Continuous thrust tangential or perpendicular to the flight direction on a given transfer trajectory in planar Circular Restricted 3-Body Problem

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Abstract: *This paper presents a systematic study for analytical determination of thrust acceleration magnitude on a given trajectory for a case where the thrust is tangential or perpendicular to the flight direction. The problem has been formulated using the planar Circular Restricted Three-Body Problem. In the numerical application for tangential thrust the transfer from low Earth orbit to planar Lyapunov orbits around L_1 and L_2 Lagrangian points by a semi-elliptic transfer path is considered. For the case in which the thrust is perpendicular to the flight direction also on a semi-elliptic trajectory, the results are compared to the ones of Miele and Mancuso (2001), and Sandro da Silva Fernandes (2010) in their studies based on a simplified version of the same restricted three-body problem of transferring a spacecraft from a circular low Earth orbit to a circular low Moon orbit.*

Key Words: *planar Circular Restricted Three-Body Problem, given trajectory, continuous thrust control, Earth-Moon system, low Moon orbit, Lagrangian points.*

1. INTRODUCTION

The preliminary trajectory design is part of the whole spacecraft preliminary design aimed at determining the path of the spacecraft transfer from a starting point to a specified target. In designing the trajectory, the features of the transfer, usually specified in terms of certain performance functions, are evaluated, and the respect of some prescribed mission constraints is also assessed. Analytical solutions have been developed for many special-case transfers (in the Newtonian central field), such as the logarithmic spiral, Forbes' spiral, the exponential sinusoid, Markopoulos' Keplerian thrust arcs, Lawden's spiral, and the analogous Bishop's and Azimov's spiral. Although the resulting trajectory is not the actual solution of an optimal control, by tuning the shaping parameters it is possible to generate solutions which are sufficiently good to be considered into a more detailed optimization process. This paper is concerned with the choosing of a method of solving the control problem of design of a given parametric trajectory. The planar Circular Restricted Three-Body Problem (pCR3BP) is used as a model for non-Keplerian trajectories. The system consisting of the Earth, the Moon and a spacecraft is the one of interest here. Two cases are studied for thrust vector orientation: when it is tangential and perpendicular to the flight direction on a given trajectory. Under these considerations analytical formulas for the variation of velocity and thrust acceleration are obtained (the time is available through quadrature). In the numerical application for tangential thrust the transfer from the low Earth orbit (LEO) to Lyapunov orbits around L_1 and L_2 Lagrangian points by a semi-elliptic transfer path is considered, and for perpendicular thrust the transfer from the Earth low orbit to the Moon low orbit (LMO) the same semi-elliptic transfer path is taken into account.

2. CIRCULAR RESTRICTED 3-BODY PROBLEM

In celestial mechanics, the 3-Body Problem consists of three masses gravitationally attracting each other in space. In the Restricted Three-Body Problem, two of the three bodies have much larger masses than the third. As a result, the motions of the two larger bodies are unaffected by the third body. The larger bodies will however govern the motion of the small body. The simplest form of the Three-Body Problem involves the primaries movements on circular paths. The primary bodies are assumed to be point-masses and no other forces or perturbations are included in the model. For this Circular Restricted Three-Body Problem (CR3BP), there are five points (the stationary solutions of this problem) in the plane of the motion of the two primaries where the forces acting on the small body are balanced. These five points are called libration or Lagrangian points (labeled L_1, …, L_5): three collinear points along the line of the two primaries, and other two equilateral points (that form an equilateral triangle with the two primaries). The collinear points are of the highest interest and in particular the L_1 point between the two primaries and the L_2 point on the far side of the smaller primary. Fig. 1 shows the basic geometry of the system consisting of two primary masses m_1 and m_2 revolving around their common center of mass $cm(0,\ 0,\ 0)$ in circular orbits, and a spacecraft m moving within the system.

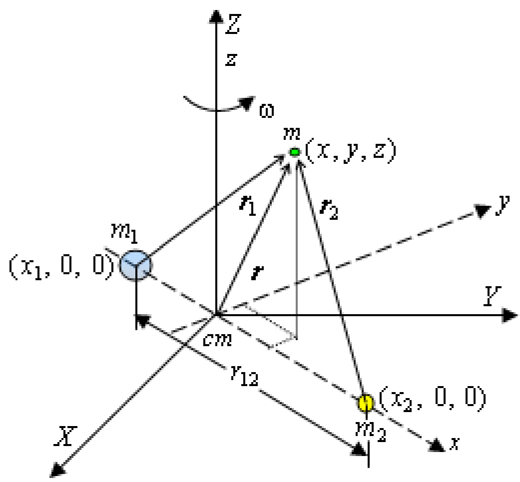


Fig. 1 – Basic geometry of the 3 -Body Problem, the pseudo-inertial (X, Y, Z) and the rotating (x, y, z) frames

At baseline: $(X, Y, Z) \equiv (x, y, z)$

Rotating frame: its origin lies at the center of mass *cm* of the two larger bodies; the *x* axis is directed towards m_2; it rotates with angular velocity ω around the *z*-axis (which is perpendicular to orbital plane); the *y* axis lies in the orbital plane and completes a trihedral dextrorsum.

