CHAOTIC VIBRATION OF BUCKLED BEAMS AND PLATES

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Abstract
The great developing of numerical analysis of the dynamic systems emphasizes the existence of a strong dependence of the initial conditions, described in the phase plane by attractors with a complicated geometrical structure. The Lyapunov exponents are used to determine if there is a real strong dependence on the initial conditions: there is at least a positive exponent if the system has a chaotic evolution and all the Lyapunov exponents are negative if the system has not such an evolution. Determining the largest Lyapunov exponent , which is easier to calculate, is sufficient to draw such conclusions. In this paper we shall use the greatest Lyapunov exponent to study two well-known problems who leads to chaotic motions: the problem of the buckled beam and the panel flutter problem. In the problem of the buckled beam we verify the results obtained with the Melnikov theorem with the maximum Lyapunov exponent [1]. The flutter of a buckled plate is also a problem characterized by strong dependence of the initial conditions, existence of attractors with complicated structure existence of periodic unstable motions with very long periods (sometimes infinite periods).

Introduction

A large class of practical problems is modeled using dynamical systems whose solutions are characterized by a strong dependence of the initial conditions, a complex representation in the phase plane (the existence of attractors with a complicate structure) or the existence of solutions with very long periods, sometime infinite periods. Such dynamical systems are for example the Duffing equations, the Van der Pol equations or the Lorentz equations and they are extensively described in monogrphies as [6], [12].

To study such problems one can use the great development of the computational technique (phase plane representations, with the possibility to build strange attractors, complicated Poincare maps or representations of the Lyapunov exponents) or analytical methods based on the particular form of the equations as for exempla the Melnikov method which uses averaging and perturbation of the homoclinical orbits.
The stability of buckled beams and plates acted upon by periodical forces is a classical problem that can be included in the dynamical systems described above. The buckled beam is usually described using the Bernoulli-Euler model [6]. However, because the system of equations that describes the problem is of Duffing type with negative rigidity it is unstable and it is interesting to analyze its solutions when one takes into account new effects as the rotator inertia effect and the shear forces effect considered by a more complex beam model, the Timoshenko model. Although the principal characters of the model are the same, the small difference between the equation systems can induce interesting mechanical signification. These differences are remarked in an analysis done using the Melnikov function in [2]. The Melnikov function evaluates the distance between the stable and the unstable manifold. In this paper we evaluate the differences between the three models with the greatest Lyapunov exponent.

A plane plate acted by axial forces in parallel flow is the usual model used to study the non-linear panel flutter problem [4, 5], [6]. Phase plane representations, Poincare maps lead Dowell to analyze different aspects of this phenomenon. In this paper we use the greatest Lyapunov exponent to discover the regions in the parameter space that lead to great sensitivity at initial conditions.

Equations of motion

a) Transversal vibration of a buckled beam

Consider a buckled beam of length l, simply supported, acted upon by an axial force $P\ \left(P_{cr1} < P < P_{cr2}\right)$ and a lateral load $F\cos\omega t_1 1$. The equation of motion in the Bernoulli-Euler model, including the membrane effect, using no dimensional variables is:

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial^4 v}{\partial x^4}+\left(\Gamma-k\int_0^1 (v''(u))^2\,du\right)\frac{\partial^2 v}{\partial x^2}+c\frac{\partial v}{\partial t}=f\cos\omega t, \tag{1}$$

where

$$\Gamma=\frac{Pl^2}{EI},\ k=\frac{\kappa l^2}{EI},c=\frac{Cl^2}{\sqrt{3AEI}},\ f=\frac{Fl^2}{EI},x=\frac{x_1}{l},\quad t=\sqrt{\frac{EI}{\rho A}}\cdot\frac{t_1}{l^2} \tag{2}$$

EI is the bending modulus, ρ is the density, A the cross sectional area and κ a coefficient which introduce the membrane effect.

