An Analytic Potential Solution for Incompressible 2D Channel Inviscid Flow with Wall Injection

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Abstract: *The present paper introduces an analytical potential solution for the incompressible flow in a 2D channel with normal wall injection. This solution is capable to support reasonable injection flow rates as compared to the general flow rate in the channel. A strategy to find solutions for longer channels in case of increasing injection velocity with the distance from the entrance, using a small number of the solution terms is pointed out. Examples of calculation are given.*

Key Words: 2D channel flow, analytic solution, wall injection.

1. INTRODUCTION

The possibility to obtain potential solutions to incompressible flow with injection was, after our knowledge, not thoroughly studied. Several papers [1], [2] take into account a vortex solution generated by the injection itself, although the superposition of eigenfunctions do not satisfy the nonlinear vortex equation of Helmholtz [3]. Of course much attention is paid to wall injection in connection with boundary layer control [4;5].

2. EQUATIONS AND BOUDARY CONDITIONS

One considers the incompressible flow in a 2D channel (Fig.1), symmetrical with respect to the channel axis. The flow is assumed incompressible, stationary and potential. By introducing the stream function $\overline{\psi}(\overline{z},\overline{y})$, [3]:

$$\overline{v}_{\overline{z}} = \frac{\partial\overline{\psi}}{\partial\overline{y}}\ ;\ \overline{v}_{\overline{y}} = -\frac{\partial\overline{\psi}}{\partial\overline{z}}\ , \tag{1}$$

one has to solve the partial differential equation for the stream function:

$$\frac{\partial^2\overline{\psi}}{\partial\overline{x}^2} + \frac{\partial^2\overline{\psi}}{\partial\overline{y}^2} = 0\,. \tag{2}$$

In order to solve the problem the boundary condition have to be specified. At the entrance, velocity is constant and parallel to channel walls:

$$\overline{v}_{\overline{z}}(0,\overline{y}) = \overline{v}_{z0} = \overline{v}_{\text{ax}} \tag{3}$$

The injection velocity is normal to the wall and variable along the wall:

$$\overline{v}_{\overline{y}}(\overline{z},a) = -\overline{u}_w(\overline{z})\,. \tag{4}$$

On the axis, the symmetry of flow imposes

$$\overline{v}_{\overline{y}}(z,0) = 0\,,\ z \geq 0 \tag{5}$$

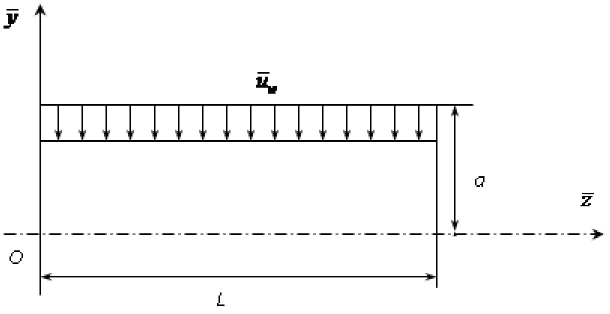

Fig.1- – Geometric configuration

The boundary conditions for the elliptical PDE (2) should include the whole frontier of the channel domain; two parts of the frontier are required by conditions (3) and (4). As condition at the channel exit, $\bar{z}=L$, one will consider the exit velocity parallel to the channel axis, an assumption corresponding to high values of *L*. Therefore:

$$\bar{v}_{\bar{y}}(\bar{L},\bar{y})=0 \tag{6}$$

Further by the change of variables:

$$z=\bar{z}/L,\ y=\bar{y}/a,\ \bar{\psi}=\bar{v}_{\text{ax}}a\psi\ ,\ \psi=\psi_1+\bar{y} \tag{7}$$

where the unbarred new variables are dimensionless, one obtains the equation:

$$\frac{a^2}{L^2}\frac{\partial^2\psi_1}{\partial z^2}+\frac{\partial^2\psi_1}{\partial y^2}=0 \tag{8}$$

the velocities being given by:

$$v_z=1+\frac{\partial\psi_1}{\partial y}\ ,\ v_y=-\frac{\partial\psi_1}{\partial z} \tag{9}$$

The boundary conditions for the new unknown ψ_1 are:

$$\frac{\partial\psi_1}{\partial y}=0 \text{ for } z=0 \text{ and } 0\le y\le 1; \tag{10a}$$

$$\frac{\partial\psi_1}{\partial z}=0 \text{ for } z=1,\ 0\le y\le 1; \tag{10b}$$

$$\frac{\partial\psi_1}{\partial z}=0 \text{ for } y=0,\ 0\le z\le 1; \tag{10c}$$

$$\frac{\partial\psi_1}{\partial z}=u_w(z) \text{ for } y=1,\ 0\le z\le 1; \tag{10d}$$

where $u_w(z)=\bar{u}_w(\bar{z})/\bar{v}_{\text{ax}}$.