$\omega = \sqrt{\dfrac{G(m_1 + m_2)}{r_{12}^{\ 3}}}$ - inertial angular velocity

r_{12} - distance between the two larger bodies

$x_1 = -\dfrac{m_2}{m_1 + m_2} r_{12}$, $x_2 = \dfrac{m_1}{m_1 + m_2} r_{12}$

G - the gravitational constant

The vector equation of motion for m relative to the common center of mass and to an inertial frame (in the absence of the thrust force, balistically) is

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{r}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{r}\right) + \boldsymbol{\omega} \times \left(\frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{r}\right) =$$

$$= \frac{\partial^2 \boldsymbol{r}}{\partial t^2} + \frac{\partial \boldsymbol{\omega}}{\partial t} \times \boldsymbol{r} + 2\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}).$$

But $\dot{\boldsymbol{\omega}} = 0$, hence

$$\ddot{\boldsymbol{r}} = \frac{\partial^2 \boldsymbol{r}}{\partial t^2} + 2\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

where $\frac{\partial^2 \boldsymbol{r}}{\partial t^2}$ denotes the acceleration relative to the rotating frame, and $\frac{\partial \boldsymbol{r}}{\partial t}$ is the relative velocity.

Also,

$$2\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial t} \text{ - Coriolis acceleration}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \text{ - centripetal acceleration.}$$

From Newton's 2nd law, the vector differential equation of motion is written

$$\ddot{\boldsymbol{r}} = \frac{1}{m}\left(-\frac{Gmm_1}{r_1^2}\frac{\boldsymbol{r}_1}{r_1} - \frac{Gmm_2}{r_2^2}\frac{\boldsymbol{r}_2}{r_2} \right),$$

so

$$-\frac{Gm_1}{r_1^3}\boldsymbol{r}_1 - \frac{Gm_2}{r_2^3}\boldsymbol{r}_2 = \frac{\partial^2 \boldsymbol{r}}{\partial t^2} + 2\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}).$$

Thus we can write

$$\frac{\partial^2 \boldsymbol{r}}{\partial t^2} = -\frac{Gm_1}{r_1^3}\boldsymbol{r}_1 - \frac{Gm_2}{r_2^3}\boldsymbol{r}_2 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) - 2\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial t}$$

where

$$\boldsymbol{r} = (x, y, z)^{\mathrm{T}}$$

$$\boldsymbol{r}_1 = (x - x_1, y, z)^{\mathrm{T}}$$

$$\boldsymbol{r}_2 = (x - x_2, y, z)^{\mathrm{T}}$$

$$\boldsymbol{\omega} = (0,\ 0,\ \omega)^{\mathrm{T}}$$

$$\frac{\partial \boldsymbol{r}}{\partial t} = (\dot{x}, \dot{y}, \dot{z})^{\mathrm{T}} = \boldsymbol{v}$$

$$\frac{\partial^2 \boldsymbol{r}}{\partial t^2} = (\ddot{x}, \ddot{y}, \ddot{z})^{\mathrm{T}}$$

$$2\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial t} = (-2\omega\dot{y},\ 2\omega\dot{x},\ 0)^{\mathrm{T}}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = (-\omega^2 x,\ -\omega^2 y,\ 0)^{\mathrm{T}}.$$

The equations of motion for the third body in the rotating frame are

$$\ddot{x} = 2\omega\dot{y} + \omega^2 x - \frac{Gm_1}{r_1^3}(x - x_1) - \frac{Gm_2}{r_2^3}(x - x_2)$$

$$\ddot{y} = -2\omega\dot{x} + \omega^2 y - \frac{Gm_1}{r_1^3}y - \frac{Gm_2}{r_2^3}y$$

$$\ddot{z} = -\frac{Gm_1}{r_1^3}z - \frac{Gm_2}{r_2^3}z,$$

where r_1 and r_2 are the distances from the third body to the larger and smaller ones, respectively

$$r_1 = \sqrt{(x - x_1)^2 + y^2 + z^2}\,; \qquad r_2 = \sqrt{(x - x_2)^2 + y^2 + z^2}\,.$$

If we define the potential function given by:

$$\Omega = \Omega(x, y, z) = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2} + \frac{1}{2}\omega^2\left(x^2 + y^2\right),$$

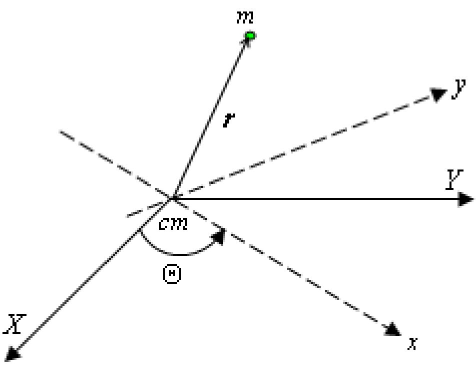
these equations have the form

$$\begin{aligned} \ddot{x} &= 2\omega\dot{y} + \Omega_x \\ \ddot{y} &= -2\omega\dot{x} + \Omega_y \\ \ddot{z} &= \Omega_z \end{aligned} \tag{1}$$

where the subscripts denote the partial derivatives of the function Ω, or in vector form:

$$\ddot{\boldsymbol{r}} = -2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \nabla\Omega\,.$$

Given the following orientation of the rotating frame (x, y, z) and inertial frame (X, Y, Z), in Fig. 2,



$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \boldsymbol{T}^{\mathrm{rel}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\boldsymbol{T}^{\mathrm{rel}} = \begin{pmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Fig. 2 – Orientation of the rotating frame (x, y, z) and inertial frame (X, Y, Z)

where $\omega = \dot{\Theta}$ is the angular velocity of the rotating frame in SI units (rad/s).
The velocity relative to the inertial frame may also be computed from

$$\dot{\boldsymbol{r}} = \frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{r}$$

