Constant-Power Engines in Flights to \( L_4 \) point of the Earth-Moon System

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Abstract: The transfer is studied in the context of the classical Circular Restricted Three-Body Problem. The initial state is on a geosynchronous orbit and the terminal state is the \( L_4 \) triangular Lagrangian point. The use of controllable acceleration engine by minimization of a time integral of the squared engine acceleration, the optimization problem is separated into two sub-problems: dynamical and parametric ones. Initial mass to constant output power ratio variation is mainly studied, depending on the transfer duration and initial thrust acceleration for fixed power jet (specific impulse is within the limits of the VASIMR magneto-plasma engine). The Two Point Boundary Value Problem is solved when the startup speed is circular or when the initial thrust acceleration is given. Time variation of the obtained state and control variables is represented relative to Earth in geocentric equatorial inertial system.

Key Words: Circular Restricted Three-Body Problem, triangular Lagrangian points, continuous thrust control, variable specific impulse, VASIMR

1. INTRODUCTION

The well-known Lagrangian points occuring in the Circular Restricted Three-Body Problem (CR3BP) are very important for astronomical applications. There are five points (Fig. 1) of equilibrium in the equations of motion. The collinear points (\( L_1 \), \( L_2 \), and \( L_3 \)) are always unstable and the triangular points (\( L_4 \) and \( L_5 \)) are stable in the present case (Earth-Moon system). The triangular points \( L_4 \) or \( L_5 \) are useful to locate a space station, since they require a small amount of fuel for station-keeping.

In this paper the problem of sending a spacecraft from geosynchronous orbit (GEO) to \( L_4 \) point with minimum fuel consumption is considered. The space vehicle is equipped with constant-power propulsion installation.

The transfer is designed in the frame of the controlled Circular Restricted Three-Body Problem (CR3BP). The control problem is solved to minimization of a time integral of the squared jet acceleration.

The variational problem for determining the laws of variation the motion parameters, is of Lagrange type, and lead to a Two Point Boundary Value Problem (2PBVP). By eliminating the multipliers the system of extremals of the spacecraft transfer is obtained. The controls profiles are determined for specified initial velocity (circular velocity GEO) or for specified initial thrust acceleration (tangent at GEO).

A performance study with the spacecraft specific mass (initial mass/constant jet power) as a basic parameter is made for the obtained profiles or for vehicles [1] of fixed engine power. By transforming the position, velocity and acceleration, the state and control variables are represented relative to Earth (Geocentric Equatorial Inertial frame).
Fig. 1 – The five Lagrangian points, with the Earth and Moon bodies rotating about their barycenter.

Thematic of transfer with the treatment of dynamic part or of the propulsion system is approached in [1-9]. A technique [5] for transferring an object like spacecraft to the stable Lagrangian points, for instance $L_4$ and $L_5$, uses a substantially negligible amount of delta-V. Another concept described in [6] is a deep space relay at Earth-Moon $L_4$ and/or $L_5$ that would serve as a high optical navigation rate and satellite communications.

2. EQUATIONS OF MOTION

The three-body problem describes the motion of three-point mass particles under their mutual gravitational interactions. This is a classical problem that covers many situations in astrodynamics. Fig. 2 shows the basic geometry of the system consisting of two primary bodies Earth and Moon, and the spacecraft within the system.

$\omega$ – the argument of the location (latitude) of the Moon relative to the node line

$\Omega$ – the longitude of ascending node of the Moon

Fig. 2 – Basic geometry of the 3-Body Problem, ($xyz$) – (EMBR), ($X_1Y_1Z_1$) – (EI) and the ($X_EY_EZ_E$) – (GEI) frames

($xyz$) – Earth-Moon Barycentric Rotating (EMBR) frame is, as its name indicates, a rotating frame with origin at the barycenter of the Earth-Moon system; the $x$-axis is directed towards Moon; it rotates with angular velocity $\omega$ around the $z$-axis (which is perpendicular to orbital plane); the $y$-axis lies in the orbital plane and completes the orthonormal triad.

($X_1Y_1Z_1$) – Earth Inertial (EI) frame. The inertial frame is centered at the Earth and has directions aligned with the directions of the rotating frame at the initial time.

