Invited Paper: Dedicated to the memory of Professor Elie Carafoli and to the 60th Anniversary of the Research Institute founded by Elie Carafoli presently named ***Institutul National de Cercetari Aerospatiale "Elie Carafoli"*** *This paper continues Professor Carafoli's tradition of theoretical studies.*

FLEXURAL OSCILLATIONS OF FLEXIBLE AIRFOILS IN SUBSONIC COMPRESSIBLE FLOWS

Dan MATEESCU
Doctor Honoris Causa, FCASI, AFAIAA, Erskine Fellow
Professor, Aerospace Program, Mechanical Engineering Department,
McGill University, Montreal, QC, Canada, dan.mateescu@mcgill.ca
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Abstract

This paper presents simple and efficient analytical solutions for unsteady subsonic compressible flows past flexible airfoils executing low frequency oscillations. These analytical solutions are obtained with a method using velocity singularities related to the airfoil leading edge and ridges. The method has been validated for the pitching and plunging oscillations of the rigid airfoils by comparison with results based on Jordan's data for compressible flows and by comparison with the solutions obtained by Theodorsen, Postel & Leppert and Mateescu & Abdo for incompressible flows. The method has been applied to obtain efficient analytical solutions for the flexural oscillations of airfoils in compressible flows, which can be efficiently used in solving aeroelastic problems.

Professor Carafoli's contributions to the aerodynamics of airfoils and wings

Starting from 1924, in Paris, Elie Carafoli made significant contributions to the aerodynamic studies of wings and airfoils, included in *Aerodynamics of airplane wings* [5], *Theory of lifting airfoils* [6], in which he presented the aerodynamics of airfoils with rounded trailing edge, named later *Carafoli airfoils*, *Influence of ailerons on the aerodynamic characteristics of lifting surfaces* [7], and in *Experimental studies on monoplane wings* [8]. After 1928, he continued his research and academic activity in Romania, bringing new contributions to the aerodynamics of wings and airfoils in incompressible flows, which were mostly included in *Aerodynamics* [9], book also translated from Romanian into German and Russian. After 1950, Elie Carafoli developed a special interest for compressible flows and published *High Speed Aerodynamics* [10], initially in Romanian and then translated in English. He made, together with his collaborators, important contributions on the aerodynamics of wings and airfoils in supersonic flows, most of them included in the monograph *Wing Theory in Supersonic Flow* [11] published by Pergamon Press, to which the present author also collaborated.

For his outstanding achievements in aerodynamics, Elie Carafoli received very numerous international distinctions and awards, such as Diploma "Paul Tisandier" of the International Aeronautical Federation, Gauss Medal attributed yearly to the best scientist in the world by the Scientific Society Braunschweig, Germany, Apollo 11 Medal, Tsiolkovski Medal and many others distinctions which can not be listed here due to the space limitation. Elie Carafoli was one of the few Honorary Fellows of the Royal Aeronautical Society, Great Britain, and one of the first members of the International Academy of Astronautics (proposed by Theodor von Karman), and he was elected first Vice-President and then President of the International Astronautic Federation. In Romania, Carafoli was elected member of the Romanian Academy in 1948, and later named President of the Technical Sciences Section of the Academy, and he received the greatest Romanian medals, prizes and awards.

One of the important realizations of Academician Carafoli was the foundation in 1949 of a research institute of the Romanian Academy, known as the *Institute of Fluid Mechanics*, which is presently named the *National Institute for Aerospace Research "Elie Carafoli"*. This institute contributed to the scientific formation of numerous former students and collaborators of Professor Carafoli, who distinguished themselves by contributions in aerodynamics and other related aerospace domains. Many of the former students and collaborators of Professor Carafoli became professors at their turn in Romania, as well as in other countries, such as Canada, France, Germany and United States of America, to name only a few.

The present author had the privilege to study Aerodynamics with Professor Carafoli, who was also the Ph. D. supervisor, and to be his close collaborator for more than two decades. He is indebted to Professor Carafoli for many research advices, which materialized not only in many publications resulted from their direct scientific collaboration, but also in his further research and academic activity, first in Romania, and since 1982 in Canada. This author, together with his Masters' and Ph. D. students, aims to continue the best scientific tradition of Professor Carafoli, with new contributions to the aerodynamics of wings and airfoils in steady and unsteady flows in incompressible, compressible subsonic and supersonic regimes and at low Reynolds numbers, in addition to other contributions in related aerospace domains (a few selected examples from the last ten years are included in references [20–37, 43].

Introduction

The analysis of the unsteady flows past oscillating airfoils and wings has been mostly motivated by the efforts made to avoid or reduce undesirable unsteady effects in aeronautics, such as flutter, buffeting, and dynamic stall. Potentially beneficial effects of these unsteady flows have also been studied, such as propulsive efficiency of flapping motion, controlled periodic vortex generation, stall delay, and optimal control of unsteady forces to improve the performance of turbomachinery, helicopter rotors, and wind turbines. The foundations of the unsteady aerodynamics of oscillating airfoils have been established by Theodorsen [44], Theodorsen & Garrick [45], Wagner [48], Kussner [18, 19], and von Karman & Sears [47], who studied the unsteady flow past an oscillating thin

flat plate and a trailing flat wake of vortices in incompressible flows. This problem has also been studied in Carafoli's *Aerodynamics* [9]. Further studies involving detailed unsteady flow solutions of oscillating airfoils have been performed by Postel and Leppert [42], Fung [14], Bisplinghoff & Ashley [4], McCroskey [39, 40], Kemp & Homicz [17], Basu & Hancock [3], Dowell [12] and Dowell et al. [13] and others.[2, 14, 16]. Some of the recent unsteady aerodynamic studies used panel methods or numerical methods based on finite difference, finite volume or spectral formulations [12, 21, 23, 27, 33, 34, 38].

The aim of this paper is to present simple and efficient analytical solutions for unsteady subsonic compressible flows past flexible airfoils executing low frequency oscillations. These analytical solutions are obtained with a method using velocity singularities related to the airfoil leading edge and ridges (defined by the changes in the boundary conditions on the airfoil). This method of velocity singularities (different from Theodorsen's method for incompressible flows based on singularities in the velocity potential) represents an extension to compressible flows of the method developed for incompressible flows by Mateescu & Abdo [35].

The analytical solutions obtained with this method are simple, efficient and in closed form (in contrast with complicated previous solutions based on Fourier series expansions, approximate polynomial fitting for each Mach number and on the numerical evaluations of several integrals [15, 46]). These efficient solutions are particularly suitable for the aeroelastic studies, in which the unsteady aerodynamic analysis is performed in conjunction with the analysis of the related structural motion involving oscillatory flexural deformations. Although potentially more accurate, a complete numerical approach to solve simultaneously the structural equations of motion and the unsteady Navier-Stokes or Euler equations governing the unsteady flows (using finite difference or finite volume formulations, which involve numerous iterations for each real time step [41]) requires a substantially large computational effort in terms of computing time and memory, even with the present computing capabilities. For this reason, efficient analytical solutions in closed form can be useful in the aeroelastic studies.

The method is validated for the case of rigid airfoil oscillations in unsteady compressible and incompressible flows. The solutions obtained with this method are found to be in good agreement with the results based on Jordan's data [15] for oscillating rigid airfoils in unsteady compressible flows, and in perfect agreement with the results obtained for the limit case of unsteady incompressible flows ($M \to 0$) by Theodorsen [44], Postel & Leppert [42] and by Mateescu & Abdo [35].