Introducing the rotatory inertia effect and the shear forces effect (1) becomes:

$$b_4\frac{\partial^4 v}{\partial t^4}+b_2\frac{\partial^2 v}{\partial t^2}+\frac{\partial^2 v}{\partial t^2}-(b_0+b_1)\frac{\partial^4 v}{\partial x^2\partial t^2}-b_3\frac{\partial^3 v}{\partial t^3}+\frac{\partial^4 v}{\partial x^4}+\left(\Gamma-k\int_0^1 (v'(u))^2\,du\right)\frac{\partial^2 v}{\partial x^2}+c\frac{\partial v}{\partial t}=f\left(\omega^2 b_5+1\right)\cos\omega t \tag{3}$$

where

$$b_0=\frac{I}{l^2A},b_1=\frac{EI}{k_1Gl^2A},b_2=\frac{Ic}{k_1A^2Gl^2}\sqrt{\frac{EI}{\rho A}},\quad b_3=\frac{c}{k_1AG}\sqrt{\frac{EI}{\rho A}},b_4=\frac{EI^2}{k_1A^2Gl^4},b_5=\frac{I\rho}{k_1AG} \tag{4}$$

To equations (1) or (4) we associate the following boundary and initial conditions:

$$\left.\frac{\partial^2 v}{\partial x^2}\right|_{x=0,x=1}=\left.v\right|_{x=0,x=1}=0,\ v(0,x)=\varphi(x),\quad \dot v(x,0)=\psi(x) \tag{5}$$

To study this problem we use the Galerkin method (in the first approximation) seeking the solution of the form:

$$v(x,t)=h(t)\sin\pi x \tag{6}$$

Introducing (6) in (1) and (3) and orthogonalizing, we obtain the following differential equations:

$$\ddot{h} - \beta h + \alpha h^3 = \varepsilon(\gamma \cos \omega t - \delta \dot{h}) \tag{7}$$

$$b_4 h^{iv} + b_2 h^{iii} + (1 + b_0\pi^2 + b_1\pi^2)\ddot{h} - \beta h + \alpha h^3 = \varepsilon(\gamma_1 \cos \omega t - \delta_1 \dot{h}) \tag{8}$$

where

$\beta = \pi^2(\Gamma - \pi^2)$, $\alpha = \kappa\pi^4 / 2$, $\gamma = 4f / \pi\varepsilon$, $\delta = c / \varepsilon$,

$\gamma_1 = (\omega^2 b_5 + 1))$, $\delta_1 = (\delta + b_3\pi^2)\varepsilon$

One neglects $b_4 h^{iv}$, $b_2 h^{iii}$, $4f\omega^2 b_5 / \pi\varepsilon$, $b_3\pi^2 / \varepsilon$ in (8) considering that they are much smaller than the unity. In this case (8) has the following approximate form:

$$(1 + b_0\pi^2 + b_1\pi^2)\ddot{h} - \beta h + \alpha h^3 = \varepsilon(\gamma \cos \omega t - \delta \dot{h}), \tag{9}$$

or

$$\ddot{h} - \beta^T h + \alpha^T h^3 = \varepsilon(\gamma^T \cos \omega t - \delta^T \dot{h}) \tag{10}$$

where

$\beta^T = \beta / (1 + \pi^2 b_0 + +\pi^2 b_1)$,

$\alpha^T = \alpha / (1 + \pi^2 b_0 + +\pi^2 b_1)$,

$\gamma^T = \gamma / (1 + \pi^2 b_0 + +\pi^2 b_1)$

Since $P_{cr}^1 < P < P_{cr}^2$ one has $\Gamma > \pi^2$ and β, β^T are positive. In this case (7) and (10) are Duffing type equations.

b) Panel flutter

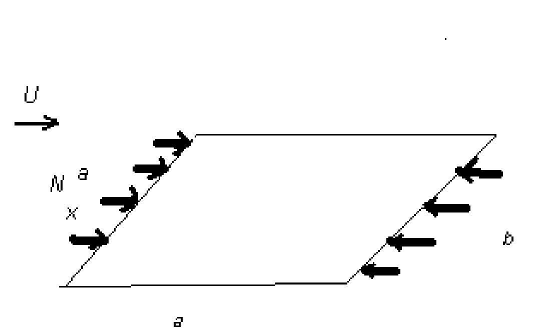


Fig. 1

The panel flutter problem is modeled sometimes in literature as a simple supported plate acted upon by in plane loads greater than the first critical load in a parallel flow as presented in fig. 1. The boundary value problem is:

$$D\frac{\partial^4 W}{\partial x^4} - (N_x + N_x^a)\frac{\partial^2 W}{\partial x^2} + \rho_m h\frac{\partial^2 W}{\partial t^2} +$$
$$(p - p_\infty) = \Delta p,$$
$$W(x,t)\Big|_{x=0,x=a} = \frac{\partial^2 W}{\partial x^2}\Bigg|_{x=0,x=a} = 0, \qquad (11)$$
$$W(x,0) = W_0(x),\ \dot{W}(x,0) = \dot{W}_0(x)$$

where the principal notations are:

$W(x,t)$ is the transversal displacement; a, b, h, are the dimensions of the plate; D is the plate rigidity, $N_x = \frac{\alpha\ Eh}{2a}\int_0^a (\frac{\partial\ W}{\partial\ x})^2 dx$ is the nonlinear in plane load; $N_x^a =$ is the in plane applied load; ρ_m is the plate density; ρ is the fluid density, $p - p_\infty = \frac{\rho\ U^2}{(M^2 - 1)}\left[\frac{\partial\ W}{\partial\ x} + \frac{M^2 - 2}{M^2 - 1}\cdot\frac{1}{U}\frac{\partial\ W}{\partial\ t}\right]$ is the aerodynamic pressure, and Δp s the static differential pressure across the panel. Introducing the following non-dimensional variables:

$$\xi = \frac{x}{a},\ w = \frac{W}{h},\ \lambda = \frac{\rho\ U^2 a^3}{(M^2 - 1)^{1/2} D},$$
$$P = \frac{\Delta p a^4}{Dh},\ \tau = t(\frac{Dha^4}{\rho_m})^{1/2},$$
$$\mu = \frac{\rho a}{\rho_m h}, R_x = \frac{N_a^x a^2}{D},\ \alpha = \frac{Ka}{Ka + Eh}$$

equation (11) becomes:

$$w'''' - \alpha\ 6\ (1 - \upsilon^2)[\int_0^1 (w')^2 d\xi\ )]w'' - R_x w'' +$$
$$\frac{\partial^2 w}{\partial \tau^2} + \lambda \qquad (12)$$
$$\left\{ w' + \left(\frac{M^2 - 2}{M^2 - 1}\right)\left(\frac{\mu}{\lambda(M^2 - 1)^{1/2}}\right)^{1/2}\frac{\partial\ w}{\partial\ \tau}\right\} = P$$

with corresponding boundary and initial conditions.

Using the Galerkin method (for P=0, α=1) one obtains from equation (12) the following system of ordinary differential equations:

$$\frac{1}{2}\ddot{a}_s(\tau\ )+\frac{\lambda}{2}\left(\frac{\mu}{M\ \ \lambda}\right)^{1/2}\dot{a}_s(\tau\ )+$$
$$\left(R_x\frac{(s\pi\ )^2}{2}+\frac{(s\pi\ )^4}{2}\right)a_s(\tau\ )+$$
$$\alpha\cdot 6\left(1-\nu^2\right)\left[\sum_r a_r^2\frac{(r\pi)^2}{2}\right]+ \tag{13}$$
$$+\lambda\sum_{m\neq s}\frac{sm}{s^2-m^2}\left[1-(-1)^{m+s}\right]a_m(s)=0,\quad a_s(0)=\alpha_s,\quad \dot{a}_s(0)=0,$$
$$s=1,2,...,N$$

This is an autonomous non linear system of ordinary differential equations extensively studied for example in [6], or [4].

Analysis and results

To study the systems (11) and (13) we use here: portraits in the phase plane, Poincare maps, Lyapunov exponents for both systems and the Melnikov function for the equations of the buckled beam.

Let $y(t;0,x_0)$ be the solution of the equation $\dot{y}=f(t,y)$,while $y\in\mathbf{R}$, $f:\mathbf{R}\to\mathbf{R}$, $[0,\tau]$ and $(0,t_1,...t_N)$ a discretization with the time step $\Delta t=\frac{\tau}{N}$ and $\delta_0>0$ a deviation of the initial condition, x_0. We consider the following rows:

$$y_i=y(i\Delta t;(i-1)\Delta t,y_{i-1}),\ \ y_0=x_0,$$
$$\left|y(i\Delta t;y_{i-1}-\delta)-y(i\Delta t;(i-1)\Delta t,y_{i-1})\right|=\delta_i, \tag{14}$$
$$L_i=\ln\frac{\delta_i}{\delta_0},\ \ i=1,2,...,N$$

From the terms of the row (1.8) one can obtain an evaluation of the difference between the perturbed solution and the unperturbed one, computed in the points of the given mesh:

$$L=\frac{1}{N}\sum_{i=1}^{N}\ln\frac{\delta_i}{\delta_0} \tag{15}$$

This expression represents an approximation of the Lyapunov exponent that, with a supplementary hypothesis has a more precise form.