($X_EY_EZ_E$) – Geocentric Equatorial Inertial (GEI) frame; its origin is the center of the Earth; the fundamental plane is the equator and the positive $X_E$-axis in the vernal equinox direction; the $Z_E$-axis in the direction of the north pole; the $Y_E$-axis completes the orthonormal triad.
In the CR3BP, two of the three bodies have much larger masses than the third. As a result, the motion of the two larger bodies are unaffected by the third body. The two primaries move circularly about their barycenter in a plane, following a counterclockwise direction with the same constant angular velocity. The Moon orbits the Earth with a period of 27.322 days (sidereal month), and the average orbit inclination ($i_{sm}$) with respect to the ecliptic is 5.145±0.15 deg varying with a period of 173 days. The inclination ($i_s$) of the equatorial plane (the inclination of the apparent orbit of the Sun about the Earth) with the ecliptic is 23.45 deg. When the ascending node of the Earth-Moon orbit is close to the vernal equinox, the inclination ($i_m$) of the moon orbit with the equator is at a maximum of 23.45+5.145 or 28.6 deg. This is called the major standstill. Conversely, when the ascending node is at the equinox, the inclination of the moon orbit with the equator is 18.3 deg. This is the minor standstill. The period of this variation is of 18.6 years. Therefore, a good launching is at a distance of 18.6 years. In this study the initial moment of the transfer is considered when the (mean) ascending node of the lunar orbit coincides with the vernal equinox, that is, when $\Omega = 0$ deg, e.g. 2025 Jan 29. The dimensionless equations of motion of a spacecraft in the Earth-Moon gravitational field are usually written in EMBR frame:

$$\ddot{x} = \Omega_x + 2\dot{y}, \quad \ddot{y} = \Omega_y - 2\dot{x}, \quad \ddot{z} = \Omega_z$$

where the subscripts denote the partial derivatives of the of the auxiliary function

$$\Omega(x, y, z) = \frac{1}{2} (x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

The mass parameter is $\mu$; then, $r_1$ and $r_2$ are the relative distances between the spacecraft and Earth and Moon, respectively:

$$r_1 = \sqrt{(x + \mu)^2 + y^2 + z^2}, \quad r_2 = \sqrt{(x - 1 + \mu)^2 + y^2 + z^2}$$

The system of units is dimensionless and is defined as follows: the mass unit is the sum of the masses of the Earth and the Moon; the distance unit is the Earth-Moon distance; and the time unit is the period of the Moon’s orbit divided by $2\pi$ so that the angular velocity of the Earth-Moon line is a unity. Conversion factors are given below:

1 UNIT OF DISTANCE = 384400 km
1 UNIT OF TIME = 375190.263 s
1 UNIT OF VELOCITY = 1024.547 m/s

In this system of units, the mass parameter $\mu = \text{mass}_{\text{Moon}}/(\text{mass}_{\text{Earth}} + \text{mass}_{\text{Moon}}) = 0.0121505844$, the gravitational parameter of the Moon is $\mu$, and the gravitational parameter of the Earth is $1 - \mu$.

The Earth is situated in the xy plane at (-$\mu$, 0, 0) and the Moon is at (1-$\mu$, 0, 0). The $L_4$ triangular Lagrangian point is at the point $(1 / 2 - \mu, \sqrt{3} / 2, 0) \equiv (0.488, 0.766, 0)$.

**3. VARIATIONAL PROBLEM**

The controlled CR3BP

$$\ddot{x} = \Omega_x + 2\dot{y} + u_x, \quad \ddot{y} = \Omega_y - 2\dot{x} + u_y, \quad \ddot{z} = \Omega_z + u_z$$

can be rewritten in the first order as

$$\dot{x} = f(x, u, t)$$
in which \( u = (u_x, u_y, u_z)^\top \) and \( x = (x, y, z, \dot{x}, \dot{y}, \dot{z})^\top \) are the control and state vectors, respectively. The basic problem is reduced to finding the control \( u \) and jet acceleration components such that the boundary conditions of transfer are satisfied and minimize the functional

\[
J = \frac{1}{2} \int_0^{t_f} u^\top u \, dt
\]

where \( t_f \) is the given transfer duration.