Then

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \dot{\boldsymbol{T}}^{\mathrm{rel}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \boldsymbol{T}^{\mathrm{rel}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \text{ with } \dot{\boldsymbol{T}}^{\mathrm{rel}} = \dot{\Theta} \begin{pmatrix} -\sin\Theta & -\cos\Theta & 0 \\ \cos\Theta & -\sin\Theta & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \omega \begin{pmatrix} -\sin(\omega t) & -\cos(\omega t) & 0 \\ \cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} =$$

$$= \begin{pmatrix} -\omega x\sin(\omega t) - \omega y\cos(\omega t) + \dot{x}\cos(\omega t) - \dot{y}\sin(\omega t) \\ \omega x\cos(\omega t) - \omega y\sin(\omega t) + \dot{x}\sin(\omega t) + \dot{y}\cos(\omega t) \\ \dot{z} \end{pmatrix} =$$

$$= \begin{pmatrix} (\dot{x} - \omega y)\cos(\omega t) - (\dot{y} + \omega x)\sin(\omega t) \\ (\dot{x} - \omega y)\sin(\omega t) + (\dot{y} + \omega x)\cos(\omega t) \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} - \omega y \\ \dot{y} + \omega x \\ \dot{z} \end{pmatrix}.$$

A useful convention in the literature of specialty is to take a step further in normalizing the equations by making them dimensionless. To normalize the equations following the standard convention one takes:

$M = m_1 + m_2$ - reference mass;

r_{12} - reference distance (length);

$\frac{1}{\omega}$ - reference time, such that

the dimensionless mass-parameter is $\mu = \frac{m_2}{m_1 + m_2}$, and we assume $\mu \le \frac{1}{2}$.

By using these new units the gravitational constant becomes equal to one, the orbital period of m_1 and m_2 about their *cm* is 2π time units, and (the new notations are regularly written):

$\omega = 1$; $m_1 = 1 - \mu$; $m_2 = \mu$; $r_{12} = 1$; $x_1 = -\mu$; $x_2 = 1 - \mu$.

The equations of motion of the CR3BP in dimensionless coordinates become:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial\Omega}{\partial x} = x - (1-\mu)\frac{x+\mu}{r_1^3} - \mu\frac{x - 1 + \mu}{r_2^3} \\ \ddot{y} + 2\dot{x} &= \frac{\partial\Omega}{\partial y} = y - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3} \\ \ddot{z} &= \frac{\partial\Omega}{\partial z} = -(1-\mu)\frac{z}{r_1^3} - \mu\frac{z}{r_2^3} \end{aligned} \tag{2}$$

where

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2} \qquad r_2 = \sqrt{(x - 1 + \mu)^2 + y^2 + z^2}$$

It should be emphasized that as indicates, the forces can indeed be derived from the scaled potential

$$\Omega = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\left(x^2 + y^2\right) \tag{3}$$

Note that the 10 integrals of the general problem do not actually hold in the Circular Restricted 3-Body Problem per se, due to the assumptions placed on the third body and the restrictions of the role it plays for the motion of the other two. There will always be an error term proportional to m with respect to exact conservation. However, conservation of energy, though violated, has a clear analogue in the conservation of the Jacobi integral, which is formulated next, and is the only known integral for the CR3BP. Formulated for the full 3-dimensional case, there exists in these rotating coordinates a further constant of the motion due to Jacobi, given in dimensionless coordinates by

$$J = v^2 - 2\Omega \text{ or equivalently}$$

$$J = \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \left(x^2 + y^2\right) - \frac{2(1-\mu)}{r_1} - \frac{2\mu}{r_2},$$

where in the planar case z may be dropped from both the v and r_1, r_2.
Multiplying (2) by $\dot{x}$, $\dot{y}$, $\dot{z}$ and summing one obtains:

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \dot{x}\frac{\partial\Omega}{\partial x} + \dot{y}\frac{\partial\Omega}{\partial y} + \dot{z}\frac{\partial\Omega}{\partial z} = \frac{d\Omega}{dt}$$

Then we observe that

$$\frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}$$

so that

$$\frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{d\Omega}{dt}.$$

Then

$$\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \Omega(x, y, z) - \frac{C}{2},$$

where C is an arbitrary constant, and the minus sign and the factor of two are just a convention. Then it is the case that

$$2\Omega(x, y, z) - \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = C.$$

This constant $C = -J$, known as Jacobi's constant, is a first integral of the system in rotating coordinates, and plays a role analogous to the energy in inertial coordinates. Note that it is only a function of the position and velocity magnitude expressed in the rotating frame. A closed-form analytic solution to the CR3BP is not currently known to exist.

3. PLANAR CIRCULAR RESTRICTED 3-BODY PROBLEM

If we further restricted the motion of the third body to be in the orbital plane of the other bodies, the problem is called the planar Circular Restricted 3-Body Problem. Besides the previous systems considered, a coordinate system with axes always parallel to the inertial frame is attached in the larger bodies Below, there are the coordinates and velocities transformations, the inertial (X, Y) and the rotating (x, y), (X_1, Y_1), (X_2, Y_2) frames, and the geometry of the pCR3BP, Fig. 3.

