi}}{s^2+a_w s+b_w}\right)\xi=\frac{k_u^{\dot{\xi}}s+k_u^{\xi}}{s^2+a_w s+b_w}\xi_D;\ \left(\frac{B}{k_w^u}s+\frac{k_u^{\dot{\eta}}s+k_u^{\eta}}{s^2+a_w s+b_w}\right)\eta=\frac{k_u^{\dot{\eta}}s+k_u^{\eta}}{s^2+a_w s+b_w}\eta_D$$
$$\left(\frac{C}{k_w^u}s+\frac{k_u^{\dot{\zeta}}s+k_u^{\zeta}}{s^2+a_w s+b_w}\right)\zeta=\frac{k_u^{\dot{\zeta}}s+k_u^{\zeta}}{s^2+a_w s+b_w}\zeta_D \tag{107}$$

Admitting certain proportionality between the coefficients and the inertial moment we can write:

$$k_1=\frac{k_u^{\dot{\xi}}k_w^u}{A}=\frac{k_u^{\dot{\eta}}k_w^u}{B}=\frac{k_u^{\dot{\zeta}}k_w^u}{C};\ k_2=\frac{k_u^{\xi}k_w^u}{A}=\frac{k_u^{\eta}k_w^u}{B}=\frac{k_u^{\zeta}k_w^u}{C}; \tag{108}$$

Using these new coefficients, the transfer function for angular size has the form:

$$H_0(s)=\frac{k_1 s+k_2}{s^3+a_w s^2+(b_w+k_1)s+k_2}; \tag{109}$$

Next we use the pole-zero allocation method [4]. For this purpose we use an optimal function quite similar with the previously obtained:

$$H_0(s)=\frac{6.35\Omega_0^2 s+\Omega_0^3}{s^3+5.1\Omega_0 s^2+6.35\Omega_0^2 s+\Omega_0^3}, \tag{110}$$

with $\Omega_0=\tau_r/t_r$, where $\tau_r=5$, and the response time is chosen.

Identifying between coefficients, we obtain the following useful relations:

$$k_1=6.35\Omega_0^2-b_w;\qquad k_2=\Omega_0^3 \tag{111}$$

Finally, choosing the response time $t_r=10\ s$ we obtain:

$$k_1=1.59;\ k_2=0.125 \tag{112}$$

Obtained values k_1 and k_2 will be used as start values for the gradient method described above.

5. INPUT DATA, CALCULUS ALGORITHM AND RESULTS

5.1 Input data for the model

As input data for our application we considered:

The eccentricity $e=0.1$; The orbital period $T=24h$

The inertial moments: $A=31.44[kgm^2]; B=34.11[kgm^2]; C=13.12[kgm^2]$

The product of inertia: $E=0.39[kgm^2];\ D=-0.02[kgm^2];\ F=-0.03[kgm^2]$

Parameters of the Schmidt Trigger element: $a_M=0.1; b_M=0.3;\ \tau_M=0.1[s]; k_M^u=2$.

Parameters of the reaction wheels: $a_w=20;\ b_w=20\ \ k_w^u=10$.

5.2 Calculus algorithm

The calculus algorithm consists in multi-step method Adams' predictor-corrector with variable step integration method: 00. Absolute numerical error was 1.e-12, and relative error was 1.e-10.

5.3 Results

First we highlight the influence of gravitational moment on the uncontrolled satellite orientation. Figure 7 presents the rotational velocity around the y axis of the mobile frame related inertial frame

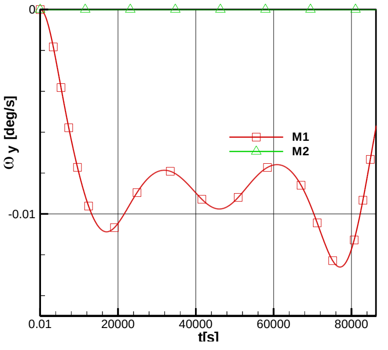

Fig. 7 Angular velocity for uncontrolled vehicle. M1 - with gravitational moment terms; M2- without gravitational moment terms

We can see that the gravitational influence leads to an additional angular velocity.

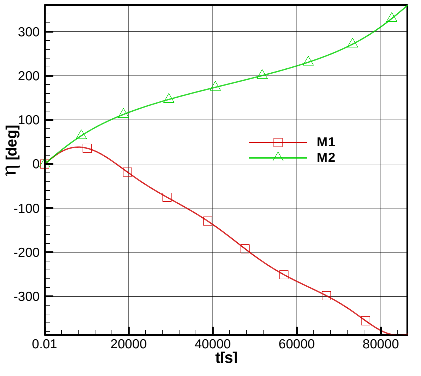

Fig. 8 angular diagram for uncontrolled vehicle. M1 - with gravitational moment terms; M2- without gravitational moment terms

Consequently, it influences the angle around the y axis, as we can see from Figure 8.

Next we compare gradient method results with Monte Carlo method presented for the same model in papers [6] and [12].