Detailed analytical solutions of the unsteady pressure coefficient distribution on the airfoil are obtained with this method for the case of flexural oscillations in compressible flows. The variations of these coefficients with the Mach number and the reduced frequency of oscillations are also presented. These analytical solutions obtained are computationally very efficient and can be successfully used in the aeroelastic studies.

Problem formulation

Consider a thin airfoil of chord c placed in a steady uniform air flow defined by the velocity U_∞ and by the corresponding pressure, p_∞, density, ρ_∞, speed of sound, a_∞,

and Mach number $M_\infty = U_\infty / a_\infty$. The airfoil executes harmonic flexural oscillations about a mean position situated along the axis *Ox*, which are defined in complex form by the general equation

$$y = e(x,t) = \hat{e}(x)\exp(\mathrm{i}\omega t), \tag{1}$$

where $\hat{e}(x)$ represents the modal amplitude of oscillations, usually expressed in the general polynomial form

$$\hat{e}(x) = \sum_{n=0}^{N+1} e_n x^n , \tag{2}$$

and where x and y are dimensionless Cartesian coordinates (with respect to the airfoil chord, c) with the origin at the airfoil leading edge, and $\exp(\mathrm{i}\omega t) = \cos\omega t + \mathrm{i}\sin\omega t$, in which ω is the radian oscillation frequency, and $\mathrm{i} = \sqrt{-1}$. In the case of the rigid airfoil oscillations, the modal amplitude of oscillation can be expressed as $\hat{e}(x) = \hat{h} - (x-a)\hat{\theta}$, where $h(t) = \hat{h}\exp(\mathrm{i}\omega t)$ defines the airfoil oscillations in translation normal to its chord, and $\theta(t) = \hat{\theta}\exp(\mathrm{i}\omega t)$ defines the pitching oscillations about an articulation situated at $x = a$. In the above expressions, the reduced quantities marked by a caret, such as $\hat{\theta}$, $\hat{h}$ and $\hat{e}$ are in general complex numbers defining the amplitude and the relative phase.

The boundary condition on this thin airfoil executing small amplitude oscillations can be expressed, as shown in our previous paper [35], in the form

$$v(x,t) = \partial e/\partial t + (1+u)(\partial e/\partial x) = \hat{F}(x)\exp(\mathrm{i}\omega t), \tag{3a}$$

$$\hat{F}(x) = \mathrm{i}2k\hat{e}(x) + \frac{\partial \hat{e}}{\partial x}, \qquad k = \frac{\omega c}{2U_\infty}, \tag{3b}$$

where $U_\infty u = U_\infty(\partial\varphi/\partial x)$ and $U_\infty v = U_\infty(\partial\varphi/\partial y)$ are the Cartesian components of the perturbation velocity, deriving from the perturbation velocity potential $U_\infty c\,\varphi(x,y,t)$, and $k = \omega c/(2U_\infty)$ is the reduced frequency of oscillations which is assumed small. In compressible flows, the equation of the perturbation velocity potential can be expressed in the linear form (Carafoli, Mateescu and Nastase [11])

$$\left(1 - M_\infty{}^2\right)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{1}{a_\infty{}^2}\left[\frac{\partial^2\varphi}{\partial t^2} + 2U_\infty\frac{\partial^2\varphi}{\partial t\,\partial x}\right], \tag{4}$$

and the unsteady pressure coefficient, $C_p(x,y,t)$, is defined by the corresponding linear equation obtained from the Bernoulli-Lagrange equation [35]

$$C_p(x,y,t) = -2u(x,y,t) - (2c/U_\infty)(\partial\varphi/\partial t). \tag{5}$$

Method of solution

Problem reduction to an equivalent steady flow

By introducing the reduced velocity potential $\hat{\varphi}(x,y)$ defined by the potential transformation

$$\varphi(x,y,t)=\hat{\varphi}(x,y)\exp(\mathrm{i}\,2k\,m\,x)\exp(\mathrm{i}\,\omega\,t), \qquad m=\frac{M_\infty{}^2}{1-M_\infty{}^2}, \tag{6}$$

equation (4) can be reduced, in the case of low frequency oscillations ($k^2 m << 1$), to the steady flow form

$$\beta^2\frac{\partial^2\hat{\varphi}}{\partial x^2}+\frac{\partial^2\hat{\varphi}}{\partial y^2}=0, \qquad \beta=\sqrt{1-M_\infty{}^2}. \tag{7}$$

The velocity potential transformation (6) leads to the reduced perturbation velocity components, $\hat{u}$ and $\hat{v}$, defined as

$$\hat{u}(x,y)=\frac{\partial\hat{\varphi}}{\partial x}, \qquad u(x,y,t)=\left[\hat{u}(x,y)+\mathrm{i}\,2k\,m\,\hat{\varphi}(x,y)\right]\exp(\mathrm{i}\,2k\,m\,x)\exp(\mathrm{i}\,\omega\,t), \tag{8}$$

$$\hat{v}(x,y)=\frac{\partial\hat{\varphi}}{\partial y}, \qquad v(x,y,t)=\hat{v}(x,y)\exp(\mathrm{i}\,2k\,m\,x)\exp(\mathrm{i}\,\omega\,t), \tag{9}$$

and the boundary condition (3) on the oscillating airfoil can be recast as

$$\hat{v}=\hat{F}(x)\exp(-\mathrm{i}\,2k\,m\,x), \tag{10}$$

By using convenient transformations of the coordinates and the reduced potential, in the form

$$X=x, \qquad Y=\beta\,y, \qquad \Phi(X,Y)=\beta\,\hat{\varphi}(x,y), \tag{11}$$

equation (7) can be reduced to the Laplace equation

$$\nabla^2\Phi=\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0, \tag{12}$$

which is satisfied also by the velocity components in this transformed plane

$$U(X,Y)=\frac{\partial\Phi}{\partial X}, \qquad \hat{u}(x,y)=\frac{1}{\beta}\left[U(X,Y)+\mathrm{i}\,2k\,m\Phi(X,Y)\right]\exp(\mathrm{i}\,2k\,m\,X), \qquad \nabla^2 U=0, \tag{13}$$

$$V(X,Y)=\frac{\partial\Phi}{\partial Y}, \qquad \hat{v}(x,y)=V(X,Y)\exp(\mathrm{i}\,2k\,m\,X), \qquad \nabla^2 V=0, \tag{14}$$

In this transformed plane, the velocity components are harmonic functions and can be expressed as the real and imaginary parts of the complex conjugate velocity

$$w(Z)=f'(Z)=U(X,Y)-\mathrm{j}V(X,Y), \qquad Z=X+\mathrm{j}Y, \tag{15}$$

where $\mathrm{j}=\sqrt{-1}$ and where $f(Z)$ is the complex potential in this plane defined as

$$f(Z)=\int w(Z)dZ, \tag{16}$$

which is related to the dimensionless perturbation velocity potential $\varphi(x,y,t)$ by the equation

$$\mathrm{Re}_{\mathrm{j}}\{f(Z)\}=\Phi(X,Y)=\beta\,\hat{\varphi}(x,y)=\beta\,\varphi(x,y,t)\exp(-\mathrm{i}\,2k\,m\,x)\exp(-\mathrm{i}\,\omega\,t). \tag{17}$$

The boundary condition on the oscillating airfoil can thus be expressed in this plane in the complex form

$$-V(X,0)=\mathrm{Im_j}\left\{w(Z)\right\}_{Z=X}=-F(X), \tag{18}$$

where

$$F(X)=\hat{F}(X)\exp(-\mathrm{i}2km\,X)=\left[\hat{e}'(X)+\mathrm{i}2k\,\hat{e}(X)\right]\exp(-\mathrm{i}2km\,X), \tag{19}$$

which can be expressed in the case of low frequency oscillations as

$$F(X)=\left[\hat{e}'(X)+\mathrm{i}2k\,\hat{e}(X)\right]\left[1-i2km\,X-2k^2m^2X^2\right]=\sum_{n=0}^{N+1}f_nX^n\ , \tag{20}$$

in which

$$f_n=(n+1)e_{n+1}+\mathrm{i}2k(1-nm)e_n+2k^2m\left[2-(n-1)m\right]e_{n-1}\ ,\quad \text{with}\quad e_n=0\ \text{for}\ n>N\ . \tag{21}$$

Thus, the unsteady problem in compressible flow has been reduced to the solution of an equivalent reduced steady incompressible flow (with more complex boundary conditions), which can be solved in a similar manner to that indicated in our previous paper [35] devoted to the solution of unsteady incompressible flows.

