Influence of drag force upon the shortest time trajectory of an aircraft

Andrei CRAIFALEANU*[,1], Roxana Alexandra PETRE[2]

*Corresponding author
*[,1]"POLITEHNICA" University of Bucharest, Department of Mechanics
313 Splaiul Independentei, room BN-01, postal code 060042, Bucharest, Romania,
ycraif@yahoo.com*
[2]"POLITEHNICA" University of Bucharest, Faculty of Aerospace Engineering
1-7 Street Polizu, postal code 011061, Bucharest, Romania,
petre.roxana.alexandra@gmail.com

DOI: 10.13111/2066-8201.2015.7.2.6

***3rd International Workshop on Numerical Modelling in Aerospace Sciences, NMAS 2015**,*
06-07 May 2015, Bucharest, Romania, (held at INCAS, B-dul Iuliu Maniu 220, sector 6)
Section 2 – Flight dynamics simulation

Abstract: *The shortest time trajectory of an aircraft between two given locations is determined using a simple mathematical model. By taking into account the drag force (viscous friction force with the air), a problem of variational calculus is obtained which consists in determining two functions that minimize a functional, subject to a non-holonomic constraint. The trajectory is determined directly, by numerical integration of Euler equations with multipliers. Three types of drag forces were considered: constant, linear and quadratic. The results are verified through comparison to the values obtained from numerical minimization of the involved functional approximate forms.*

Key Words: *extremal of a functional, non-holonomic constraint, constant drag, linear drag, quadratic drag, brachistochrone.*

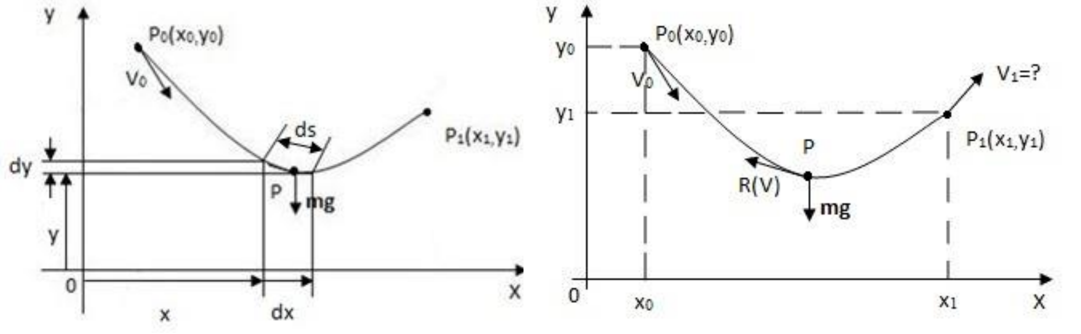
1. INTRODUCTION

An important problem of flight optimization consists in determining the shortest time trajectory of an aircraft, between two given locations, P_0 and P_1.

In a first order approximation, an aircraft in constant thrust flight can be modeled as a material point moving in a non-resistant medium, while an aircraft in glided flight can be considered as a material point moving in a resistant medium. This approximation reduces the optimization problem to a classical one, treated by the variational calculus, which consists in determining the curve between two fixed points, that is traveled in the minimum possible time by a material point moving under the action of gravity, starting with a given initial velocity, v_0 [1]. Such a curve is called brachistochrone (from the Greek words "brahistos", meaning *the shortest* and "chronos", which means *time*).

As shown in the literature, the brachistochrone problem in a non-resistant medium can be studied analytically [1, 2, 4, 5], while in a resistant medium, a semi-analytical solution is available [4]. However, in the later case, because of the complexity of the algebraic calculations, a numerical solution can prove more convenient in practical applications. Such an approach was used in [3], in order to determine parameterized forms of the optimal curves.

In the present paper, the brachistochrone in a resistant medium is determined directly, by numerically integrating the Euler equations with multipliers. Three types of drag forcers (friction forces dependent on the velocity) were considered: constant, linear and quadratic. The influence of the drag coefficient is evaluated in each case. The results are verified through comparison to the values obtained from numerical minimization of the involved functionals approximate forms.



a) b)

Fig. 1 Brachistochrone in non-resistant (a) and resistant (b) medium, respectively

From the mathematical point of view, the brachistochrone problem is equivalent to the determination of the function $y(x)$ that minimizes the functional (Fig. 1)

$$t_{01} = \int_{x_0}^{x_1} \mathrm{d}\,t = \int_{x_0}^{x_1} \frac{\mathrm{d}\,s}{v} = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{v}\,\mathrm{d}\,x\,, \tag{1}$$

where v is the velocity of the material point.

