An Analytic Potential Solution for Incompressible 2D Channel Inviscid Flow with Wall Injection

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Abstract: The present paper introduces an analytical potential solution for the incompressible flow in a 2D channel with normal wall injection. This solution is capable to support reasonable injection flow rates as compared to the general flow rate in the channel. A strategy to find solutions for longer channels in case of increasing injection velocity with the distance from the entrance, using a small number of the solution terms is pointed out. Examples of calculation are given.

Key Words: 2D channel flow, analytic solution, wall injection.

1. INTRODUCTION

The possibility to obtain potential solutions to incompressible flow with injection was, after our knowledge, not thoroughly studied. Several papers [1], [2] take into account a vortex solution generated by the injection itself, although the superposition of eigenfunctions do not satisfy the nonlinear vortex equation of Helmholtz [3]. Of course much attention is paid to wall injection in connection with boundary layer control [4;5].

2. EQUATIONS AND BOUNDARY CONDITIONS

One considers the incompressible flow in a 2D channel (Fig.1), symmetrical with respect to the channel axis. The flow is assumed incompressible, stationary and potential. By introducing the stream function \( \Psi(\bar{z}, \bar{y}) \), [3]:

\[
\bar{v}_z = \frac{\partial \Psi}{\partial \bar{y}}; \quad \bar{v}_y = -\frac{\partial \Psi}{\partial \bar{z}},
\]

one has to solve the partial differential equation for the stream function:

\[
\frac{\partial^2 \Psi}{\partial \bar{x}^2} + \frac{\partial^2 \Psi}{\partial \bar{y}^2} = 0.
\]

In order to solve the problem the boundary condition have to be specified. At the entrance, velocity is constant and parallel to channel walls:

\[
\bar{v}_z(0, \bar{y}) = \bar{v}_z^0 = \bar{v}_{ax}
\]

The injection velocity is normal to the wall and variable along the wall:

\[
\bar{v}_y(\bar{z}, \alpha) = -\bar{u}_w(\bar{z}).
\]

On the axis, the symmetry of flow imposes

\[
\bar{v}_y(\bar{z}, 0) = 0, \quad \bar{z} \geq 0
\]
The boundary conditions for the elliptical PDE (2) should include the whole frontier of the channel domain; two parts of the frontier are required by conditions (3) and (4). As condition at the channel exit, $z = L$, one will consider the exit velocity parallel to the channel axis, an assumption corresponding to high values of $L$. Therefore:

$$\bar{v}_y(L, \bar{y}) = 0$$  \hfill (6)

Further by the change of variables:

$$z = \bar{z} / L, \quad y = \bar{y} / a, \quad \bar{y} = \bar{v}_w a \psi, \quad \psi = \psi_1 + \bar{y}$$  \hfill (7)

where the unbarred new variables are dimensionless, one obtains the equation:

$$\frac{a^2}{L^2} \frac{\partial^2 \psi_1}{\partial z^2} + \frac{\partial^2 \psi_1}{\partial y^2} = 0$$  \hfill (8)

the velocities being given by:

$$v_z = 1 + \frac{\partial \psi_1}{\partial y}, \quad v_y = - \frac{\partial \psi_1}{\partial z}$$  \hfill (9)

The boundary conditions for the new unknown $\psi_1$ are:

$$\frac{\partial \psi_1}{\partial y} = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad 0 \leq y \leq 1;$$

$$\frac{\partial \psi_1}{\partial z} = 0 \quad \text{for} \quad z = 1, \quad 0 \leq y \leq 1;$$

$$\frac{\partial \psi_1}{\partial z} = 0 \quad \text{for} \quad y = 0, \quad 0 \leq z \leq 1;$$

$$\frac{\partial \psi_1}{\partial z} = u_w(z) \quad \text{for} \quad y = 1, \quad 0 \leq z \leq 1;$$  \hfill (10)

where $u_w(z) = \bar{u}_w(\bar{z}) / \bar{v}_w$.