The boundary condition upstream of the airfoil on the extension of its chord [35] is

$$U(X,0)=\mathrm{Re_j}\left\{w(Z)\right\}_{Z=X}=0\,,\qquad \text{for}\quad Z=X<0\,. \tag{22}$$

Shedding vortices analysis and boundary condition on the wake

The intensity of the free vortices shed at the trailing edge ($x=1$) of the oscillating airfoil at time t can be determined from Kelvin's theorem for a closed material contour in the form

$$\gamma_{\mathrm{f}}(1,t)=-\frac{d\,\Gamma_C}{c\,dx}=-\frac{1}{U_\infty}\frac{d\,\Gamma_C}{d\,t}\,, \tag{23}$$

where $\Gamma_C(t)$ is the circulation around the oscillating airfoil [14] defined as

$$\Gamma_C(t)=2cU_\infty\int_0^1 u(x,0,t)\,dx=cU_\infty\,\hat{\Gamma}_C\exp(\mathrm{i}\,\omega\,t), \tag{24}$$

in which the reduced circulation $\hat{\Gamma}_C$ is defined as

$$\hat{\Gamma}_C=\frac{2}{\beta}\int_0^1\exp(\mathrm{i}2km\,X)\,\mathrm{Re_j}\left\{w(Z)+\mathrm{i}2k\,f(Z)\right\}_{Z=X}\,dX\ . \tag{25}$$

Based on Helmholtz' circulation theorem, the free vortices maintain their intensity while they are transported downstream by the moving fluid, and thus the intensity of a distributed free vortex situated at time t in the wake at the location $c\,x=c\,\sigma$ is equal to that issued at the trailing edge at a previous time $t-\Delta t$, where the time lag is $\Delta t=c(\sigma-1)/U_\infty$. Hence,

$$\gamma_{\mathrm{f}}(\sigma,t)=-\mathrm{i}2kU_\infty\,\hat{\Gamma}_C\,\exp\left[\mathrm{i}\,\omega\,t-\mathrm{i}2(\sigma-1)\right]. \tag{26}$$

By considering an elementary contour around an infinitesimal length $c\,d\sigma$ of the wake, as shown in our previous paper [35], the distributed free vortices intensity can be expressed as

$$\gamma(\sigma,t)=2U_\infty\, u(x,0,t)=2U_\infty \frac{1}{\beta}\left[U(X,0)+\mathrm{i}\,2k\,m\,\Phi(X,0)\right]\exp(\mathrm{i}\,2k\,m\,x)\exp(\mathrm{i}\,\omega\,t), \quad (27)$$

and thus the resulting boundary condition on the wake of the oscillating airfoil is

$$\left[U(X,0)+\mathrm{i}\,2k\,m\,\Phi(X,0)\right]_{X=\sigma}=-\mathrm{i}\,k\,\beta\,\hat{\Gamma}_C\,\exp(-\mathrm{i}\,2k\,m\,X)\exp\left[-\mathrm{i}\,2K(\sigma-1)\right], \quad (28)$$

where $K=k(1+m)$. Since $U(X,0)=(\partial/\partial X)\Phi(X,0)$, equation (28) can be viewed as a differential equation in $\Phi(X,0)$, with the following solution which represents the boundary condition on the wake:

$$U(\sigma,0)=-\mathrm{i}\,K\,\beta\,\hat{\Gamma}_C\,\exp(-\mathrm{i}\,2k\,m)\exp\left[-\mathrm{i}\,2K(\sigma-1)\right], \qquad K=k(1+m)\,. \quad (29)$$

Contribution of the free vortices in the expression of $w(Z)$

The boundary conditions for the contribution $\hat{\Gamma}_C\,w_w(Z)$ of the free vortices in the wake defined by the reduced circulation $\hat{\Gamma}_C$ around the airfoil, can be expressed, taking into account (22) and (29), in the complex form

$$\mathrm{Re_j}\left\{\hat{\Gamma}_C\,w_w(Z)\right\}_{Z=X}=-\mathrm{i}\,K\,\beta\,\hat{\Gamma}_C\,\exp(-\mathrm{i}\,2k\,m)\exp\left[-\mathrm{i}\,2K(\sigma-1)\right], \quad \text{for } Z=X>1, \quad (30a)$$

$$\mathrm{Re_j}\left\{\hat{\Gamma}_C\,w_w(Z)\right\}_{Z=X}=0, \quad \text{for } Z=X<0, \quad (30b)$$

$$\mathrm{Im_j}\left\{\hat{\Gamma}_C\,w_w(Z)\right\}_{Z=X}=0, \quad \text{for } Z=X\in[0,1]. \quad (30c)$$

The solution of this problem, $\hat{\Gamma}_C\,w_w(Z)$, can be obtained in a similar manner to that presented in our previous paper [35] in the form

$$w_w(Z)=-\frac{1}{2}\,\beta\,\exp(-\mathrm{i}\,2k\,m)\int_1^\infty \exp\left[-\mathrm{i}\,2K(\sigma-1)\right]\widetilde{H}(Z,\sigma)\,d\sigma, \quad (31)$$

where

$$\widetilde{H}(Z,S)=\frac{2}{\pi}\cos^{-1}\sqrt{\frac{(1-Z)S}{S-Z}}\,. \quad (32)$$

Solution of the prototype unsteady problem

Consider first the prototype unsteady problem, corresponding to a sudden change δF_0 in the boundary condition on the airfoil at $Z=S$, which is defined by the boundary conditions

$$\mathrm{Im_j}\left\{\delta w(Z)\right\}_{Z=X}=-f_0\,, \qquad \text{for } Z=X\in[0,S]\,, \quad (33a)$$

$$\mathrm{Im_j}\left\{\delta w(Z)\right\}_{Z=X}=-f_0+\delta F_0\,, \qquad \text{for } Z=X\in[S,1], \quad (33b)$$

$$\mathrm{Re_j}\left\{\delta w(Z)\right\}_{Z=X}=-\mathrm{i}\,K\,\beta\,\delta\hat{\Gamma}_C\,\exp(-\mathrm{i}\,2k\,m)\exp\left[-\mathrm{i}\,2K(\sigma-1)\right], \quad \text{for } Z=X>1, \quad (33c)$$

$$\mathrm{Re_j}\left\{\delta w(Z)\right\}_{Z=X}=0, \qquad \text{for } Z=X<0, \quad (33d)$$

where δF_0 defines a jump in the airfoil boundary condition located at the ridge $Z=X=S$, and $\delta\hat{\Gamma}_C$ represents the reduced circulation around the airfoil in this case.