If $y,x_0\in\mathbf{R^n}$, $f:\mathbf{R^n}\to\mathbf{R^n}$, the above result can be extended to n Lyapunov exponents. It offers useful information about the dependence of the initial conditions of the solution of the system. If at least there is one positive Lyapunov exponent then the solution is strong dependent of the initial conditions, and if all the Lyapunov exponents are negative then the solution is not chaotic, it may be periodic or almost periodic.

Equations (7) and (10) have for $\varepsilon = 0$ a hyperbolic saddle point and a homoclinical orbit. For $\varepsilon > 0$ one can associate to such an equation a real valued function, called the Melnikov function, a sort of distance between the stable and the unstable manifolds of the fixed point of the perturbed equation. If the Melnikov function has simple zeroes independent of ε, then for $\varepsilon > 0$ sufficiently small, the stable and the unstable manifolds intersect transversely and this means via the Smale-Birkhoff theorem that the motion described by (7) becomes “chaotic” [6]: it is sensitive dependent on the initial conditions, and some iterate of the Poincare map contains a countable infinite unstable periodic orbits, un uncountable set of bounded, nonperiodic orbits, and a dense orbit. To draw the conclusions above one integrates the differential equations obtained in the precedent paragraph and represents the solutions in the phase plane for the beams, or makes a choice of two directions for the plates.

Analysis of the chaotic vibration of a buckled beam

The possible relations between the mathematical beam model and the chaotic dynamics were numerically analyzed considering a steel beam with a circular cross section. The homoclinic orbit corresponding to (7) is:

$$q^0(t)=\left[\sqrt{\frac{2}{\alpha}}\beta\,\mathrm{sec}h\sqrt{\beta}t,-\sqrt{\frac{2}{\alpha}}\beta\tanh\sqrt{\beta}t\,\mathrm{sec}h\sqrt{\beta}t\right] \tag{16}$$

The Melnikov function for (7) is:

$$M(t_0)=\sqrt{\frac{2}{\alpha}}\gamma\frac{\pi\omega\sin\omega t_0}{\cosh\frac{\pi\omega}{2\sqrt{\beta}}}-4\frac{\delta\beta^{3/2}}{3\alpha} \tag{17}$$

$M(t_0)$ has simple zeroes if $\gamma/\delta > R_0(\omega)$ where:

$$R_0(\omega)=4\beta^{3/2}\cosh\frac{\pi\omega}{2\sqrt{\beta}}\Bigg/\frac{3\pi\omega}{\sqrt{2\alpha}} \tag{18}$$

For (10) we have:

$$R_0^T(\omega)=4\beta^{T\,3/2}\cosh\frac{\pi\omega}{2\sqrt{\beta^T}}\Bigg/\frac{3\pi\omega}{\sqrt{2\alpha^T}} \tag{19}$$

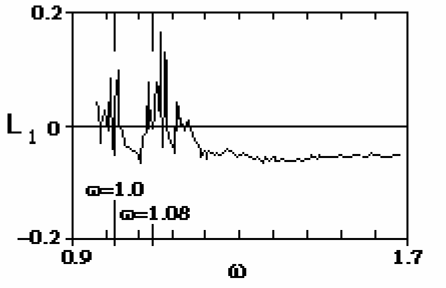
Taking into account the form of the coefficients $\alpha,\ \beta,\ \gamma,\ \delta$ one obtains the following relation between the bifurcation values:

$$R_0 > R_0^T \tag{20}$$

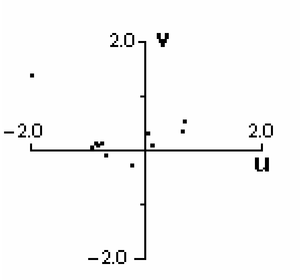
This relation reflects the fact that the two motions perform chaotic motions for different values of the parameters of the problem (it is possible that under the same initial conditions and external loads one model performs a periodic motion and the other a chaotic one).

We calculate the maximum Lyapunov exponent for (7) and (10) for the same initial conditions $h_0 = \dot{h}_0 = 0.1$ which are in a small neighborhood of the homoclinical saddle point of (7) when $\varepsilon = 0$. The evolutions are analyzed representing the maximum Lyapunov exponents, Poincare maps and portraits in the phase plane.