### 3. THE ANALYTIC SOLUTION

One looks for the eq.(7) solutions of the form:

$$\psi_1(z, y) = Z(z)Y(y)$$  \hfill (11)

$Z(z)$ and $Y(y)$ being obtained by separation of variables $z, y$ [6]. By introducing (9) in eq. (7) one obtains two ordinary differential equations which are easily solved under the form:
\[ Z(z) = A_1 \cos\left(\frac{L\lambda}{a} z\right) + A_2 \sin\left(\frac{L\lambda}{a} z\right); \quad (12) \]
\[ Y(y) = B_1 \exp(\lambda y) + B_2 \exp(-\lambda y) \]

\[ A_i, B_i, i = 1; 2 \] and \( \lambda \) being real arbitrary constants. The stream function and the corresponding velocity components are:

\[ \psi_1(z, y) = A_1 \cos\left(\frac{L\lambda}{a} z\right) + A_2 \sin\left(\frac{L\lambda}{a} z\right) \left[ B_1 \exp(\lambda y) + B_2 \exp(-\lambda y) \right]; \quad (13a) \]

\[ \frac{\partial \psi_1}{\partial y} = A_1 \cos\left(\frac{L\lambda}{a} z\right) + A_2 \sin\left(\frac{L\lambda}{a} z\right) \left[ B_1 \lambda \exp(\lambda y) - B_2 \lambda \exp(-\lambda y) \right]; \quad (13b) \]

\[ \frac{\partial \psi_1}{\partial z} = -A_1 \frac{L\lambda}{a} \sin\left(\frac{L\lambda}{a} z\right) + A_2 \frac{L\lambda}{a} \cos\left(\frac{L\lambda}{a} z\right) \left[ B_1 \exp(\lambda y) + B_2 \exp(-\lambda y) \right]; \quad (13c) \]

Imposing the boundary condition (10c):

\[ \left[ -A_1 \frac{L\lambda}{a} \sin\left(\frac{L\lambda}{a} z\right) + A_2 \frac{L\lambda}{a} \cos\left(\frac{L\lambda}{a} z\right) \right] (B_1 + B_2) = 0, \quad (14) \]
yields:

\[ B_1 = -B_2. \quad (15) \]

At the entrance, \( z = 0 \), from (10a):

\[ A_1 \left[ B_1 \lambda \exp(\lambda y) - B_2 \lambda \exp(-\lambda y) \right] = 0, \quad (16) \]
one obtains

\[ A_1 = 0. \quad (17) \]

Replacing (15) and (17) in (13) results:

\[ \psi_1(z, y) = A \sin(\Lambda z) \sinh(\lambda y); \quad (18a) \]

\[ \frac{\partial \psi_1}{\partial y} = A \Lambda \sin(\Lambda z) \cosh(\lambda y); \quad (18b) \]

\[ \frac{\partial \psi_1}{\partial z} = A \Lambda \cos(\Lambda z) \sinh(\lambda y). \quad (18c) \]

where \( \Lambda = \lambda L / a \).

At the exit, \( z = 1 \), the transversal velocity should vanish. So, from (18c) results the equation:

\[ \cos(\Lambda) = 0, \quad (19) \]
having the solution:

\[ \Lambda_n = (2n - 1)\frac{\pi}{2}, \quad n = 1, 2, ..., \quad (20) \]

Because the eqs. (2) and (8) are linear, the general solution is:

\[ \psi = y + \sum_{n=1}^{\infty} A_n \sinh(\lambda_n y) \sin(\Lambda_n z) \quad (21a) \]

\[ v_z = \sum_{n=1}^{\infty} \lambda_n A_n \cosh(\lambda_n y) \sin(\Lambda_n z); \quad (21b) \]

\[ v_y = -\sum_{n=1}^{\infty} \lambda_n A_n \sinh(\lambda_n y) \cos(\Lambda_n z) \quad (21c) \]
where:
\[v_y = -\frac{a}{L} \frac{\partial \psi}{\partial z}, \quad \Lambda_n = \frac{L}{a} \lambda_n = (2n-1) \frac{\pi}{2}, \quad n = 1, 2, \ldots\]  
(22)

The constants \(A_n, \quad n = 1, 2, \ldots\) are given by the injection condition:
\[\sum_{n=1}^{\infty} \lambda_n A_n \sinh(\lambda_n) \cos(\Lambda_n z) = u_w(z),\]  
(23)

which will be written as follows:
\[\sum_{n=1}^{\infty} B_n \cos(\Lambda_n z) = \beta_w(z)\]  
(24a)

\[\beta_w(z) = u_w(z) / u_{wmed}, \quad B_n = \lambda_n A_n \sinh(\lambda_n) / u_{wmed}\]  
(24b)