The solution for this problem can be expressed in a similar form to that obtained in our previous paper [35]

$$\delta w(Z) = -\mathrm{j}\, f_0 + \delta A \sqrt{\frac{1-Z}{Z}} - \delta F_0\, \widetilde{G}(Z,S) + \delta \hat{\Gamma}_C\, w_w(Z), \tag{34}$$

in which $\widetilde{G}(Z,S)$ represents the contribution of the ridge situated at $Z = S$ (where the boundary condition changes) defined as

$$\widetilde{G}(Z,S) = \frac{2}{\pi} \cosh^{-1} \sqrt{\frac{(1-Z)S}{S-Z}}\,. \tag{35}$$

The constant δA can be determined from the condition at infinity where the perturbation velocity components become zero [35] resulting

$$\delta A = -f_0 - \delta F_0\, C(S) + 2K^2 \beta\, \delta \hat{\Gamma}_C \exp(-\mathrm{i}\, k\, m) \int_1^\infty \exp\left[\mathrm{i}\, 2K(\sigma - 1)\right] \frac{2}{\pi} \cosh^{-1} \sqrt{\sigma}\; d\sigma\,, \tag{36}$$

where

$$C(S) = \frac{2}{\pi} \cos^{-1} \sqrt{S}\,. \tag{37}$$

The solution for this prototype unsteady problem can thus be expressed as

$$\delta w(Z) = -f_0 \left(\sqrt{\frac{1-Z}{Z}} + \mathrm{j} \right) - \delta F_0 \left[\widetilde{G}(Z,S) - C(S) \sqrt{\frac{1-Z}{Z}} \right] - \mathrm{i}\, K \beta\, \delta \hat{\Gamma}_C \exp(-\mathrm{i}\, 2k\, m)\, J(Z), \tag{38a}$$

$$J(Z) = \int_1^\infty \exp\left[\mathrm{i}\, 2K(\sigma - 1)\right] \frac{1}{\pi} \frac{1}{\sigma - Z} \sqrt{\frac{1-Z}{Z}} \sqrt{\frac{\sigma}{\sigma - 1}}\; d\sigma\,. \tag{38b}$$

The solution for $\delta f(Z) = \int \delta w(Z) dZ$, defined by an equation similar to (16), is then obtained in the form

$$\delta f(Z) = -\left[f_0 + \delta F_0\, C(S)\right] \left[-\frac{\pi}{2} C(Z) + \sqrt{(1-Z)Z} \right] - \mathrm{j}\, f_0\, Z$$
$$- \delta F_0 \left[(Z - S) \widetilde{G}(Z,S) - \sqrt{(1-S)S}\; C(Z) \right] - \mathrm{i}\, K \beta\, \delta \hat{\Gamma}_C \exp(-\mathrm{i}\, 2k\, m) \widetilde{J}(Z)\,, \tag{39a}$$

$$\widetilde{J}(Z) = \int_1^\infty \exp\left[\mathrm{i}\, 2K(\sigma - 1)\right] \left[-\widetilde{H}(Z,\sigma) - \sqrt{\frac{\sigma}{\sigma - 1}}\; C(Z) \right] d\sigma\,. \tag{39b}$$

Determination of the reduced circulation $\delta \hat{\Gamma}_C$

The reduced circulation for the prototype unsteady problem is defined, similarly to (25), as

$$\delta \hat{\Gamma}_C = \frac{2}{\beta} \int_0^1 \exp(\mathrm{i}\, 2k\, m\, X)\, \mathrm{Re}_{\mathrm{j}} \left\{ \delta w(Z) + \mathrm{i}\, 2k\, \delta f(Z) \right\}_{Z=X} d\, X\,, \tag{40}$$

which leads, after performing the integration in the case of low frequency oscillations, to the expression

$$\delta\hat{\Gamma}_C = \frac{\pi}{\beta}\frac{\mathrm{i}\exp(\mathrm{i}\,2k\,m)}{2K\,B_1(K)}\left[\,f_0 + \delta F_0\,C(S) + \delta F_0\,\frac{2}{\pi}\sqrt{(1-S)S}\,\right], \tag{41}$$

where

$$B_1(K) = -\frac{\pi}{4}\exp(\mathrm{i}K)\left[H_1^{(2)}(K) + \mathrm{i}\,H_0^{(2)}(K)\right]$$

$$= -\frac{\pi}{4}\exp(\mathrm{i}K)\left\{\left[J_1(K) + Y_0(K)\right] + \mathrm{i}\left[J_0(K) - Y_1(K)\right]\right\}, \tag{42}$$

in which $H_n^{(2)}(K)$ denotes the Hankel functions of second kind and of order n, and $J_n(K)$ and $Y_n(K)$ are, respectively, the Bessel functions of the first and second kind and order n.

Reduced pressure coefficient

The unsteady pressure coefficient for the flow past the oscillating airfoil can be expressed, using (5), (8), (13), (15) and (17), in the form

$$C_p(x,y,t) = \hat{C}_p(x,y)\exp(\mathrm{i}\omega t), \tag{43}$$

where $\hat{C}_p(x,y)$ is the reduced pressure coefficient defined as

$$\hat{C}_p(x,y) = -\frac{2}{\beta}\exp(\mathrm{i}\,2k\,m\,X)\mathrm{Re}_{\mathrm{j}}\left\{g(Z)\right\}_{Z=X}, \qquad g(z) = w(Z) + \mathrm{i}\,2K\,f(Z). \tag{44}$$

For the prototype unsteady problem, one results

$$\delta g(Z) = -\left[f_0 + \delta F_0\,C(S)\right]\left[1 + \mathrm{i}\,2K\,Z + D(K)\right]\sqrt{\frac{1-Z}{Z}} - \mathrm{j}\,f_0\left(1 + \mathrm{i}\,2K\,Z\right)$$

$$-\delta F_0\left\{\left[1 + \mathrm{i}\,2K\,(Z-S)\right]\widetilde{G}(Z,S) + \frac{2}{\pi}D(K)\sqrt{(1-S)S}\sqrt{\frac{1-Z}{Z}}\right\}, \tag{45}$$

and hence

$$\delta C_p(x) = \frac{2}{\beta}\exp(\mathrm{i}\,2k\,m\,x)\left\{\left[f_0 + \delta F_0\,C(S)\right]\left[1 + \mathrm{i}\,2K\,x + D(K)\right]\sqrt{\frac{1-x}{x}}\right.$$

$$\left. + \delta F_0\left\{\left[1 + \mathrm{i}\,2K\,(x-S)\right]G(x,S) + \frac{2}{\pi}\sqrt{(1-S)S}\,D(K)\sqrt{\frac{1-x}{x}}\right\}\right\}, \tag{46}$$

where

$$D(K) = \frac{-\mathrm{i}\,H_0^{(2)}(K)}{H_1^{(2)}(K) + \mathrm{i}\,H_0^{(2)}(K)} = C_T(K) - 1, \qquad K = \frac{k}{1 - M_\infty{}^2}, \tag{47a}$$

$$G(x,S)=\mathrm{Re}_{\mathrm{j}}\left\{\widetilde{G}(Z,S)\right\}=\begin{cases}\dfrac{2}{\pi}\cosh^{-1}\sqrt{\dfrac{(1-x)S}{S-x}} & \text{for} \quad x\in[0,S]\\ \dfrac{2}{\pi}\sinh^{-1}\sqrt{\dfrac{(1-x)S}{x-S}} & \text{for} \quad x\in[S,1]\\ 0 & \text{for} \quad x<0 \quad \text{and} \quad x>1\end{cases} \tag{47b}$$

in which $C_T(K)$ denotes here the Theodorsen function, but of a different argument than in incompressible flows, $K=k(1+m)=k/\left(1-M_\infty{}^2\right)$.