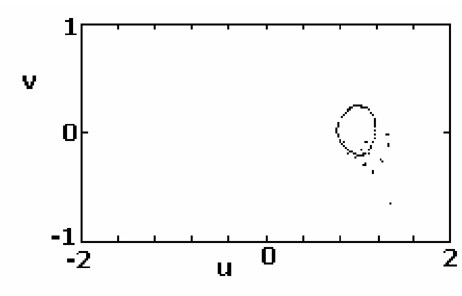
In figure 2 we represented Lyapunov exponents, Poincare maps and phase plane portraits for the Bernoulli-Euler beam and in figure 3 Poincare maps and phase plane portraits for theTimoshenko beam.



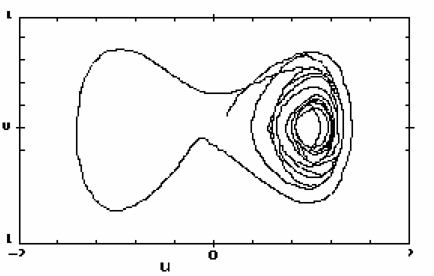
$\gamma / \delta = 1$



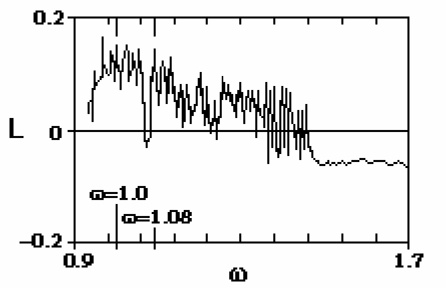
$\gamma / \delta = 1, \ \omega = 1$



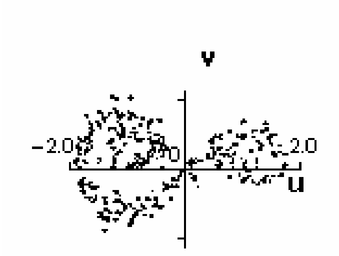
$\gamma / \delta = 1, \ \omega = 1.08$



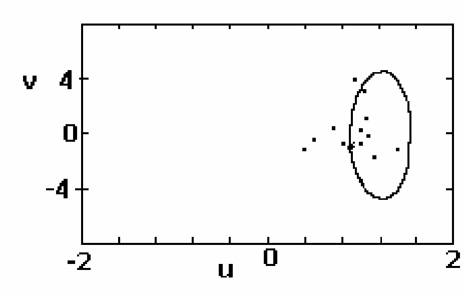
$\gamma / \delta = 1, \ \omega = 1.08$



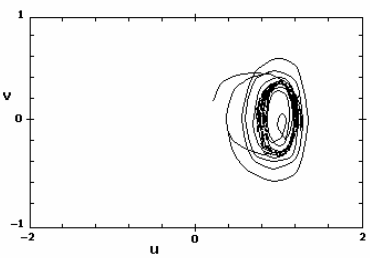
$\gamma / \delta = 2$



$\gamma / \delta = 2, \ \omega = 1$

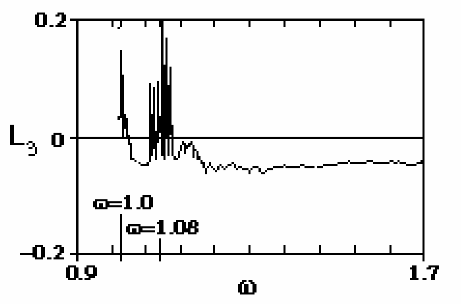


$\gamma / \delta = 2, \ \omega = 1.7$

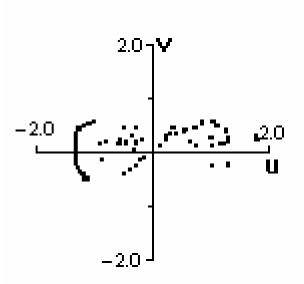


$\gamma / \delta = 1, \ \omega = 1.08$

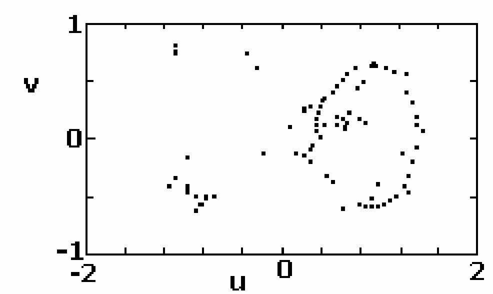
Fig.2 Bernoulli-Euler Beam



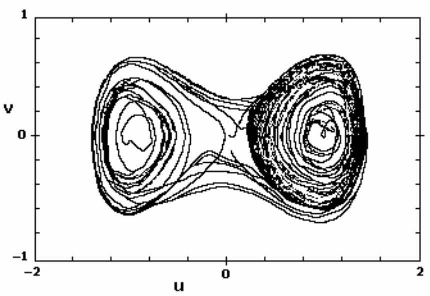
$\gamma / \delta = 1$



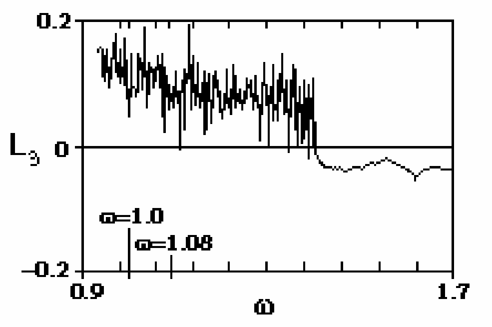
$\gamma / \delta = 1, \ \omega = 1$



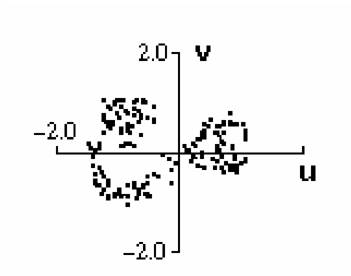
$\gamma / \delta = 1, \ \omega = 1.08$



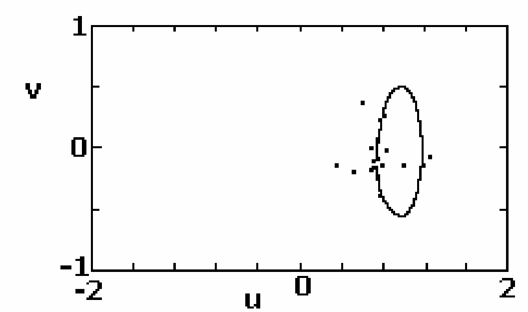
$\gamma / \delta = 1, \ \omega = 1.08$



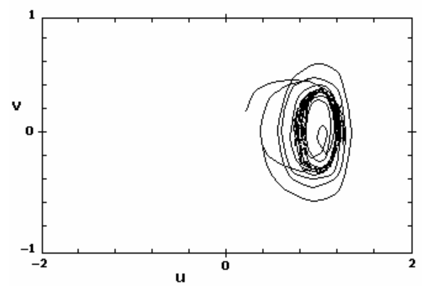
$\gamma / \delta = 2$



$\gamma / \delta = 2, \ \omega = 1$



$\gamma / \delta = 2, \ \omega = 1.7$



$\gamma / \delta = 1, \ \omega = 1.08$

Fig. 3 Timoshenko Beam

Analysis for the panel flutter problem