In the above relations, the mean injection velocity \(u_{wmed}\) was introduced, defined by:
\[u_{wmed} = \frac{1}{L} \int u_w(z) dz\]  
(25)

In order to determine the coefficients \(B_n\) from (24a), one uses the orthogonality properties of cosine functions under the form:
\[\int_0^1 \cos(\Lambda_m z) \cos(\Lambda_n z) dz = \begin{cases} 0, & \text{for } m \neq n; \\ 1 / 2, & \text{for } m = n. \end{cases}\]  
(26)

4. RESULTS

In the following several applications for different channel parameters \(L/a\) are presented. Taking into account that the injected flow rate divided by the general flow, \(r_Q\), rate can be defined as:
\[r_Q = \frac{\bar{u}_w L}{v_{ax} a} = u_{wmed} \frac{L}{a} = \frac{r_Q}{u_{wmed}}\]  
(27)

one can obtain an expression for the mean injection velocity:
\[u_{wmed} = \frac{a}{L} r_Q.\]  
(28)

In Table 1 are given the expressions of the coefficients \(B_n, A_n\) for several distributions of the injection velocity, as well as the limit forms of the stream function \(\psi\) for \(L / a \to \infty\).

<table>
<thead>
<tr>
<th>(\beta_w(z))</th>
<th>(B_n)</th>
<th>(A_n / u_{wmed})</th>
<th>(\lim_{L/a \to \infty} \psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{2(-1)^{(n-1)}}{\Lambda_n})</td>
<td>(\frac{2(-1)^{(n-1)} L}{\Lambda_n^2 \sinh \lambda_n a})</td>
<td>(y \left(1 + r_Q \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{\Lambda_n^2} \sin(\Lambda_n z)\right))</td>
</tr>
<tr>
<td>2(1 - z)</td>
<td>(\frac{4}{\Lambda_n^2})</td>
<td>(\frac{4 L}{\Lambda_n^3 \sinh \lambda_n a})</td>
<td>(y \left(1 + 2r_Q \sum_{n=1}^{\infty} \frac{\sin(\Lambda_n z)}{\Lambda_n^3}\right))</td>
</tr>
</tbody>
</table>
The distribution $\beta(z)=1$, although very simple, has an undefined value for $v_z$ at $z=1$. This is due to the disagreement between the boundary conditions at this only point. The uncertainty disappears in the limit $L/a \to \infty$ (long channel).

The other two injection laws, having null injection velocity at $z=1$, are concordant and besides lead to more rapid convergence rate for the series.

**A solution using three terms.** Let us consider the injection velocity of the form:

$$\beta_w(z,k,m) = \sum_{n=1}^{3} C_n(k,m) \cos(\Lambda_n z),$$

where the coefficients are:

$$C_3 = \frac{15}{8} \left( k + \frac{m}{12} - \frac{\pi}{8} \right),$$

$$C_2 = C_3 - \frac{m-k}{4},$$

$$C_1 = k - C_2 - C_3.$$

The parameter $k$ represents the value of $\beta_w(z,k,m)$ at channel entrance whereas the product $-\pi m/2$ represents the curve slope at $z=1$.

In Fig.2 the curves $\beta_w(z,0.5,3)$, $\beta_w(z,0.3,5)$ and $\beta_w(z,0,8)$ are represented, suggesting a gradually increasing of the injection up to a maximum as one moves to the channel end. This configuration can be appropriate to some combustion problems [1;2].

**5. CONCLUSIONS**

A potential solution (satisfying identically the Helmholtz vortex equation unlike some the existing vortical solutions) is capable to take over an injection (or suction) normal to wall in a 2D incompressible flow. An analytical solution was obtained for stationary inviscid flow that can be used further as a frontier condition for calculation of the boundary layer. Limit behaviours for very long channels are obtained considering a finite ratio of the injection flow rate to the general one.
Interesting injection velocity profiles with gradually increasing up to a maximum in the last part of the channel can be obtained using a small number of series development.

6. ACKNOWLEDGEMENTS

The present work has been supported by the CNCSIS –UEFISCUS, project number PNII – IDEI 1030/2007 (contract number 109/2007).

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