Introducing the Lagrange multipliers \( \lambda_i, \ i = x, y, z, \dot{x}, \dot{y}, \dot{z} \) the Euler-Lagrange equations are

\[
\dot{x} = \frac{\partial L}{\partial \dot{x}}, \quad \dot{\lambda} = \frac{\partial L}{\partial \dot{\lambda}^i}, \quad 0 = \frac{\partial L}{\partial u}, \quad (1)
\]

in which

\[
L = \frac{1}{2} u^\top u + \lambda^\top (\dot{x} - f)
\]

is the Lagrangian function. System (1) represents a set of differential algebraic equations to be solved by substitution, exploiting the last equation which provides the values of the control functions in terms of Lagrange multipliers

\[
u_x = -\lambda_x, \quad u_y = -\lambda_y, \quad u_z = -\lambda_z
\]

The multiplier equations and the expressions corresponding to each of the partial derivatives (as [1], [13]) are:

\[
\dot{\lambda}_x = -U_{xx}\lambda_x - U_{xy}\lambda_y - U_{xz}\lambda_z
\]

\[
\dot{\lambda}_y = -U_{yx}\lambda_x - U_{yy}\lambda_y - U_{yz}\lambda_z
\]

\[
\dot{\lambda}_z = -U_{zx}\lambda_x - \lambda_y U_{yz} - \lambda_z U_{zz}
\]

\[
\dot{\lambda}_x = -\lambda_x + 2\lambda_y
\]

\[
\dot{\lambda}_y = -\lambda_y - 2\lambda_x
\]

\[
\dot{\lambda}_z = -\lambda_z
\]

\[
\Omega_{xx} = 1 - \frac{1 - \mu}{r_1^3} - \frac{\mu}{r_2^3} + \frac{3(1 - \mu)(x + \mu)^2}{r_1^5} + \frac{3\mu(x - 1 + \mu)^2}{r_2^5}
\]

\[
\Omega_{xy} = \frac{3(1 - \mu)(x + \mu)y}{r_1^5} + \frac{3\mu(x - 1 + \mu)y}{r_2^5}
\]

\[
\Omega_{xz} = \frac{3(1 - \mu)(x + \mu)z}{r_1^5} + \frac{3\mu(x - 1 + \mu)z}{r_2^5}
\]

\[
\Omega_{yx} = \Omega_{xy}
\]

\[
\Omega_{yy} = 1 - \frac{1 - \mu}{r_1^3} - \frac{\mu}{r_2^3} + \frac{3(1 - \mu)y^2}{r_1^5} + \frac{3\mu y^2}{r_2^5}
\]

\[
\Omega_{yz} = \Omega_{zy}
\]

\[
\Omega_{zz} = 1 - \frac{1 - \mu}{r_1^3} - \frac{\mu}{r_2^3} + \frac{3(1 - \mu)z^2}{r_1^5} + \frac{3\mu z^2}{r_2^5}
\]
\[ \Omega_{yz} = \frac{3(1 - \mu)yz}{r_1^5} + \frac{3\mu yz}{r_2^5} \]
\[ \Omega_{xz} \equiv \Omega_{xz} \]
\[ \Omega_{zy} \equiv \Omega_{zy} \]
\[ \Omega_{zz} = -\frac{1 - \mu}{r_1^5} - \frac{\mu}{r_2^5} + \frac{3(1 - \mu)z^2}{r_1^5} + \frac{3\mu z^2}{r_2^5} \]

By eliminating the multipliers the system of differential equations of spacecraft motion is obtained in the following form:
\[ \begin{align*}
\ddot{x} &= \Omega_x + 2\dot{y} + u_x \\
\ddot{y} &= \Omega_y - 2\dot{x} + u_y \\
\ddot{z} &= \Omega_z + u_z \\
\dot{u}_x &= \Omega_{xx}u_x + \Omega_{xy}u_y + \Omega_{xz}u_z + 2\dot{u}_y \\
\dot{u}_y &= \Omega_{yx}u_x + \Omega_{yy}u_y + \Omega_{yz}u_z - 2u_x \\
\dot{u}_z &= \Omega_{zx}u_x + \Omega_{zy}u_y + \Omega_{zz}u_z \\
\end{align*} \tag{2} \]