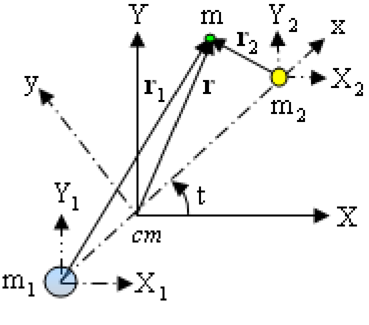


Fig. 3 – Basic geometry of the pCR3BP

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \mu\cos t \\ \mu\sin t \end{pmatrix}$$

$$\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} -(1-\mu)\cos t \\ -(1-\mu)\sin t \end{pmatrix}$$

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} \dot{x} - y \\ \dot{y} + x \end{pmatrix}.$$

For the numerical application we need the quantities:

$$\left(\dot{X}_1, \dot{Y}_1\right)^T = \left(\dot{X}, \dot{Y}\right)^T + \mu(-\sin t, \cos t)^T$$

$$\left(\dot{X}_2, \dot{Y}_2\right)^T = \left(\dot{X}, \dot{Y}\right)^T + (1-\mu)(\sin t, -\cos t)^T$$

Taking into account the first two equations (2)

$$\ddot{x} = 2\dot{y} + \Omega_x$$
$$\ddot{y} = -2\dot{x} + \Omega_y \tag{4}$$

where the auxiliary function Ω depends this time on the two coordinates (x, y) only and has the same expression as in equation (3) but for the spacecraft distances from the primaries (z = 0)

$$r_1 = \sqrt{(x+\mu)^2 + y^2}\,; \qquad r_2 = \sqrt{(x-1+\mu)^2 + y^2}\,.$$

The Jacobi constant reduces to

$$C = 2\Omega(x, y) - \left(\dot{x}^2 + \dot{y}^2\right) \tag{5}$$

and can be evaluated for a set of initial or final conditions.

By expansions, the coordinates and velocities are obtained relatively to the primary bodies in the forms

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x+\mu \\ y \end{pmatrix}$$

$$\begin{pmatrix} \dot{X}_1 \\ \dot{Y}_1 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \dot{x} - y \\ \dot{y} + x + \mu \end{pmatrix}$$

and

$$\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x-1+\mu \\ y \end{pmatrix}$$

$$\begin{pmatrix} \dot{X}_2 \\ \dot{Y}_2 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \dot{x} - y \\ \dot{y} + x - 1 + \mu \end{pmatrix}.$$

4. THE CONTROL ON A GIVEN PARAMETRIC TRAJECTORY

The control problem has been formulated using the pCR3BP. Two cases are studied for the thrust vector orientation: 1. when tangential and 2. when perpendicular to the flight direction on a given trajectory. The motion equation with a continuous thrust density $\mathbf{u} = \left(u_x, u_y\right)^T$ is

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} - \nabla\Omega = \mathbf{u}$$

or

$$u_x = \ddot{x} - 2\dot{y} - \Omega_x$$
$$u_y = \ddot{y} + 2\dot{x} - \Omega_y \tag{6}$$

1. The first case: $\mathbf{u} \parallel \dot{\mathbf{r}}$ $(\dot{\mathbf{r}} = \mathbf{v})$

involves that

$$\mathbf{u} \times \mathbf{v} = 0 \qquad \mathbf{u} \cdot \mathbf{v} = uv,\ u \ge 0, u \le 0$$

hence

$$\dot{y}u_x - \dot{x}u_y = 0 \tag{7}$$

and

$$\dot{x}u_x + \dot{y}u_y = uv \tag{8}$$

Replacing the relations (6) in (7), we obtain the next equation for the velocity in the rotating frame

$$2v^2 - \dot{x}\Omega_y + \dot{y}\Omega_x + \dot{x}\ddot{y} - \dot{y}\ddot{x} = 0. \tag{9}$$

The trajectory is given in parametric form

$$\begin{cases} x = x(\theta) \\ y = y(\theta) \end{cases}$$

where $\theta_0 \le \theta \le \theta_f$; θ_0 - initial value of θ; θ_f -final value of θ, for which

$$\begin{aligned} \dot{x} &= \frac{\partial x}{\partial \theta}\frac{d\theta}{dt} = x'\dot{\theta}; \\ \dot{y} &= \frac{\partial y}{\partial \theta}\frac{d\theta}{dt} = y'\dot{\theta}; \\ \ddot{x} &= \left(x'\dot{\theta}' + x''\dot{\theta}\right)\dot{\theta}; \\ \ddot{y} &= \left(y'\dot{\theta}' + y''\dot{\theta}\right)\dot{\theta}. \end{aligned} \tag{10}$$

From the first two relations (10) the time variation of the parameter θ is obtained

$$\dot{\theta} = \frac{v}{\sqrt{x'^2 + y'^2}} = \frac{d\theta}{dt} \tag{11}$$

By replacing relations (10) and (11) in (9), we obtain the following expression for the velocity

$$v = v(\theta) = \frac{\left(x'^2 + y'^2\right)^{3/2}}{x'y'' - y'x''}\left[-1 \pm \sqrt{1 - \frac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^2}\left(y'\Omega_x - x'\Omega_y\right)}\right] \tag{12}$$

where

$$\begin{aligned} &x'y'' - y'x'' \ne 0; \\ &1 - \frac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^2}\left(y'\Omega_x - x'\Omega_y\right) \ge 0; \\ &v(\theta) \ge 0. \end{aligned}$$