As shown in the following, the velocity is determined in different ways, according to the considered case.

2. THE CASE OF A NON-RESISTANT MEDIUM

If the motion takes place in a non-resistant medium, i.e. if the material point is acted only by the weight, neglecting the drag force (Fig. 1 a), the mechanical energy is conserved,

$$\mathrm{d}\left(\frac{mv^2}{2} + mgy\right) = 0\,, \tag{2}$$

where m is the mass of the material point.

The velocity can be determined as a function of the y coordinate:

$$v^2 = v_0^2 + 2g(y_0 - y)\,. \tag{3}$$

By replacing (3), functional (1) becomes

$$t_{01} = \frac{1}{2g}\int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}\,x\,, \tag{4}$$

where

$$F(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{e - y}}, \tag{5}$$

$$e = \frac{v_0^2}{2g} + y_0, \tag{6}$$

As known from the literature [1, 2, 4, 5], the function $y(x)$ that extremizes a functional of the form

$$J = \int_{x_0}^{x_1} F(x, y, y') \,\mathrm{d}\, x, \tag{7}$$

is the solution of the Euler equation:

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{8}$$

By replacing expression (5) in equation (8), the following second order differential equation is obtained:

$$y'' = \frac{1 + y'^2}{2(e - y)}. \tag{9}$$

This equation must be integrated with the boundary conditions

$$x = x_0 \quad \Rightarrow \quad y = y_0, \qquad x = x_1 \quad \Rightarrow \quad y = y_1. \tag{10}$$

3. THE CASE OF A RESISTANT MEDIUM

If the motion takes place in a resistant medium, i.e. if the material point is acted by the weight, as well as by a drag force, $R(v)$ (Fig. 1 b), the mechanical energy is not conserved, which means that, in order to obtain an equation determining the velocity, the theorem of the kinetic energy and work should be applied in its general form:

$$\mathrm{d}\left(\frac{mv^2}{2}\right) = -mg\,\mathrm{d}\, y - R(v)\,\mathrm{d}\, s. \tag{11}$$

Equation (11) is equivalent to

$$vv' + gy' + r(v)\sqrt{1 + y'^2} = 0, \tag{12}$$

where

$$r(v) = \frac{R(v)}{m}. \tag{13}$$

Since relation (12) is non-integrable, an expression of the velocity in terms of variable y and its derivative, y’, cannot be obtained. Therefore, in this case, the brachistochrone problem consists in determining the functions $y(x)$ and $v(x)$ that minimize the functional (1) subject to constraint (12).

The functional and the constraint, respectively, can be written as

$$t_{01} = \int_{x_0}^{x_1} F(x, y, y', v)\,\mathrm{d}\,x\,, \tag{14}$$

$$G(x, y, y', v, v') = 0\,, \tag{15}$$

where

$$F(x, y, y', v) = \frac{\sqrt{1 + y'^2}}{v}\,, \tag{16}$$

$$G(x, y, y', v, v') = vv' + gy' + r(v)\sqrt{1 + y'^2}\,. \tag{17}$$

As known from the literature [2, 4], the functions $y(x)$ and $v(x)$ that extremize a functional of the form

$$J = \int_{x_0}^{x_1} F(x, y, y', v, v')\,\mathrm{d}\,x\,, \tag{18}$$

subject to a non-holonomic constraint (a non-integrable constraint, dependent on the derivatives of the variables), of the form (15) represent the solution of the Euler equations with multipliers,

$$\begin{cases} \dfrac{\partial F^*}{\partial y} - \dfrac{\mathrm{d}}{\mathrm{d}\,t}\left(\dfrac{\partial F^*}{\partial y'}\right) = 0 \\ \dfrac{\partial F^*}{\partial v} - \dfrac{\mathrm{d}}{\mathrm{d}\,t}\left(\dfrac{\partial F^*}{\partial v'}\right) = 0\,, \\ G(x, y, y', v, v'), \end{cases} \tag{19}$$

where

$$F^*(x, y, y', v, v') = F(x, y, y', v, v') + \lambda(x)G(x, y, y', v, v'), \tag{20}$$

while $\lambda(x)$ is a supplementary unknown function, called Lagrange multiplier.