Lift and Pitching Moment Coefficients

The lift and pitching moment (with respect to the leading edge) coefficients for the oscillating airfoil are defined as

$$C_L(t)=\hat{C}_L\ \exp(\mathrm{i}\,\omega\, t), \qquad \hat{C}_L=-2\int_0^1\hat{C}_p(x)dx\,, \tag{48a}$$

$$C_m(t)=\hat{C}_m\exp(\mathrm{i}\,\omega\, t), \qquad \hat{C}_m=-2\int_0^1 x\hat{C}_p(x)dx\,. \tag{48b}$$

For the prototype unsteady problem in the case of low frequency oscillations, one obtains thus

$$\delta\hat{C}_L=-\frac{2\pi}{\beta}\left[f_0+\delta F_0\,C(S)+\delta F_0\frac{2}{\pi}\sqrt{(1-S)S}\right]\left\{1+D(K)+\mathrm{i}\frac{k}{2}\left[1+2m+mD(K)\right]-\right.$$
$$\left.-\frac{m}{4}k^2\left[2+3m+mD(K)\right]\right\}+\frac{4}{\beta}\,\delta F_0\sqrt{(1-S)S}\left[\mathrm{i}\,k+\frac{m}{3}k^2(2+3m)S(1+2S)\right], \tag{49a}$$

$$\delta\hat{C}_m=-\frac{\pi}{2\beta}\left[f_0+\delta F_0\,C(S)+\delta F_0\frac{2}{\pi}\sqrt{(1-S)S}\right]\left\{1+D(K)+\mathrm{i}\,k+\mathrm{i}\,k\,m\left[2+D(K)\right]-\right.$$
$$\left.-\frac{5}{8}k^2m\left[2+3m+mD(K)\right]\right\}-$$
$$-\frac{1}{\beta}\delta F_0\sqrt{(1-S)S}\,S\left\{2-\mathrm{i}\,k\frac{2}{3}(1-m)(1-2S)-k^2m(2+3m)\left(\frac{3}{4}+S+2S^2\right)\right\} \tag{49b}$$

Solution of the complete problem

The solution of the complete compressible flow problem of an oscillating airfoil can be obtained by considering a continuous distribution of elementary ridges along the airfoil chord to model the boundary condition (18) – (20). For each of these elementary ridges situated at $Z=S$, the change δF_0 in the boundary conditions is thus

$$\delta F_0=\left[\frac{dF(X)}{dX}\right]_{X=S}dS=F'(S)\,dS=\sum_{n=0}^{N+1}n\,f_n\,S^{n-1}\,dS\,, \tag{50}$$

with the coefficients f_n defined by (21) in function of modal oscillation amplitude $\hat{e}(x)$.

By introducing the expression (50) of δF_0 in the unsteady prototype solutions (46) and (49) for the pressure, lift and moment coefficients and performing the integration in S between $S=0$ and $S=1$, one obtains the solutions of the unsteady pressure difference coefficient between the lower and upper sides of the oscillating airfoil, in the form

$$\Delta C_p(x,t) = -2C_p(x,t) = \Delta\hat{C}_p(x)\exp(\mathrm{i}\,\omega\, t), \tag{51}$$

as well as the unsteady lift and pitching moment (with respect to the leading edge) coefficients, expressed in the form

$$C_L(t) = \hat{C}_L\ \exp(\mathrm{i}\,\omega\, t), \qquad C_m(t) = \hat{C}_m\ \exp(\mathrm{i}\,\omega\, t). \tag{52}$$

The resulting general expressions for the reduced coefficients $\Delta\hat{C}_p(x)$, $\hat{C}_L$ and $\hat{C}_m$ are

$$\Delta\hat{C}_p(x) = -\frac{4}{\beta}\exp(\mathrm{i}\,2k\,m\,x)\sqrt{\frac{1-x}{x}}\sum_{n=0}^{N+1} f_n\left\{\frac{2n+1}{n+1}g_n D(K) + \sum_{q=0}^{n} g_{n-q}\left[1+\frac{\mathrm{i}\,2k(1+m)}{n+1}x\right]x^q\right\}, \tag{53}$$

$$\hat{C}_L = -\frac{2\pi}{\beta}\sum_{n=0}^{N+1} f_n g_n \frac{2n+1}{n+1}\left\{1+\frac{\mathrm{i}\,k}{n+2}+\mathrm{i}\,m\,k+\left[1+\frac{m}{4}k(2\mathrm{i}-m\,k)\right]D(K)-\frac{3}{4}m\,k^2(2+3m)\frac{n+1}{n+3}\right\}, \tag{54}$$

$$\hat{C}_m = -\frac{\pi}{2\beta}\sum_{n=0}^{N+1} f_n g_n \frac{2n+1}{n+1}\left\{\frac{2n+2}{n+2}+\mathrm{i}\frac{3k}{n+3}\left[1+m(n+2)\right]+\left[1+\mathrm{i}\,m\,k-\frac{5}{8}m^2k^2\right]D(K)-\right.$$
$$\left.-5m\,k^2\left(1+\frac{3}{2}m\right)\frac{n+1}{n+4}\right\}, \tag{55}$$

where f_n are defined by Eq. (21) in function of the coefficients e_n of the modal amplitude of oscillations, $\hat{e}(x)$, and where $D(K)$ of $K=k(1+m)$ is defined by Eq. (47a), and $g_n = (2n)!/\left[2^{2n}(n!)^2\right]$ is also defined by the recurrence formula

$$g_n = g_{n-1}(2n-1)/(2n), \qquad g_0 = 1, \qquad g_1 = 1/2. \tag{56}$$

Thus, simple algebraic expressions in closed form (53) - (55) have thus been obtained for the unsteady aerodynamic coefficients in the general case of oscillating rigid or flexible airfoils in compressible flows. These general expressions of the reduced coefficients $\Delta\hat{C}_p(x)$, $\hat{C}_L$ and $\hat{C}_m$ are also valid in the limit case of incompressible flows ($m=0$, $\beta=1$ and $K=k$) when they become identical to the corresponding expressions derived for incompressible flows in our previous paper, Mateescu & Abdo [35].

Method validation for rigid airfoil oscillations in pitching rotation and translation

For rigid airfoil oscillations in pitching rotation, $\theta(t)=\hat{\theta}\exp(\mathrm{i}\,\omega\, t)$, about an articulation situated at $x=a$, and in normal-to-chord translation, $h(t)=\hat{h}\exp(\mathrm{i}\,\omega\, t)$, the modal amplitude of oscillation is

$$\hat{e}(x)=\hat{h}-(x-a)\hat{\theta}, \qquad e_0=\hat{h}+a\hat{\theta}, \qquad e_1=-\hat{\theta}, \qquad e_n=0 \text{ for } n>1, \tag{57}$$

and hence the values of the coefficients f_n appearing in the solutions (53) - (55) are in this case

$$f_0=e_1+\mathrm{i}2ke_0=-\hat{\theta}+\mathrm{i}2k\left(\hat{h}+a\hat{\theta}\right), \qquad f_2=2k^2m[2-m]e_1=-2k^2m[2-m]\hat{\theta}, \tag{58a}$$

$$f_1=\mathrm{i}2k(1-m)e_1+4k^2me_0=-\mathrm{i}2k(1-m)\hat{\theta}+4k^2m\left(\hat{h}+a\hat{\theta}\right), \qquad f_n=0 \text{ for } n>2. \tag{58b}$$

For unsteady compressible flows, the present solutions are compared with the previous results given by Jordan [15], which are based on the method presented in [46] for $M=0.7$. This is a rather complicated procedure [15] which uses a Fourier series expansions (similar to Kussner [18]) and requires the approximate solutions of a succession of integral equations with kernel $K(s,0)$ from the incompressible flow problem; it also uses an approximating 9th order polynomial with coefficients determined separately for each Mach number by fitting the polynomial to tabulated values, as well as the numerical evaluations of several integrals.