It is noted in (12) that in the stationary points of the pCR3BP the velocity is zero; such a variation may be suitable for direct transfer in the Lagrangian points. The magnitude of the thrust density is obtained from (8) as

$$u = \frac{\dot{x}u_x + \dot{y}u_y}{v} \tag{13}$$

being directed along the velocity vector. By replacing in (13) the thrust density components (6), using relations (10) and considering the derivative

$$\dot{\theta}' = \frac{\left(x'^2 + y'^2\right)v' - \left(x'x'' + y'y''\right)v}{\left(x'^2 + y'^2\right)^{3/2}};$$

the magnitude of the thrust density is finally obtained

$$u = u(\theta) = \frac{vv' - \Omega_x x' - \Omega_y y'}{\sqrt{x'^2 + y'^2}}$$
$$\left(u = \dot{v} - \frac{\dot{\Omega}}{v}\right) \tag{14}$$

being directed along ($u > 0$) the velocity vector. The variation of time is obtained from (11)

$$t = t(\theta) = \int_{\theta_0}^{\theta} \frac{\sqrt{x'(\theta)^2 + y'(\theta)^2}}{v(\theta)} d\theta \tag{15}$$

available by quadrature.

2. The second case: $\mathbf{u} \perp \dot{\mathbf{r}} \quad (\dot{\mathbf{r}} = \mathbf{v})$

involves that

$$\mathbf{u} \cdot \mathbf{v} = 0$$

hence

$$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow u_x \dot{x} + u_y \dot{y} = 0 \Rightarrow \dot{x}\ddot{x} + \dot{y}\ddot{y} - \dot{\Omega} = 0$$
$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = \mathbf{v} \cdot \dot{\mathbf{v}} = \frac{d}{dt}\left(\frac{\mathbf{v} \cdot \mathbf{v}}{2}\right) = \frac{d}{dt}\left(\frac{v^2}{2}\right)$$
$$\frac{d}{dt}\left(\frac{v^2}{2}\right) - \frac{d}{dt}(\Omega) = 0$$

i.e. the same variation as that given by (5)

$$v^2 = 2\Omega - C \tag{16}$$

The magnitude of thrust acceleration is

$$u = \frac{|\mathbf{u} \times \mathbf{v}|}{v}$$
$$u = \frac{|u_x \dot{y} - u_y \dot{x}|}{v} = \frac{|-2v^2 + \dot{y}\ddot{x} - \dot{x}\ddot{y} + \dot{x}\Omega_y - \dot{y}\Omega_x|}{v} =$$
$$= \frac{|-2v^2 + (y'x'' - x'y'')\dot{\theta}^3 + x'\dot{\theta}\,\Omega_y - y'\dot{\theta}\,\Omega_x|}{v},$$

resulting from algebraic manipulation in

$$u = \left| \frac{y'x'' - x'y''}{(x'^2 + y'^2)^{\frac{3}{2}}} v^2 - 2v + \Omega_x \frac{x'}{(x'^2 + y'^2)^{\frac{1}{2}}} - \Omega_y \frac{y'}{(x'^2 + y'^2)^{\frac{1}{2}}} \right| \tag{17}$$

The variation of time is obtained as for (15).

5. NUMERICAL APPLICATIONS

In this study, the primary bodies are assumed to be the Earth (m_1) and the Moon (m_2), while the motion of a spacecraft is modeled in term of (m). For the Earth-Moon system, the dimensionless mass-parameter is $\mu = 0.012150868$, the unit of distance r_{12} equals 384400 km, the unit of time t equals 375190.371 s, the unit of velocity equals 1024.547 m/s and the unit of acceleration equals 2.7307 mm/s^2.

The value of the dimensionless mass-parameter is considered for the following data

$\mu_1 = Gm_1 = 3.986 \cdot 10^5$ km^3/s^2 – Earth standard gravitational parameter [1];

$\mu_2 = Gm_2 = 4.903 \cdot 10^3$ km^3/s^2 – Moon standard gravitational parameter [1].

Therefore

$$\mu = m_2/(m_1+m_2) = (\mu_2/G)/[(\mu_1/G)+(\mu_2/G)] = \mu_2/(\mu_1+\mu_2) = 0.012150868$$

and the radii of the primary bodies

$$R_1 = 6378 \text{ km – Earth radius; } R_2 = 1738 \text{ km – Moon radius.}$$

1. The first case (tangential thrust)

The transfer from LEO(463 km) to Lyapunov orbits (semiminor axis 1000 km) around L_1 and L_2 Lagrangian points by a semi-elliptic transfer path (counterclockwise) is considered. In Table 1 and Figures 4-9 the characteristics of the two transfers are presented.