Solving the 2PBVP of system (2) means finding the functions \( x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t), u_x(t), u_y(t), u_z(t) \) for \( t \in [0, t_f] \). By successively transforming \cite{14} the position, velocity and acceleration, the state and control variables are represented relative to Earth in GEI. Conversion between references frames is determined by relations:
\[ \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} \cos(u) & \sin(u) & 0 \\ -\sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\Omega) & \sin(\Omega) & 0 \\ -\sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x - x_1 \\ y \\ z \end{pmatrix} \]
\[ \begin{pmatrix} \dot{X}_1 \\ \dot{Y}_1 \\ \dot{Z}_1 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} - \omega y \\ \dot{y} + \omega(x - x_1) \\ \dot{z} \end{pmatrix} \]
\[ \begin{pmatrix} \ddot{X}_1 \\ \ddot{Y}_1 \\ \ddot{Z}_1 \end{pmatrix} = \begin{pmatrix} \cos(u) & \sin(u) & 0 \\ -\sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\Omega) & \sin(\Omega) & 0 \\ -\sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{X}_E \\ \dot{Y}_E \\ \dot{Z}_E \end{pmatrix} \]
and for controls
\[ \begin{pmatrix} u_{X_E} \\ u_{Y_E} \\ u_{Z_E} \end{pmatrix} = \begin{pmatrix} u_x \cos(\omega t) - u_x \sin(\omega t) \\ u_x \sin(\omega t) \cos(i_m) + u_y \cos(\omega t) \cos(i_m) - u_z \sin(i_m) \\ u_x \sin(\omega t) \sin(i_m) + u_y \cos(\omega t) \sin(i_m) + u_z \cos(i_m) \end{pmatrix} \]

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4. STUDY OF PERFORMANCES

The spacecraft is assumed to be initially on GEO of 28.6 deg on the line Departure-Earth-L₄. The orbit is situated in Moon orbital plane, and is characterized by the following parameters:
- Altitude: 35786 km
- Circular velocity: 3.075 km/s
- Longitude of the ascending node: 0 deg
- Inclination: 28.6 deg

The endpoint trajectory is mainly characterized by the following:
- Position (EMBR): \((1/2 - \mu, \sqrt{3}/2, 0) \approx (0.488, 0.866, 0)\) [dimensionless]
- Velocity (EMBR): 0
- Velocity (GEI): 1.025 km/s
- Distance from Earth: 384400 km

An equation for constant-power engine is obtained combining the fundamental equations of thrust

\[ T = (-\dot{m})(g_0 I_{sp}) \]

and jet power

\[ P_j = \frac{1}{2} (-\dot{m})(g_0 I_{sp})^2 \]

with the expression of the thrust acceleration

\[ u = \frac{T}{m} \]

The expression for mass consumption can be represented in the form

\[ \dot{m} = \frac{dm}{dt} = -\frac{m^2 u^2}{2P_j} \] (3)

Integrating (3) over time from zero initial moment up to current time \(t\), we obtain the followings:

\[ \frac{1}{m(t)} = \frac{1}{m(0)} + \frac{1}{P_j} \int_0^t u^2 dt \] (4)

where they assume that the power jet is constant, \(m_0 = m(0)\) is the initial mass of the spacecraft. The initial mass to power jet ratio- spacecraft specific mass-is considered and is noted by \(\alpha\):

\[ \alpha = \frac{m_0}{P_j} \] (5)

Using (3), (4) and the above fundamental relations for impulse specific the following relation is obtained

\[ I_{sp} = \frac{2}{g_0 u} \left( \frac{1}{\alpha} + \frac{1}{2} \int_0^t u^2 dt \right) \]

for the mass to initial mass ratio.
\[ m(t) = \frac{1}{m_0} \left( 1 + \alpha \frac{1}{2} \int_0^t u^2 \, dt \right) \]

and for the thrust to initial thrust ratio

\[ \frac{T(t)}{T_0} = \frac{u(t)}{u_0} \left( 1 + \alpha \frac{1}{2} \int_0^t u^2 \, dt \right) \]

By specific impulse constraints:

\[ I_{sp} \in [I_{sp_{min}}, I_{sp_{max}}] \]

It results the constraints needed for the “alpha”- parameter

\[ \alpha_{min} \leq \alpha \leq \alpha_{max} \]

where

\[ \alpha_{min} = \max \left( \frac{1}{\frac{1}{2} u_0 I_{sp_{max}} - \frac{1}{2} \int_0^t u^2 \, dt} \right), \quad \alpha_{max} = \min \left( \frac{1}{\frac{1}{2} u_0 I_{sp_{min}} - \frac{1}{2} \int_0^t u^2 \, dt} \right) \]

**1. The first case** (initial circular GEO velocity)

This case corresponds to the departure from GEO with circular orbital velocity. In Figs. 3-4, required initial thrust acceleration, the admissible values of specific mass when specific impulse is within the limits 1000-30000 s are presented for transfer durations between 5-19 days. For the maximum duration possible of the transfer (19 days) corresponding to specific mass of 26 kg/kW, feasible performances are represented in Figs 5-11.