In order to write the boundary conditions for this system, it should be remarked that the velocity at the end of the flight is not imposed, i.e. $v_1 = v(x_1)$ is arbitrary. It can be shown [4] that this condition is equivalent to $\left.\dfrac{\partial F^*}{\partial v'}\right|_{x=x_1} = 0$.

Consequently, system (19) must be integrated with the boundary conditions

$$x = x_0 \quad \Rightarrow \quad \begin{cases} y = y_0 \\ v = v_0\,, \end{cases} \qquad x = x_1 \quad \Rightarrow \quad \begin{cases} y = y_1 \\ \dfrac{\partial F^*}{\partial v'} = 0\,. \end{cases} \tag{21}$$

By replacing expressions (16) and (17) in function (20) and this function in system (19), the following differential equations are obtained:

$$\begin{cases} y'' = \dfrac{g\left(1-\lambda v^2 r'\right)\left(1+y'^2\right)}{v^2\left(1+\lambda v r\right)} \\ v' = -\dfrac{g y' + r\sqrt{1+y'^2}}{v} \\ \lambda' = \dfrac{\left(\lambda v^2 r' - 1\right)\sqrt{1+y'^2}}{v^3}. \end{cases} \tag{22}$$

By denoting

$$z_1 = y, \quad z_2 = y', \quad z_3 = v, \quad z_2 = \lambda \tag{23}$$

system (22) can be transformed into a first order one, with four equations,

$$\begin{cases} z_1' = z_2 \\ z_2' = \dfrac{g\left(1 - r' z_3^2 z_4\right)\left(1+z_2^2\right)}{z_3^2\left(1 + r\, z_3 z_4\right)} \\ z_3' = -\dfrac{g\, z_2 + r\sqrt{1+z_2^2}}{z_3} \\ z_4' = \dfrac{\left(r' z_3^2 z_4 - 1\right)\sqrt{1+z_2^2}}{z_3^3}. \end{cases} \tag{24}$$

while boundary conditions (21) become:

$$x = x_0 \quad \Rightarrow \quad \begin{cases} u_1 = y_0 \\ u_3 = v_0, \end{cases} \qquad x = x_1 \quad \Rightarrow \quad \begin{cases} u_1 = y_1 \\ u_4 = 0. \end{cases} \tag{25}$$

4. AN APPROXIMATE METHOD

An alternative method for determining the optimal trajectory, i.e. the brachistochrone in a resistant medium, consists in approximating the function $y(x)$, by interpolation between a number of points $\left(\tilde{x}_0, \tilde{y}_0\right)$, $\left(\tilde{x}_1, \tilde{y}_1\right), \ldots,$ $\left(\tilde{x}_{n+1}, \tilde{y}_{n+1}\right)$, with the abscissas chosen in the interval $[x_0 , x_1]$,

$$x_0 = \tilde{x}_0 < \tilde{x}_1 < \ldots < \tilde{x}_{n+1} = x_1, \tag{26}$$

and with the ordinates determined so that functional (14) takes on its minimum value.

The interpolation can be made using spline functions.

It follows that

$$y(x) \cong \tilde{y}\left(x, \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n\right), \tag{27}$$

where the approximation function $\tilde{y}\left(x, \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n\right)$ satisfies the conditions

$$\tilde{y}\left(x_i, \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n\right) = \tilde{y}_i \quad \left(i = 1,2,\ldots,n\right). \tag{28}$$

For a given set of values $\tilde{y}_1$, $\tilde{y}_2$,…, $\tilde{y}_n$, an approximate form of the derivative $y'(x)$ can be found:

$$y'(x) \cong \tilde{y}'(x, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n) = \frac{\partial \tilde{y}}{\partial x}(x, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n). \quad (29)$$