Table 1 – Trajectories characteristics: LEO – L_1, L_2 planar (Lyapunov) orbits

Lagrangian Point	**L_1**	**L_2**
periapsis radius (km)	6,841	6,841
apoapsis radius (km)	325,380	447,916
semi-major axis (km)	166,111	227,378
semi-minor axis (km)	47,180	55,355
eccentricity	0.959	0.970
transfer time (days)	15	25
initial velocity (km/s)	10.648	10.678
final velocity (m/s)	12.6	8.9
maximum thrust acceleration (m/s^2)	0.034	0.036

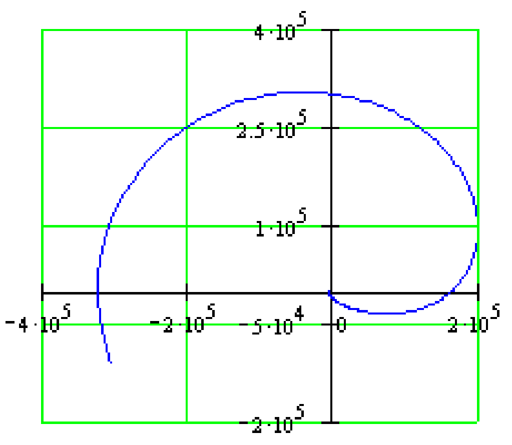


Fig. 4 – LEO-L_1 trajectory, in the Earth reference system (both coordinates in km)

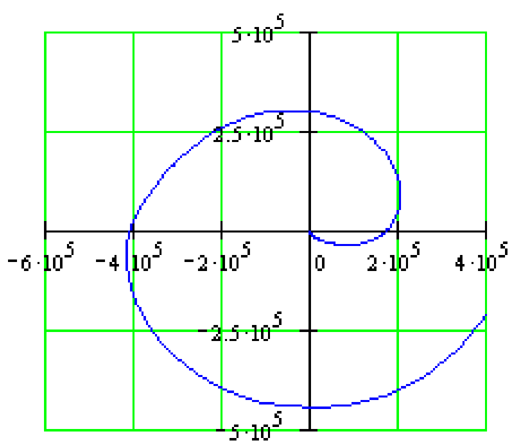


Fig. 5 – LEO-L_2 trajectory, in the Earth reference system (both coordinates in km)

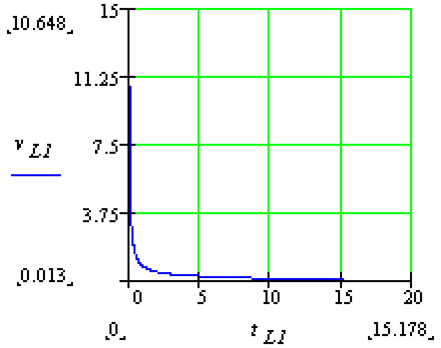


Fig. 6 – LEO-L$_1$: Velocity v_{L1} (km/s) vs. time t_{L1} (day), in the rotating system

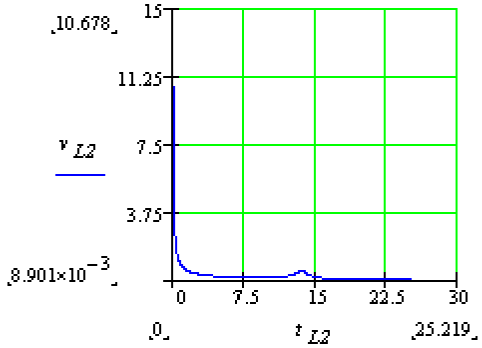


Fig. 7 – LEO-L$_2$: Velocity v_{L2} (km/s) vs. time t_{L2} (day), in the rotating system

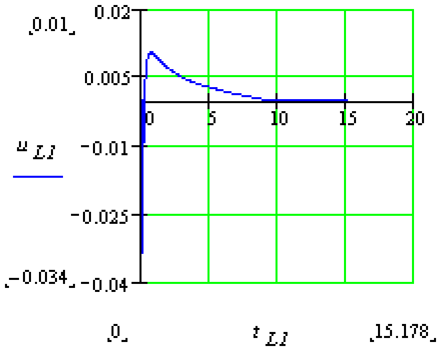


Fig. 8 – LEO-L$_1$: Needed thrust density (m/s^2) vs. time (day)

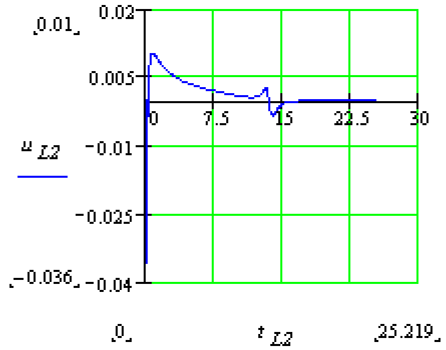


Fig. 9 – LEO-L$_2$: Needed thrust density (m/s^2) vs. time (day)

2. The second case (perpendicular thrust)

The transfer from LEO to LMO (sheltered side) by semi-elliptic transfer path is considered. The following data are used:

$$h_{LEO} = 463 \text{ km}, \; V_{LEO} = 7.633 \text{ km/s};$$

$$h_{LMO} = 100 \text{ km}, \; V_{LMO} = 1.633 \text{ km/s}.$$

Jacobi's constant which gives the velocity variation is calculated as the solution of the equation

$$t_f - t(\theta_f) = 0 \tag{18}$$

for specified time t_f.

The value of transfer time is considered [1], [3], $t_f = 4.370$ day. Using (15) and (16), equation (18) has the form:

$$t_f - \int_{\theta_0}^{\theta_f} \frac{\sqrt{x'(\theta)^2 + y'(\theta)^2}}{\sqrt{2\Omega(\theta) - C}} d\theta = 0, \; (\theta_0 = \pi, \; \theta_f = 2\pi).$$

The resulting numerical solution of Jacobi's constant value is C = 2.844. The results are presented in Figures 10-17.