3. THE ANALYTIC SOLUTION

One looks for the eq.(7) solutions of the form:

$$\psi_1(z,y)=Z(z)Y(y) \tag{11}$$

$Z(z)$ and $Y(y)$ being obtained by separation of variables *z*, *y* [6]. By introducing (9) in eq. (7) one obtains two ordinary differential equations which are easily solved under the form:

$$Z(z) = A_1 \cos(\frac{L\lambda}{a} z) + A_2 \sin(\frac{L\lambda}{a} z); \quad Y(y) = B_1 \exp(\lambda y) + B_2 \exp(-\lambda y) \tag{12}$$

$A_i, B_i, i = 1;2$ and λ being real arbitrary constants. The stream function and the corresponding velocity components are:

$$\psi_1(z,y) = \left[A_1 \cos(\frac{L\lambda}{a} z) + A_2 \sin(\frac{L\lambda}{a} z) \right] \left[B_1 \exp(\lambda y) + B_2 \exp(-\lambda y) \right] \tag{13a}$$

$$\frac{\partial \psi_1}{\partial y} = \left[A_1 \cos(\frac{L\lambda}{a} z) + A_2 \sin(\frac{L\lambda}{a} z) \right] \left[B_1 \lambda \exp(\lambda y) - B_2 \lambda \exp(-\lambda y) \right]; \tag{13b}$$

$$\frac{\partial \psi_1}{\partial z} = \left[-A_1 \frac{L\lambda}{a} \sin(\frac{L\lambda}{a} z) + A_2 \frac{L\lambda}{a} \cos(\frac{L\lambda}{a} z) \right] \left[B_1 \exp(\lambda y) + B_2 \exp(-\lambda y) \right]. \tag{13c}$$

Imposing the boundary condition (10c):

$$\left[-A_1 \frac{L\lambda}{a} \sin(\frac{L\lambda}{a} z) + A_2 \frac{L\lambda}{a} \cos(\frac{L\lambda}{a} z) \right] (B_1 + B_2) = 0, \tag{14}$$

yields:

$$B_1 = -B_2. \tag{15}$$

At the entrance, $z = 0$, from (10a):

$$A_1 \left[B_1 \lambda \exp(\lambda y) - B_2 \lambda \exp(-\lambda y) \right] = 0, \tag{16}$$

one obtains

$$A_1 = 0. \tag{17}$$

Replacing (15) and (17) in (13) results:

$$\psi_1(z,y) = A \sin(\Lambda z) \sinh(\lambda y); \tag{18a}$$

$$\frac{\partial \psi_1}{\partial y} = A\lambda \sin(\Lambda z) \cosh(\lambda y); \tag{18b}$$

$$\frac{\partial \psi_1}{\partial z} = A\Lambda \cos(\Lambda z) \sinh(\lambda y). \tag{18c}$$

where $\Lambda = \lambda L / a$.

At the exit, $z = 1$, the transversal velocity should vanish. So, from (18c) results the equation:

$$\cos(\Lambda) = 0, \tag{19}$$

having the solution:

$$\Lambda_n = (2n-1)\frac{\pi}{2}, \ n = 1,2,.... \tag{20}$$

Because the eqs. (2) and (8) are linear, the general solution is:

$$\psi = y + \sum_{n=1}^{\infty} A_n \sinh(\lambda_n y) \sin(\Lambda_n z) \tag{21a}$$

$$v_z = 1 + \sum_{n=1}^{\infty} \lambda_n A_n \cosh(\lambda_n y) \sin(\Lambda_n z); \tag{21b}$$

$$v_y = -\sum_{n=1}^{\infty} \lambda_n A_n \sinh(\lambda_n y) \cos(\Lambda_n z) \tag{21c}$$

where:

$$v_y = -\frac{a}{L}\frac{\partial \psi}{\partial z} \;,\; \Lambda_n = \frac{L}{a}\lambda_n = (2n-1)\frac{\pi}{2} \;,\; n = 1,2,.... \tag{22}$$

The constants A_n , $n = 1,2,....$ are given by the injection condition:

$$\sum_{n=1}^{\infty} \lambda_n A_n \sinh(\lambda_n)\cos(\Lambda_n z) = u_w(z), \tag{23}$$

which will be written as follows:

$$\sum_{n=1}^{\infty} B_n \cos(\Lambda_n z) = \beta_w(z) \tag{24a}$$

$$\beta_w(z) = u_w(z)/u_{wmed} \;,\; B_n = \lambda_n A_n \sinh(\lambda_n)/u_{wmed} \tag{24b}$$

In the above relations, the mean injection velocity u_{wmed} was introduced, defined by:

$$u_{wmed} = \int_0^1 u_w(z)\,dz \tag{25}$$

In order to determine the coefficients B_n from (24a), one uses the orthogonality properties of cosine functions under the form:

$$\int_0^1 \cos(\Lambda_m z)\cos(\Lambda_n z)dz = \begin{cases} 0, \text{for } m \neq n; \\ 1/2, \text{for } m = n. \end{cases} \tag{26}$$

4. RESULTS

In the following several applications for different channel parameters *L/a* are presented. Taking into account that the injected flow rate divided by the general flow , r_Q, rate can be defined as:

$$r_Q = \frac{\overline{u}_w L}{v_{\text{ax}}\, a} = u_{wmed}\frac{L}{a} \;,\; \frac{L}{a} = \frac{r_Q}{u_{wmed}} \tag{27}$$

one can obtain an expression for the mean injection velocity:

$$u_{wmed} = \frac{a}{L} r_Q \,. \tag{28}$$

In Table 1 are given the expressions of the coefficients B_n A_n for several distributions of the injection velocity, as well as the limit forms of the stream function ψ for $L/a \to \infty$.