Dowell studied this system using time histories, spectral densities, portraits in the phase plane and Poincaré maps. In [5] the principal values of the parameters are: $\lambda = 150,\ \mu / M = 0.1,\ \xi = 3/4,\ N = 2.$ The bifurcation parameter is R_x. In his analysis he emphasizes the the possible appearance of chaotic motions of all kind (beginning of chaos, mature chaos, etc) for different values of the parameter R_x.

In this paper we consider the following three cases, according to the well-known diagram of Dowell [1].

1. $\lambda = 150,\ \mu / M = 0.1,\ \xi = 3/4,\ N = 2.$
2. $\lambda = 150,\ \mu / M = 0.01,\ \xi = 3/4,\ N = 2.$
3. $\lambda = 300,\ \mu / M = 0.1,\ \xi = 3/4,\ N = 2.$

In figure 4 and 5 we represent the maximum Lyapunov exponent diagrams in the first two cases, followed by phase plane portraits in significant situations 3a), b), c) d) e), and f)for case 1.

As a first remark on these diagrams one can note easier the time intervals described by Dowell:

When the maximum Lyapunov exponent is in a small neighborhood of the origin the evolution is in a phase called the beginning and the onset of chaos (the portraits in the phase plane shows periodic, but not simple motions); when the maximum Lyapunov exponent is negative, but well separated of the origin , the evolution is periodic; when the maximum Lyapunov exponent is positive, well separated of the origin , the evolution is in a phase called the maturing process of chaos (the portraits in the phase plane is characterized by a diffusion of periodic curves in such a way that a complete region is filled).