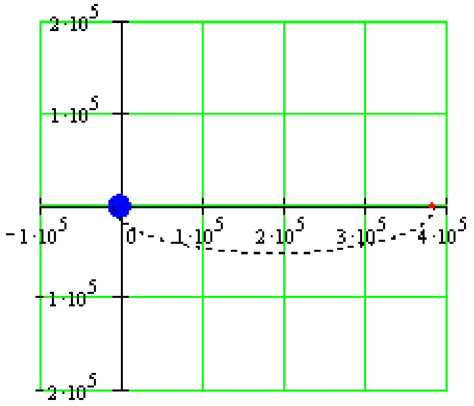


Fig. 10 – LEO - LMO trajectory in the rotating system (coordinates in km)

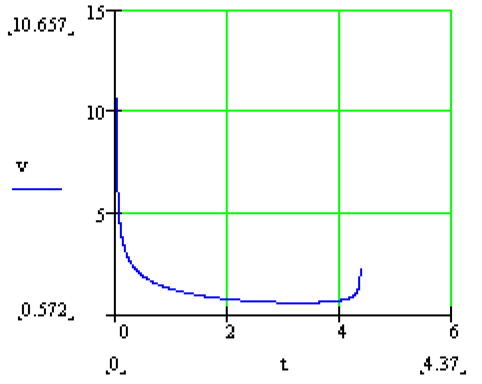


Fig. 11 – Velocity *v* (km/s) vs. time *t* (day), in the rotating system

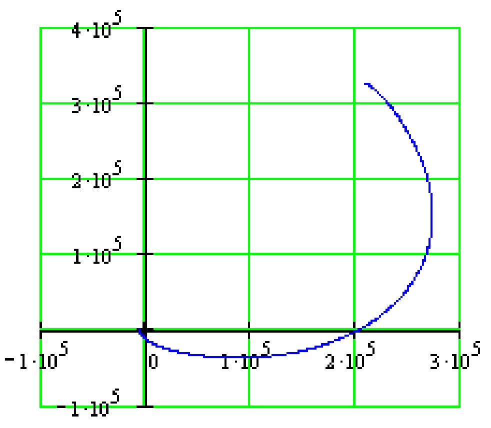


Fig. 12 – LEO - LMO trajectory, in the Earth reference system (both coordinates in km)

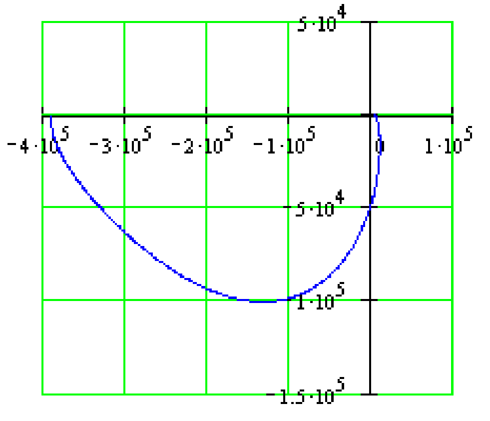


Fig. 13 – LEO - LMO trajectory, in the Moon reference system (both coordinates in km)

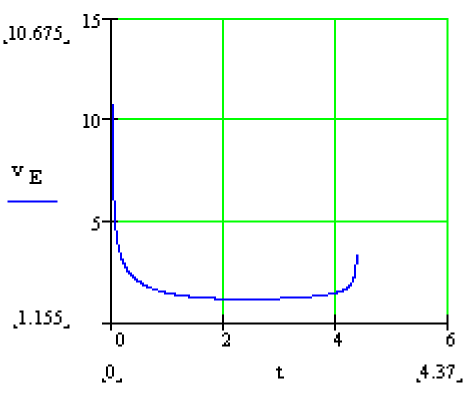


Fig. 14 – Velocity v_E (km/s) vs. time *t* (day), in the Earth reference system

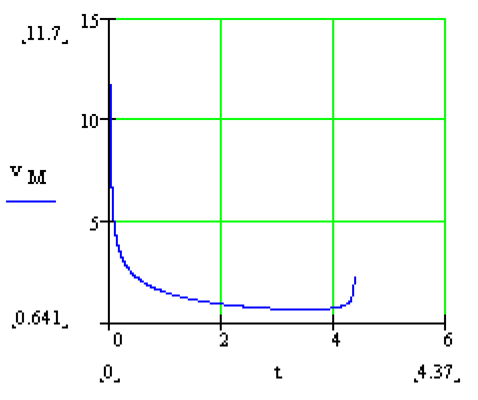


Fig. 15 – Velocity v_M (km/s) vs. time *t* (day), in the Moon reference system

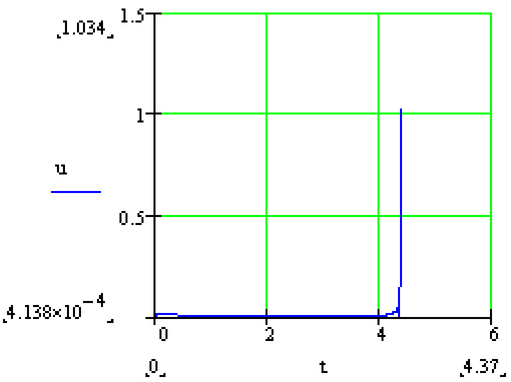


Fig. 16 – Thrust acceleration u (m/s^2) vs. time t (day)