The comparison with previous unsteady compressible flow results is shown in Figures 2 and 3, for the variations of the unsteady lift and pitching moment coefficients (with respect to the leading edge), $\widetilde{C}_L(t)=\mathrm{Re}\left\{\hat{C}_L\right\}\cos(\omega t)-\mathrm{Im}\left\{\hat{C}_L\right\}\sin(\omega t)$ and $\widetilde{C}_m(t)=\mathrm{Re}\left\{\hat{C}_m\right\}\cos(\omega t)-\mathrm{Im}\left\{\hat{C}_m\right\}\sin(\omega t)$, with the airfoil position during the oscillatory cycle.

Case of Pitching Oscillations.

The typical variations with the Mach number and the reduced oscillation frequency of the real and imaginary components of the pressure difference coefficient distribution on the airfoil are shown in Figure 1. In this figure, and in the following, M_∞ is replaced by M for convenience.

The influence of the Mach number and of the reduced frequency on the variations with the airfoil position, $\widetilde{\theta}(t)/\theta_A$, of the unsteady lift and pitching moment coefficients, $\widetilde{C}_L(t)/\theta_A$ and $\widetilde{C}_m(t)/\theta_A$, is illustrated in Figure 2 for the case of pitching oscillations, $\widetilde{\theta}(t)=\theta_A\cos(\omega t)$, with respect to the leading edge ($a=0$), for various Mach numbers and reduced frequencies of oscillations. One can notice that the present solutions are in good agreement with the results for unsteady compressible flows based on Jordan's data [15] (especially for low oscillation frequencies), and in excellent agreement with the solutions in the limit case of incompressible flows ($M\to 0$) obtained by Theodorsen [44], Postel & Leppert [42] and Mateescu & Abdo [35].

Case of Plunging Oscillations.

The influence of the Mach number and of the reduced frequency on the variations with the airfoil position, $\widetilde{h}(t)/h_A$, of the unsteady lift and pitching moment coefficients, $\widetilde{C}_L(t)/h_A$ and $\widetilde{C}_m(t)/h_A$, is illustrated in Figure 3 for the case of normal-to-chord

oscillatory translations $\tilde{h}(t) = h_A \cos(\omega t)$. The present solutions are found again in good agreement for unsteady compressible flows with the results based on Jordan's data [15] and with the incompressible flow solutions ($M \to 0$) obtained by Theodorsen [44], Postel & Leppert [42] and Mateescu & Abdo [35].

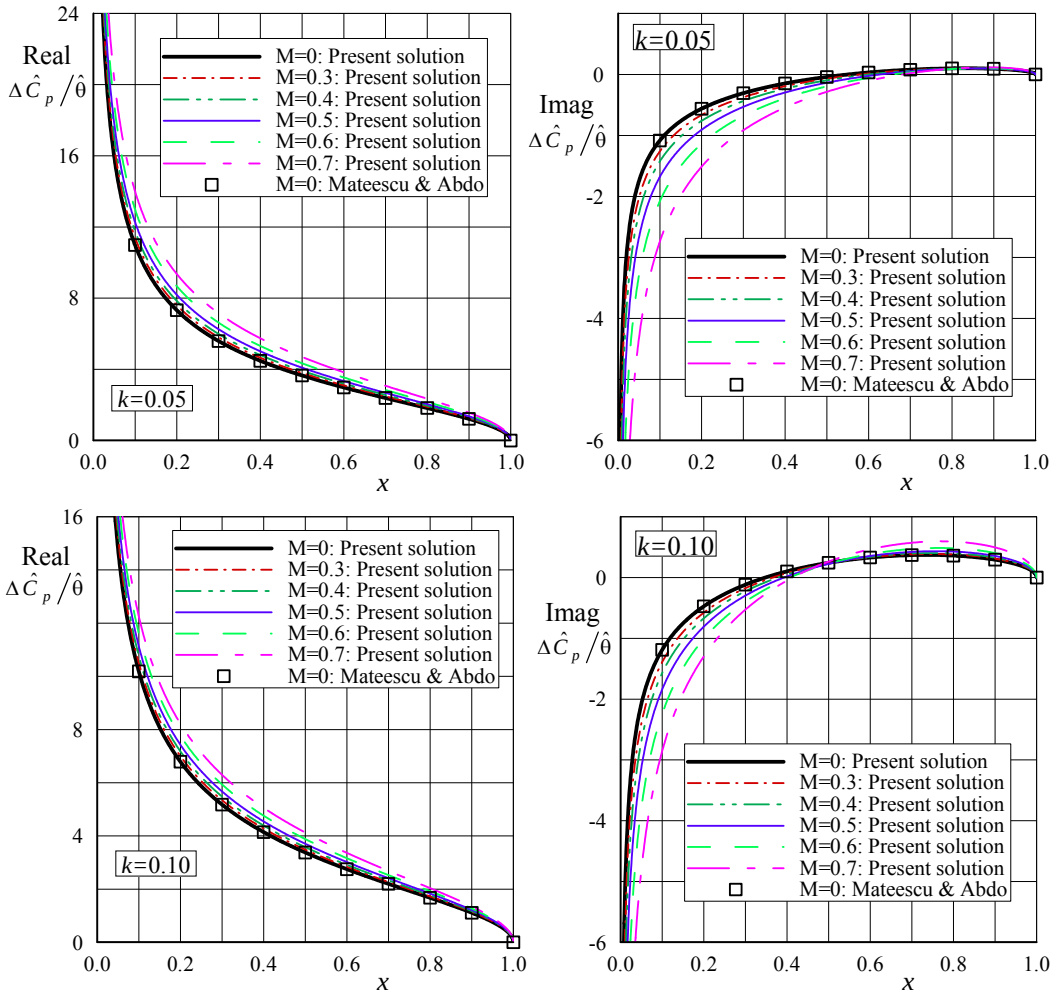

Fig. 1: Typical variations with the Mach number of the chordwise distributions of the real and imaginary components of the reduced pressure difference coefficient, $\Delta\hat{C}_p(x)/\hat{\theta}$, for a thin airfoil executing pitching oscillations, $\theta(t) = \hat{\theta}\exp(\mathrm{i}\,\omega\, t)$, about an articulation located at $a = 0.25$, at two values of the reduced frequency of oscillations, $k = 0.05$ and $k = 0.10$. For $M = 0$, comparison between: Present solutions (——), and Mateescu & Abdo [35] solutions (□).