An approximate form of the velocity can be also found, by replacing $\tilde{y}(x, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n)$ and $\tilde{y}'(x, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n)$ in condition (12) and by integrating with respect to x the resulted differential equation

$$\tilde{v}' = -\frac{g\tilde{y}' + r(v)\sqrt{1+\tilde{y}'^2}}{\tilde{v}}. \quad (30)$$

By substituting the approximate functions in (16) and the resulted function in (14), the approximate total flight time t_{01} results as a function of the ordinates of the interpolation points:

$$\tilde{t}_{01}(\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n) = \int_{x_0}^{x_1} F(x, \tilde{y}(x_i, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n), \tilde{y}'(x_i, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n), v(x, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n)) \mathrm{d}\, x. \quad (31)$$

These ordinates and, consequently, the approximate optimal trajectory, can be determined by finding the minimum of the unconstrained multivariable function (31).

5. NUMERICAL APPLICATION

The brachistochrones in a resistant medium were determined by numerical integration of Euler equations with multipliers, i.e. of system (24), as well as by minimization of the function (31).

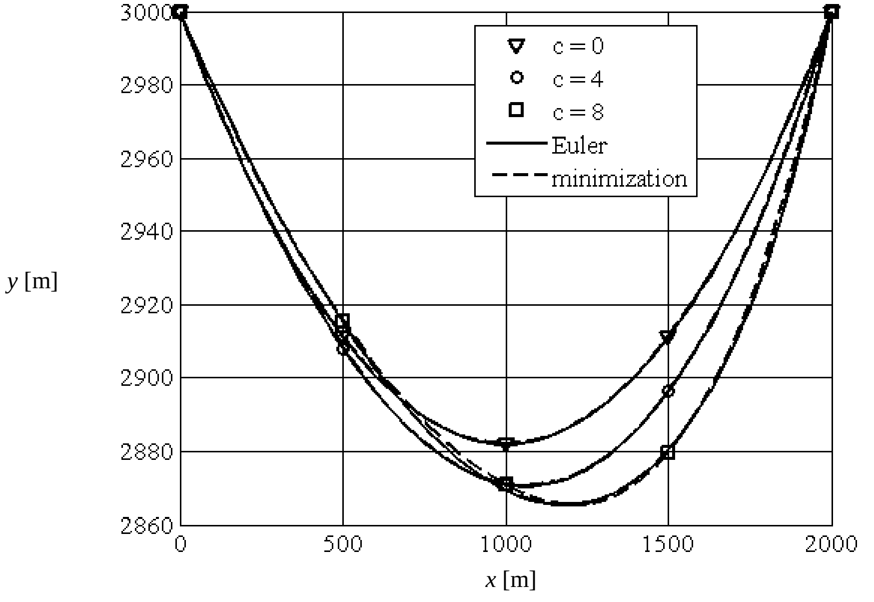


Fig. 2 The brachistochrone in a resistant medium, with constant drag force

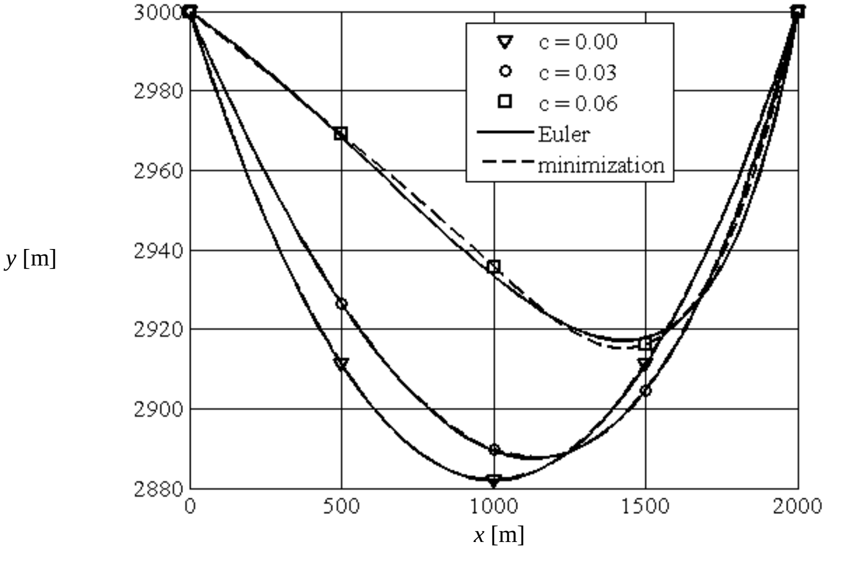


Fig. 3 The brachistochrone in a resistant medium, with linear drag force

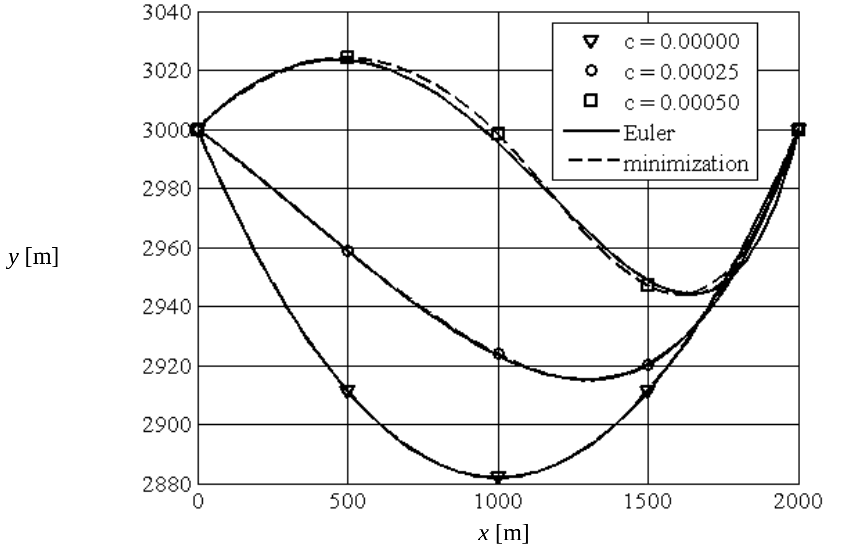


Fig. 4. The brachistochrone in a resistant medium, with quadratic drag force

Three types of drag forces were considered, namely:

- constant (Fig. 2),

$$r(v) = c, \tag{32}$$

- linear (Fig. 3),

$$r(v) = cv; \tag{33}$$

- quadratic (Fig. 4),

$$r(v) = cv^2. \tag{34}$$