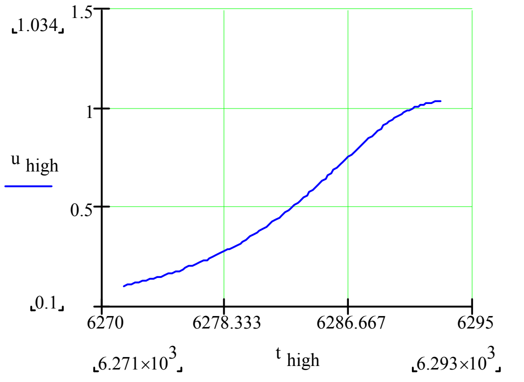


Fig. 17 – High thrust acceleration u_{high} (m/s^2) vs. time t (minute)

With the acceleration levels [4], ("Very low thrust": $u \approx 10^{-5}$, "Low thrust": $u \approx 10^{-2}$ to 10^{-1}, "High thrust": $u \approx 0.5$ to 1.0 m/s^2) the propulsion system characteristics are shows in Table 2.

Table 2 – Thrust characteristics for the LEO - LMO transfer on a semi-elliptic path

Time t (days)	$0 \cdots 4.355$	$4.355 \cdots 4.370$ ($\Delta t = 22$ minutes)
Thrust	Very low, Low	High

The velocity changes at each thrust impulse can be determined from:

$$\Delta V_1 = \left| \left(\dot{X}_1(0), \dot{Y}_1(0) \right)^{\mathrm{T}} - \left(0, -V_{\mathrm{LEO}} \right)^{\mathrm{T}} \right|$$

$$\Delta V_2 = \left| \left(\dot{X}_2(t_f), \dot{Y}_2(t_f) \right)^{\mathrm{T}} - V_{\mathrm{LMO}} \left(-\frac{Y_2(t_f)}{R_2 + h_{\mathrm{LMO}}}, \frac{X_2(t_f)}{R_2 + h_{\mathrm{LMO}}} \right)^{\mathrm{T}} \right|$$

and the total characteristic velocity is then given by $\Delta V = \Delta V_1 + \Delta V_2$.

Table 3 shows the results for lunar mission with a counterclockwise LEO departure and a counterclockwise LMO arrival.

The major parameters that are presented in the table are the velocity changes ΔV_1 and ΔV_2 at each impulse, and the total characteristic velocity ΔV. The results are presented for the same transfer time t (4.37 days) and are compared to the results obtained in [1] and [2].

Table 3 – The major parameters of a Lunar mission, for LEO departure and LMO arrival, both counterclockwise (comparison)

	Model adopted		time t (day)	ΔV (km/s)	ΔV_1 (km/s)	ΔV_2 (km/s)
	paper's author	problem type				
LMO altitude (km)	Miele and Mancuso (2001) [1]	free flight of the 3rd body (on a ballistic trajectory); a transfer optimization problem	4.370	3.876	3.065	0.811
	Sandro Fernandes da Silva (2010) [2]		5.563	3.8758	3.0649	0.8109
	Fazelzadeh and Varzandian (2010) [3]		4.370	-	-	-
	Dumitrache - present (2011)	controlled motion of the 3rd body (on a given parametric trajectory) continuous perpendicular thrust	4.370	3.748	3.042	0.706

The orthogonality of the velocity and position vectors is assured at baseline and at final point.

6. CONCLUSION

In this paper a systematic study of determination of acceleration magnitude for continuous thrust directed tangential or perpendicular to the flight direction on a given trajectory is presented. The problem has been formulated in the framework of the pCR3BP (planar circular restricted three-body problem) theory and has been analytically solved. For the case of a thrust directed perpendicularly to the flight direction the results are presented for the same lunar mission (LEO-LMO) as that described in [1] and [2] (where the problem is formulated using a simplified version of the restricted three-body model). The difference between the values in Table 3 is due to the followings:

- in [1] and [2] is assumed a fixed Earth, the Moon and the third body are moving around it; the thrust is applied to the 3rd body, only on departure and arrival; the transfer trajectory is purely ballistic;
- in this work it is assumed that the Earth and the Moon are moving around their common center of mass; the thrust is applied to the 3rd body throughout the given transfer trajectory.

The given trajectories (half ellipse) in Earth-Moon system are characterized by low-thrust acceleration level ($10^{-3}g_0$) and for short period of time by high low-thrust acceleration ($10^{-1}g_0$), in which we denoted by g_0 the Earth gravitational acceleration at the sea level (the Earth is considered spherical, homogeneous and isotropic).

The paper is the first to analyze (based on the specialized literature known by the author) the performances made by the transfer with the continuous thrust (parallel or perpendicular to the velocity direction, in the revolving plan) on a trajectory given by parameters on pCR3BP through analytical assessments that can be made to:

- thrust acceleration (density) limits;
- velocity variation;
- transfer time.

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[2] S. Fernandes, *Optimal Trajectories for Earth-Moon Flight*, Proceedings of the 9th Brazilian Conference on Dynamics, Control and their Applications, Serra Negra, SP-ISSN 2178-3667, pp. 1219-1224, 2010.

[3] S. A. Fazelzadeh, G.A. Varzandian *Minimum-time Earth-Moon and Moon-Earth orbital maneuvers using time-domain finite element method*, Acta Astronautica, Volume **66**, Issues 3-4, pp. 528-538, February-March 2010.

[4] V. A. Chobotov (ed.), *Orbital Mechanics*, 3rd ed., AIAA Education Series, pp. 129, 2002.