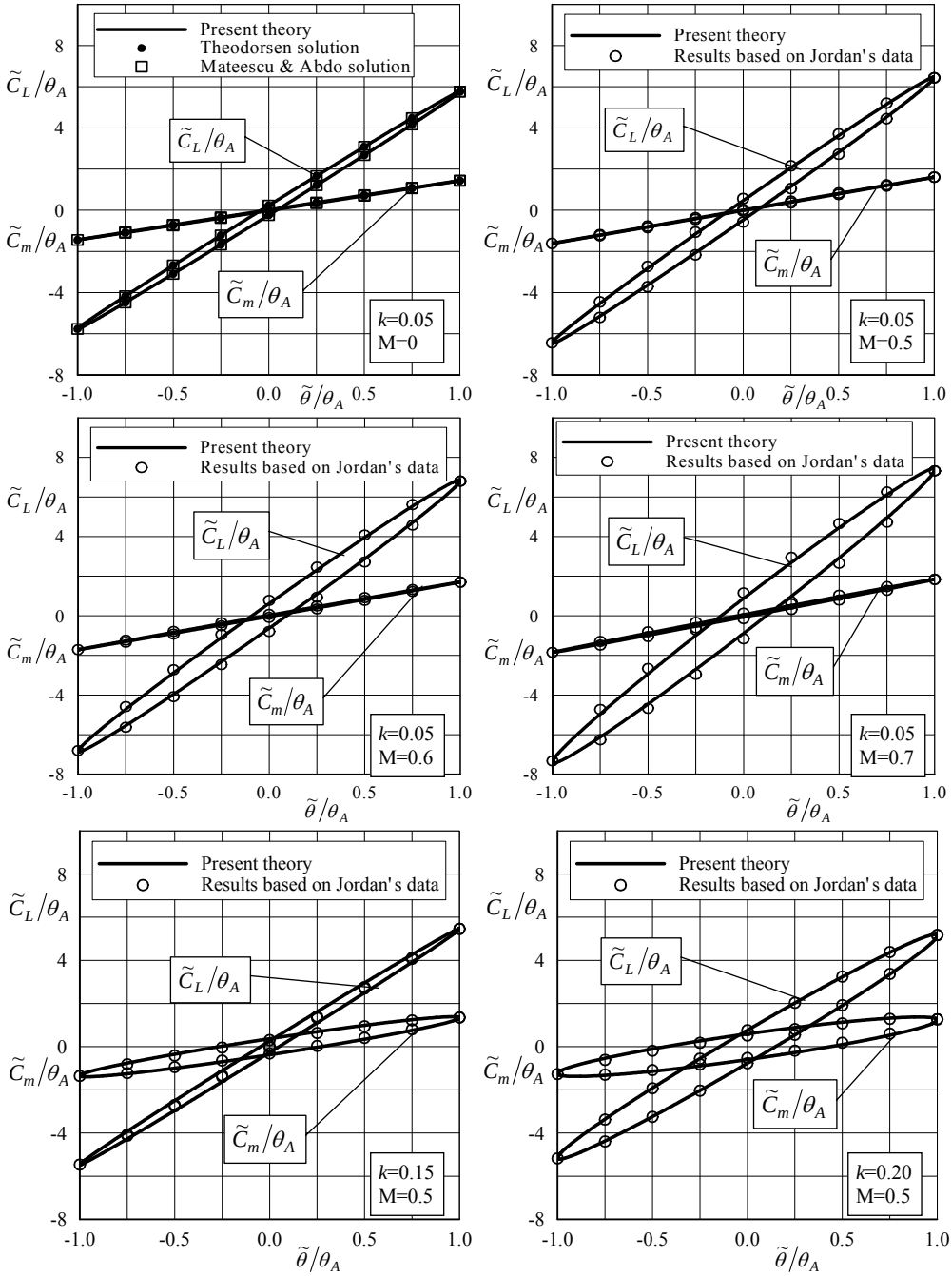

Fig. 2: *Pitching oscillations*: Typical variations with the airfoil position during the oscillatory cycle, $\tilde{\theta}(t)/\theta_A$, of the unsteady lift and pitching moment coefficients, $\tilde{C}_L(t)/\theta_A$ and $\tilde{C}_m(t)/\theta_A$ for various Mach numbers and reduced frequencies of oscillations. Comparison between the present solutions (——) and the results based on Jordan's data (○), and for $M = 0$ with the incompressible flow solutions derived by Theodorsen (●) and by Mateescu & Abdo (□).

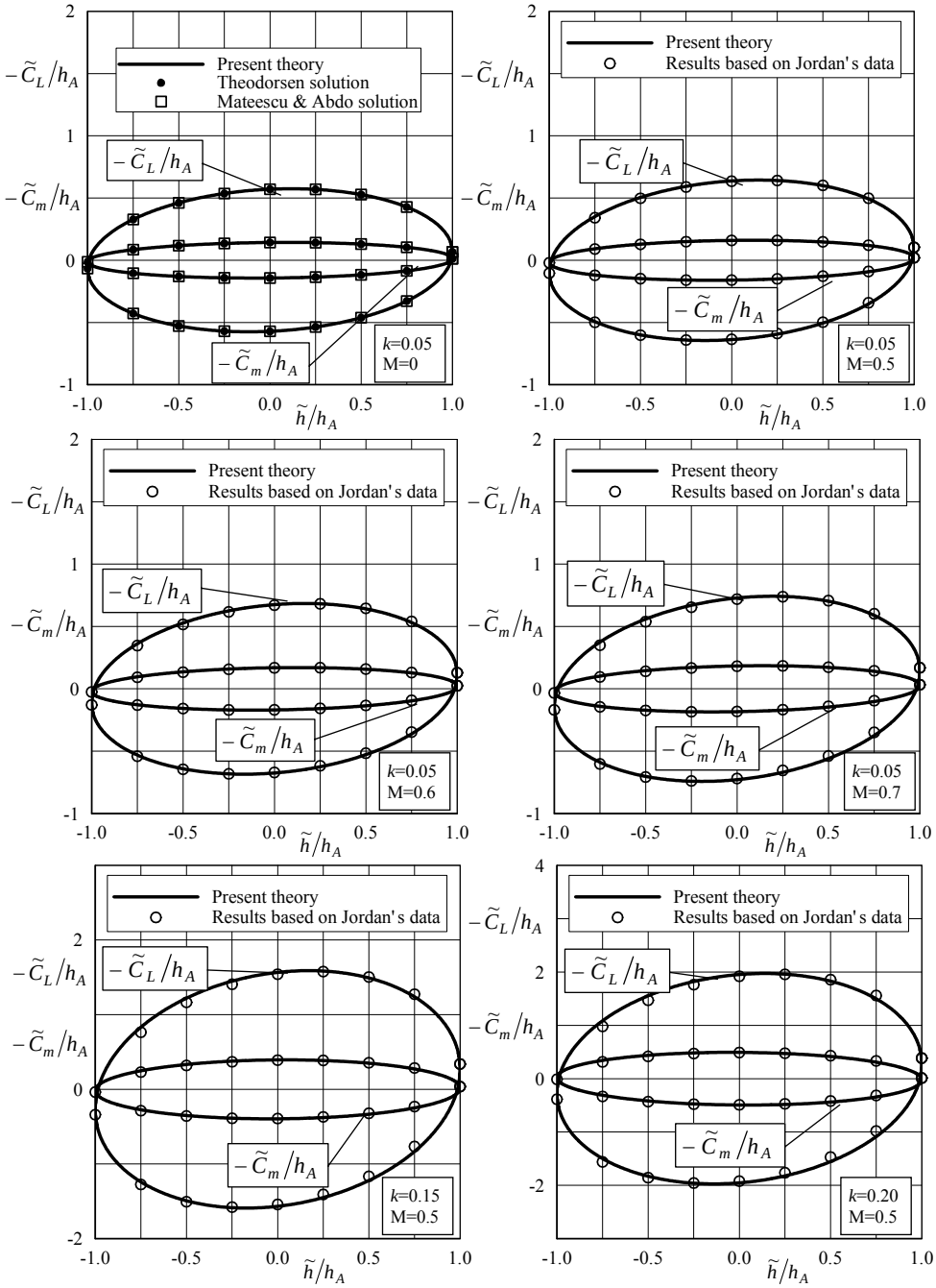

Fig. 3: *Oscillatory translations*: Typical variations with the airfoil position during the oscillatory cycle, $\tilde{h}(t)/h_A$, of the unsteady lift and pitching moment coefficients, $\tilde{C}_L(t)/h_A$ and $\tilde{C}_m(t)/h_A$ for various Mach numbers and reduced frequencies of oscillations. Comparison between the present solutions (——) and the results based on Jordan's data (○), and for $M = 0$ with the incompressible flow solutions derived by Theodorsen (●) and by Mateescu & Abdo (□).