More precisely analyzing fig. 1 we have the following situations:

1. For R_x / π^2 less than 3 the motion is periodic.
2. For $R_x / \pi^2 \approx 3$ there is a strong dependence on the initial values and in the same time with a positive maximum Lyapunov exponent we have a diffusion in the phase plane (fig. 3a).
3. For $R_x / \pi^2 \in (3,\ 5.5)$ the motion is periodic, but not simple (fig. 3b).
4. For $R_x / \pi^2 \in (5.5,\ 5.9)$ the motion is characterized by a strong dependence on the initial conditions and a positive maximum Lyapunov exponent (3c).
5. For $R_x / \pi^2 \approx 6$ again we have a periodic, but not simple motion (fig. 3d).
6. For $R_x / \pi^2 \in (6,7)$ the motion is chaotic (fig. 3e).
7. For $R_x / \pi^2 > 7$ the motion is simple periodic (3f).

In the second case (fig. 5) we have almost the same principal regions, for R_x, with small differences - situation described at 3) does not occur, the situation described at 4) covers a larger region while 5) is smaller.

As we have expected in the third case the maximum Lyapunov exponent is very small for the whole domain of R_x [4].

Conclusions

Analyzing the results of the two examples discussed above one can draw the following conclusions:

1. Both the beams and the plates acted upon by axial loads and lateral loads can, in specific situations, lead to differential equations with chaotic solutions characterized by great sensivity to the initial conditions and complicated attractors in the phase plane.
2. When the maximum Lyapunov exponent is negative the evolution is periodic or cuasi-periodic.
3. When the maximum Lyapunov exponent is positive the evolution is rather complex, we may say it is chaotic.
4. There are cases in which for the same initial conditions and the same external loads we obtain periodic or cuasi-periodic motions for the Bernoulli beam and chaotic for the Timoshenko model.
5. The intervals discussed by Dowell (on set of chaos, beginning of chaos and mature chaos) can be noted from the Lyapunov exponent

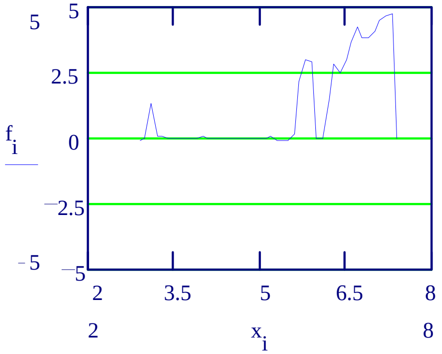


Fig. 4. Largest Lyapunov exponent diagram for panel flutter problem μ=0.1, λ=150.