Table 1 – Coefficients of series developments

$\beta_w(z)$	B_n	A_n / u_{wmed}	$\lim\limits_{L/a\to\infty} \psi$
1	$\frac{2(-1)^{(n-1)}}{\Lambda_n}$	$\frac{2(-1)^{(n-1)}}{\Lambda_n^2 \sinh\lambda_n}\frac{L}{a}$	$y\left(1 + r_Q \sum_{n=1}^{\infty}\frac{(-1)^{(n-1)}}{\Lambda_n^2}\sin(\Lambda_n z)\right)$
$2(1-z)$	$\frac{4}{\Lambda_n^2}$	$\frac{4}{\Lambda_n^3 \sinh\lambda_n}\frac{L}{a}$	$y\left(1 + 2r_Q \sum_{n=1}^{\infty}\frac{\sin(\Lambda_n z)}{\Lambda_n^3}\right)$

$\frac{3}{2}(1-z^2)$	$\frac{6(-1)^{(n-1)}}{\Lambda_n^3}$	$\frac{6(-1)^{(n-1)}}{\Lambda_n^4 \sinh\lambda_n}\frac{L}{a}$	$y\left(1+3r_Q\sum_{n=1}^{\infty}\frac{(-1)^{(n-1)}}{\Lambda_n^4}\sin(\Lambda_n z)\right)$

The distribution $\beta_w(z)=1$, although very simple, has an undefined value for v_z at z =1.This is due to the disagreement between the boundary conditions at this only point. The uncertainty disappears in the limit $L/a\to\infty$ (long channel).

The other two injection laws, having null injection velocity at z = 1, are concordant and besides lead to more rapid convergence rate for the series.

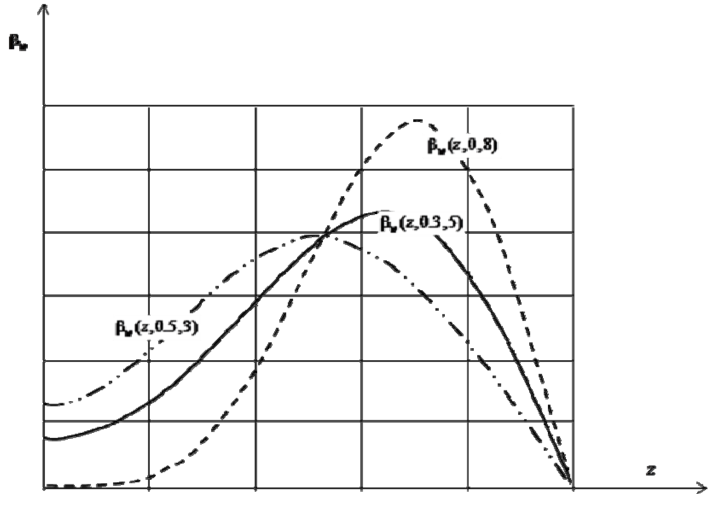

Fig. 2.-Injection velocity profiles considering 3 terms in series developments.

A solution using three terms. Let us consider the injection velocity of the form:

$$\beta_w(z,k,m)=\sum_{n=1}^{3}C_n(k,m)\cos(\Lambda_n z), \tag{29}$$

where the coefficients are:

$$C_3=\frac{15}{8}\left(\frac{k}{6}+\frac{m}{12}-\frac{\pi}{8}\right),\ C_2=C_3-\frac{m-k}{4},\ C_1=k-C_2-C_3. \tag{30}$$

The parameter k represents the value of $\beta_w(z,k,m)$ at channel entrance whereas the product $-\pi m/2$ represents the curve slope at z = 1.

In Fig.2 the curves $\beta_w(z,0.5,3)$, $\beta_w(z,0.3,5)$ and $\beta_w(z,0,8)$ are represented, suggesting a gradually increasing of the injection up to a maximum as one moves to the channel end. This configuration can be appropriate to some combustion problems [1;2].

5. CONCLUSIONS

A potential solution (satisfying identically the Helmholtz vortex equation unlike some the existing vortical solutions) is capable to take over an injection (or suction) normal to wall in a 2D incompressible flow. An analytical solution was obtained for stationary inviscid flow that can be used further as a frontier condition for calculation of the boundary layer. Limit behaviours for very long channels are obtained considering a finite ratio of the injection flow rate to the general one.

Interesting injection velocity profiles with gradually increasing up to a maximum in the last part of the channel can be obtained using a small number of series development.

6. ACKNOWLEDGEMENTS

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