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**Fig. 3** – Initial thrust acceleration required vs. transfer durations [day]
initial velocity = circular velocity
transfer durations = 5 ÷ 19 days

**Fig. 4** – \( \alpha \)-parameter admissible values envelope
initial velocity = circular velocity
transfer durations = 5 ÷ 19 days
\[ \alpha = 26 \text{ kg/kW}, \text{ specific mass} \]
\[ I_{sp} \in [1000, 30000] \text{ s}, \text{ specific impulse} \]
\[ 19 \text{ days, transfer duration} \]
\[ 3075 \text{ m/s, initial velocity} \]
\[ 7.2 \text{ mm/s}^2, \text{ initial thrust acceleration} \]
2. The second case (tangential initial thrust at GEO)

This case corresponds to departure from GEO with specified tangential initial thrust. In Fig. 12 is shown the values of admissible “alpha” parameter for initial thrust acceleration \( u_0 \in [0.1, 1] \text{ mm/s}^2 \) and transfer durations \( t_f \in [6, 9] \text{ days} \).

In the representations of Figs. 13-19 the spacecraft model with variable specific impulse engines and fixed engine power adopted here is a vehicle [1] with a total initial mass of 1500 kg and a constant output power of 10 kW (\( \alpha = 150 \text{ kg/kW} \) or \( \approx 7 \text{ W/kg} \) total specific power). It is to mention that presently for the engine only, typical values [15] of the specific power range between 100 and 200 W/kg or 5 to 10 kg/kW.

\[
\alpha_{\text{admissible}} \in [25, 170] \text{ kg/kW}
\]

Fig. 12 – \( \alpha \)-parameter admissible values envelope
- initial thrust acceleration = 0.1 \( \div 1 \text{ mm/s}^2 \)
- transfer durations = 6 \( \div 9 \text{ days} \)

\[
\alpha = 150 \text{ kg/kW}
9 \text{ days} – \text{transfer duration}
0.1 \text{ mm/s}^2 – \text{initial thrust acceleration}
\]

Fig. 13 – Specific Impulse vs. time [day]
- initial thrust acceleration = 0.1 mm/s²
- transfer duration = 9 days
Fig. 14 – Spacecraft mass vs. time [day]

Fig. 15 – Thrust vs. time [day]

Fig. 16 – Spacecraft trajectory, Moon’s orbit in GEI frame

Fig. 17 – Velocity vs. time [day] in GEI frame

Fig. 18 – Controls vs. time [day] in GEI frame

Fig. 19 – Control vs. time [day]
5. CONCLUSIONS

The research discussed above represents a preliminary study of possible ways for a spacecraft to reach the $L_4$ point of the Earth-Moon CR3BP. The approach is open to different models and propulsion types. Table 1 summarizes the main performances realizable in the two considered cases.

<table>
<thead>
<tr>
<th>Table 1 – Performances realized in cases (1) and (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cases (1) (2)</td>
</tr>
<tr>
<td>transfer duration [day]</td>
</tr>
<tr>
<td>initial velocity [km/s]</td>
</tr>
<tr>
<td>impulsive delta-v [km/s]</td>
</tr>
<tr>
<td>overall thrust delta-v [km/s]</td>
</tr>
<tr>
<td>initial thrust acceleration [mm/s$^2$]</td>
</tr>
<tr>
<td>initial mass / power jet [kg/kW]</td>
</tr>
<tr>
<td>specific impulse [s]</td>
</tr>
<tr>
<td>thrust / initial thrust [-]</td>
</tr>
<tr>
<td>final mass / initial mass [%]</td>
</tr>
<tr>
<td>power jet [kW]</td>
</tr>
<tr>
<td>initial mass [kg]</td>
</tr>
<tr>
<td>thrust [N]</td>
</tr>
<tr>
<td>fuel consumed [kg]</td>
</tr>
</tbody>
</table>

The transfer trajectories are characterized by the following:

- # fast transfer- up to 19 days
- # low-thrust accelerations
- # hight initial mass- up to 170 times Power
- # small fuel consumption- up to 2% of initial mass

The Variable Specific Impulse Magnetoplasma Rocket (VASIMR), is a new interesting type of engine, Fig. 16, [16], [17] capable of specific impulse/thrust modulation at constant power.

The findings suggest that, at least under approximations, models and parameterizations used in this paper, by adopting of a low-thrust accelerations constant power, the VASIMR engine represents the favorable option.
REFERENCES