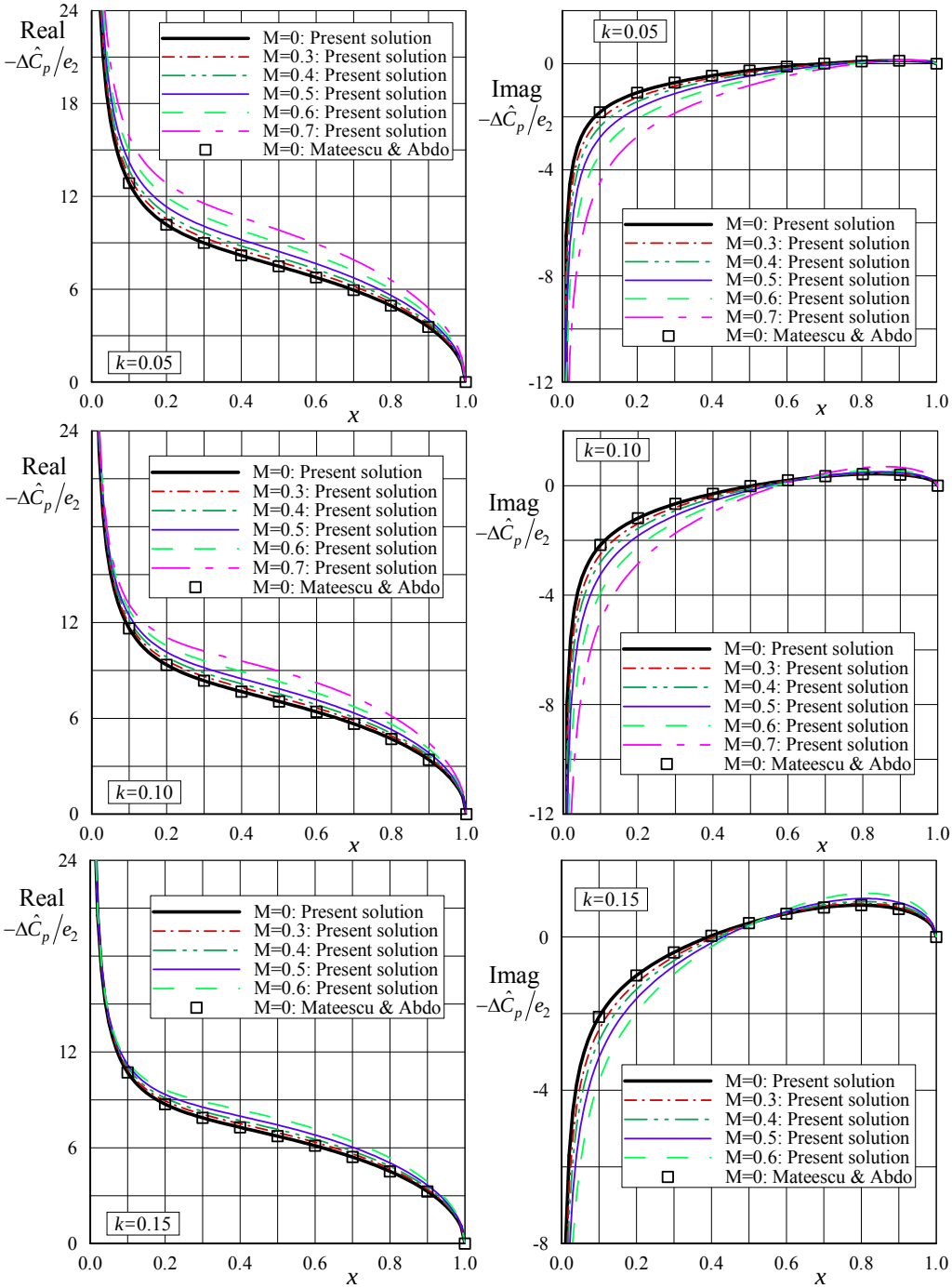

Fig. 4: Variations with the Mach number and frequency of oscillation of the chordwise distributions of the real and imaginary components of the reduced pressure difference coefficient, $\Delta\hat{C}_p(x)/e_2$, for a thin airfoil executing parabolic flexural oscillations.

Solutions for flexural oscillations of flexible airfoils

Examples of results are presented for the case of parabolic flexural oscillations defined by the modal amplitude of oscillations

$$e(x,t) = e_2\, x^2\, \exp(\mathrm{i}\,\omega t), \qquad \hat{e}(x) = e_2\, x^2, \qquad e_n = 0 \text{ for } n \neq 2. \tag{59}$$

In this case the values of the coefficients f_n defined by equation (21) are

$$f_0 = 0, \quad f_1 = 2e_2, \quad f_2 = \mathrm{i}\,2k(1-2m)e_2, \quad f_3 = 4k^2 m[1-m]e_2, \quad f_n = 0 \text{ for } n > 3. \tag{60}$$

The variations with the Mach number of the chordwise distributions of the real and imaginary components of the reduced pressure difference coefficients, $\Delta\hat{C}_p(x)/e_2$, for the case of parabolic flexural oscillations are illustrated in Figure 4 for various values of the reduced frequency of oscillations. The solutions obtained with the present method for the limit case of incompressible flows ($M \to 0$) were found to be in excellent agreement with the previous solutions obtained by Mateescu and Abdo [35] for incompressible flows (no data were found for comparison in the case of the flexural oscillations of flexible airfoils in compressible flows).

Conclusions

Efficient analytical solutions in closed form are presented in this paper for unsteady subsonic compressible flows past rigid and flexible airfoils executing low frequency flexural oscillations. These analytical solutions are obtained with a method using velocity singularities related to the airfoil leading edges and ridges (defined by the changes in the boundary conditions), in contrast to Theodorsen's method for incompressible flows which uses singularities in the expression of the velocity potential. The present solutions in closed form obtained for rigid airfoils executing pitching oscillations and normal-to-chord oscillatory translations in compressible flows have been found to be in good agreement with the results based on Jordan's data (obtained by a very complicated procedure based on Fourier series expansions, approximate polynomial fitting for each Mach number and on the numerical evaluations of several integrals). These solutions have also been found in perfect agreement in the limit case $M \to 0$ with the solutions obtained for unsteady incompressible flows by Theodorsen, Postel & Leppert and by Mateescu & Abdo.

Simple and efficient analytical solutions in closed form are also presented for the chordwise distribution of the unsteady pressure difference coefficient in the general case of flexural oscillations of flexible airfoils.

A detailed study of the variations of the unsteady aerodynamic coefficients with the Mach number and with the reduced frequency of oscillations is also presented.

The analytical solutions obtained in closed form were found to be very efficient computationally and can be successfully used in the aeroelastic studies.

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