The following numerical data were used: x_0 = 0m, y_0 =3000m, x_1 = 2000m, y_1 = 3000m, v_0 = 200m. For each type of drag force, three values of the drag coefficient c were considered, as shown in the figures above.

6. CONCLUSIONS

In order to study the influence of the drag upon the shortest time trajectory of an aircraft between two given locations, a simple mathematical model has been developed, leading to the problem of finding the brachistochrone in a resistant and in a non-resistant medium, respectively.

The brachistochrones were determined by numerically integrating the Euler equations with multipliers, as well as by searching for the parameters of an interpolated curve that minimizes the total flight time functional.

In all analyzed cases, the two methods provided close results.

For the constant and for the linear drag force, the trough of the optimal trajectory deepens as the drag coefficient increases, while for the quadratic drag force an opposite tendency is exhibited.

The aircraft model can be improved in a further study by taking into account the variation of the thrust and of the aerodynamic characteristics (drag and lift) with the flight parameters (velocity, height and attack angle).

REFERENCES

[1] B. Dacorogna, *Introduction to the calculus of variations*, Imperial College Press, London, ISBN: 186094499X, 1860945082, 2004.

[2] I. M. Gelfand, S. V. Fomin, *Calculus of variations*, Prentice-Hall, Englewood Cliffs, New Jersey, ISBN 10: 0486414485, ISBN 13: 9780486414485, 1963.

[3] P. Negrón–Marrero, Bárbara Santiago–Figueroa, *The Nonlinear Brachistochrone Problem with Friction*, University of Puerto Rico (Humacao), Department of Mathematics, Technical Rep. PR 00791-4300, 2005.

[4] V. Smirnov, *Cours de mathematiques superieures*, tome IV, premiere partie, Ed. Mir, Moscou, 1975.

[5] R. Weinstock, *Calculus of Variations with Applications to Physics and Engineering*, Dover Publications Inc., New York, ISBN-13: 000-0486630692, ISBN-10: 0486630692, 1974.