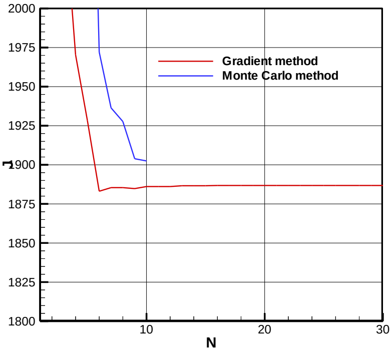

Fig. 9 Performance index evolution related steps iteration (N)

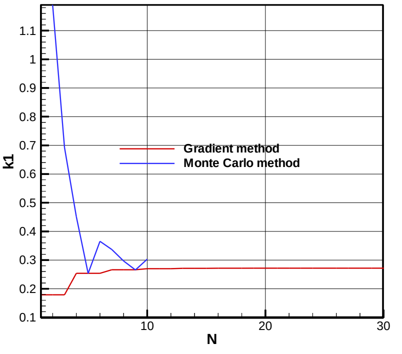

Fig. 10 K1 parameter evolution related steps iteration (N)

Figure 9 compares the evolution of the performance index in the case of the iterative methods: gradient and Monte- Carlo, and figure 10 shows the evolution of “k1” parameter during the iterations, starting from the value obtained by pole-zero allocation method.

Next we analyze the three types of orientation control systems described above. For starters, the thrust control using a trigger Schmidt element is presented.

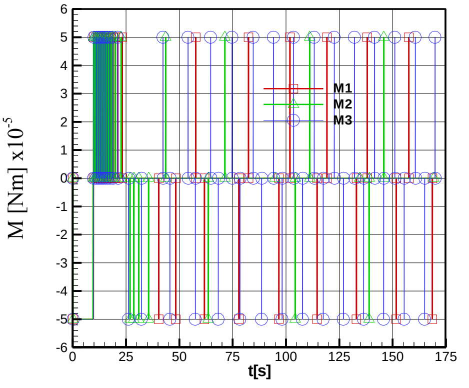

Fig. 11 Command moment for controlled vehicle. M1- Optimal control using uncoupled states ; M2- Optimal control using coupled states ; M3- Optimal control using Kalman filter ;

Note that after achieving control system synthesis, the model uses the nonlinear switching element. Because at the beginning we have an angular velocity jump, the command is more active in this moment. By applying the above presented control systems, the absolute angular velocity is stabilized at the base, which provides to the satellite a rotation velocity around its y axis synchronous with the motion around the Earth, as we can see in figure 12.

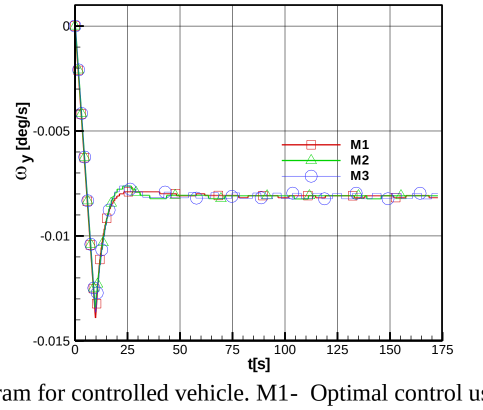

Fig. 12 angular velocity diagram for controlled vehicle. M1- Optimal control using uncoupled state vector ; M2- Optimal control using coupled state vector ; M3 – Kalman filter;

Finally, figure 13 shows the rotation angle around the y axis, which is stabilized at null value, and providing the overlap of the mobile frame over reference frame.

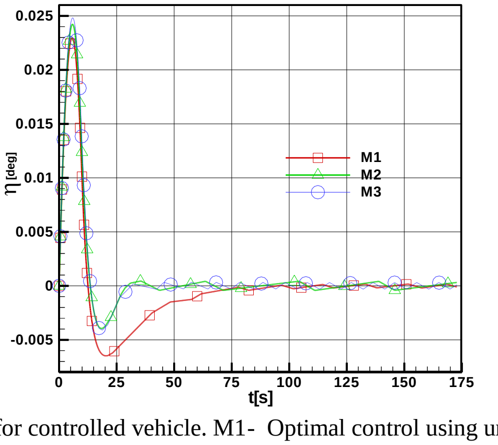

Fig. 13 angular diagram for controlled vehicle. M1- Optimal control using uncoupled state vector ; M2- Optimal control using coupled state vector ; M3 – Kalman filter;

6. CONCLUSIONS

The paper presents some synthesis aspects of the simulation model, developed for the calculation of Attitude Control System- ACS of the small satellite which uses as command a micro jet engine. The application is made for three ACS variants, the first one using a control system for uncoupled state, the second using a control system for coupled state and the third using the Kalman filter. From the results obtained one can observe that the last two solutions, although are more complicated, they give better results than the previous ones, providing an ACS with the shortest response time and a smaller override. As a general conclusion we must underline three novelty aspects introduced by the paper:

- We achieved the description of the model by using the rotation angles, which lead to polynomial forms for the rotation and connection matrix and which eliminate the singularities of the connection matrix in the case of Euler's angles. On the other hand, these 3 values are independent and on the same time they have an angular dimension, and so they are measurable. This creates a great advantage as compared to the use of the Hamilton quaternion.

- By the linearization of the Trigger-Schmidt element we have constructed a homogenous linear system and we made the ACS synthesis. With all the simplifications introduced by the Fourier transformation, the result obtained is valid, this being verified by testing the system in its non-linear form.

- The use of the adjoint analysis methods for the synthesis of optimal controller based on uncoupled state. The advantage of this method consists in allowing optimization of the controller and also of the other system parameters as mass distribution or Trigger Schmidt parameters.

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