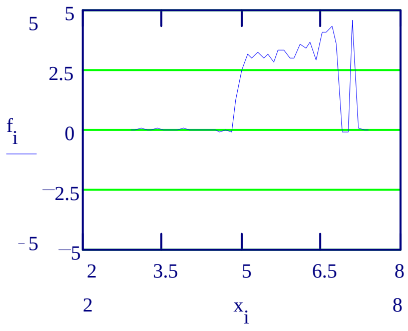
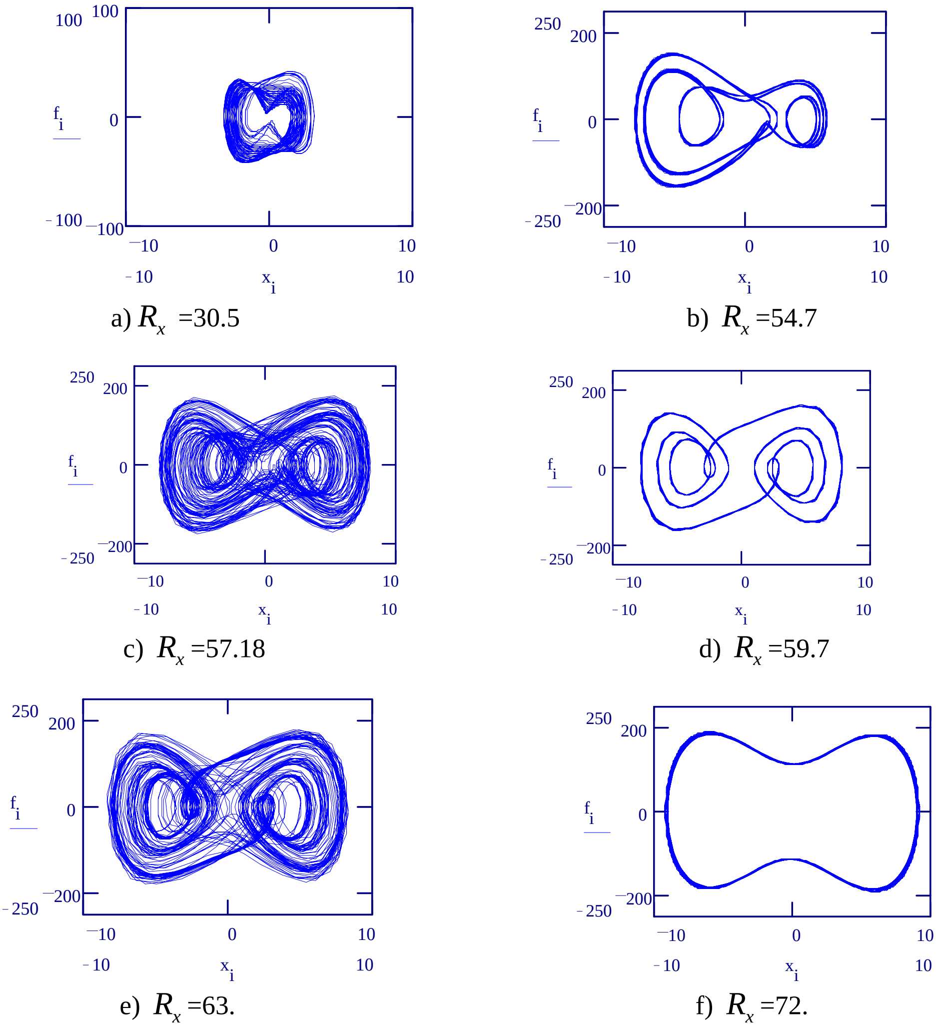


Fig. 5. Largest Lyapunov exponent diagram for panel flutter problem μ=0.01, λ=150



a) R_x =30.5 b) R_x =54.7

c) R_x =57.18 d) R_x =59.7

e) R_x =63. f) R_x =72.

Fig. 6

REFERENCES

[1.] *V. I. ARNOLD*, ***Ecuatii diferentiale ordinare***, Edit. Stiintifica si Enciclopedica, Bucuresti, 1979.

[2.] *DANIELA BARAN*, ***Vibratii aleatoare si haotice ale structurilor mecanice cu aplicatii in aviatie si energetica nucleara,*** Teza de doctorat, Bucuresti,1995.

[3.] *S. H. CHEN, S. G. CHEN*, ***Chaotic Vibration in Fluid Elastic Instability of a Tube Row in Cross Flow***, PVP-vol. 258, *Flow induced Vibration and Fluid Structure Interaction*, ASME, 1999.

[4.] *E. H. DOWELL*, ***Chaotic Oscillation in Mechanical Systems***, Computers and Structures, **30**, 1988.

[5.] *E. H. DOWELL*, ***Observation and Evolution of Chaos for an Autonomous System,*** J. of Appl. Mech.,**51**, 1984.

6. *J. GUCKENHEIMER, P. HOLMES*, ***Nonlinear Oscillations Dynamical Systems and Bifurcation of Vector Fields***, New York, Berlin, Heidelberg, Tokyo, Springer Verlag, 1984.

7 *A. HALANAY*, ***Teoria calitativa a ecuatiilor diferentiale***, Edit. Academiei, Bucuresti, 1963.

8. *I. HAYASHY*, ***Nonlinear Vibration in physics (in Russian)***, Izdatelstvo Mir, Moscow, 1968

9. *M. W. HIRSH, S. SMALE*, ***Differential Equations Dynamical Systems and Linear Algebra***,Academic Press, New-York, San Francisco, London, *1974.*

10. *V. K. MELNIKOV*, ***On the stability of the Centre for time periodic perturbations***, Trans. Moscow Math. Soc., **12**, 1963.

11. *Y. UEDA*, ***Explosion of a strange attractor exhibited by Duffing's equations***, Nonlinear Dynamics, New York Academy of Sciences, 1981.

12. *R. VOINEA, I. STROE*, ***Sisteme Dinamice,*** Universitatea "Politehnica", Bucuresti, 1993