A Wave Theory of Incompressible Fluid Turbulence: Wall/ Elastic Turbulence

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Abstract: *The true problem of turbulence dynamics is the problem of its origin and successive development from some initial conditions (IC) at some boundary conditions (BC) to an ultimate state restored, in a statistical sense, as fully developed turbulence. Since this intricate route is unknown up to now, a second approach is used quite frequently. That is, turbulent flows are studied as they are disregarding their origin, and without looking into some of the details of the mechanisms of turbulence production and sustenance. The last approach has been mainly represented by Kolmogorov's papers which have kept the turbulence community quite busy until now. This fractional/ local approach aiming to describe theoretically the extremely complicated details of the visible fluctuating motion superimposed on a main motion, i.e. its "noise", to the detriment of the essential aspects of turbulence (the origin and its self-sustaining mechanism), was however worth developing both mathematical and experimental analysis methods for the representation/ decomposition of the flow field. On the other hand, the extremely detailed decompositions became to some extent less useful by obscuring the physics of turbulence. In this paper, we propose a holistic approach for the entire evolution of turbulence phenomenon as being created and governed by boundary vorticity dynamics, which in our opinion mostly responds to Leonardo da Vinci's questions: where the turbulence is generated, where the turbulence maintains for long, and where the turbulence comes to rest. In contrast to the previous approaches, here we will try firstly to understand the hidden causes of various existing visible facts and then to explain them.*

Key Words: *Boundary layer, Vorticity waves, Laminar-turbulent transition, boundary vorticity dynamics.*

1. INTRODUCTION

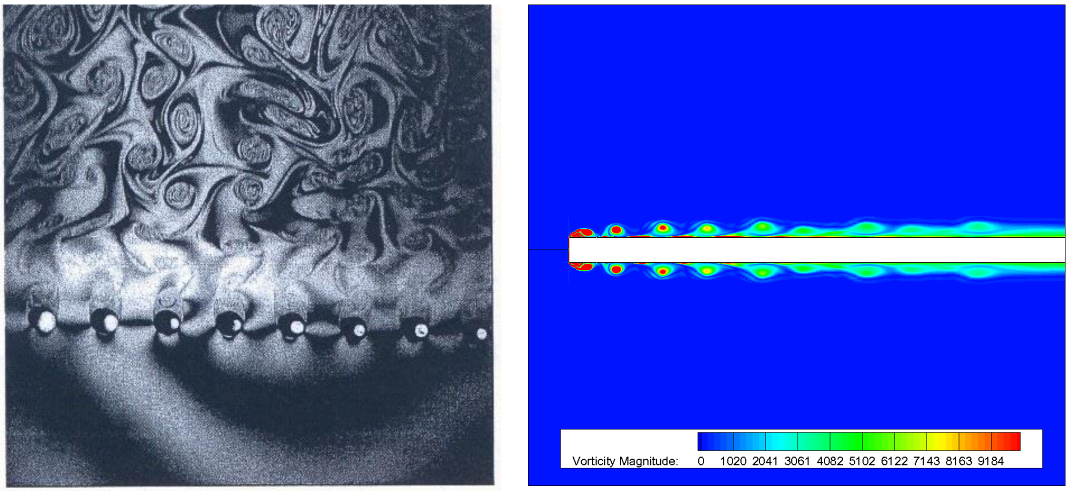
The fluid as a deformable continuum without own shape its motion must be guided by some physical surfaces/boundaries, which are subjected to an active action from fluid under the form of the wall pressure, equal to the reaction of the boundaries on the fluid. The fundamental law of equal action and reaction was easily misleading by assuming that the wall pressure is equal to the static pressure from the main motion. This result known as Prandtl's hypothesis associated to the concept of a Newtonian fluid [1], i.e. a viscous fluid obeying a linear law of frictional shear stress and constant viscosity, could resolved the no-drag crisis (d'Alembert's paradox [2]), and have dominated the paradigmatic development in research of Fluid Dynamics during the last century. However, the motions fluctuating in a disordered manner observed experimentally by Reynolds [3] and known as turbulent/noisy motions or simply turbulence remain a long-lasting and continuing paradigmatic crisis [4]. The issue involves a set of questions concerning the various theories of turbulence depending crucially on the boundary and initial conditions which are poorly known so that the physical/mathematical problem of turbulence is an ill-posed one [5].

From the physical/ mechanical point of view, the all previous theories are based on the idealized Prandtl-hypothesis, i.e. constant static pressure across the wall-bounded flow and a Newtonian fluid concept, which more or less violate the law of equal action and reaction failing to elucidate the nebulous aspects of turbulence concerning its origin, self-sustenance and fading. Concretely, the lack of a suited boundary reaction is reflected by the existence of a gap between the scales of molecular/laminar motions and the scales of the smallest relevant scales in fluid flows including turbulence. The complex interplay between the flowing fluid and its boundaries results in an oscillating behavior of the intrinsic properties of the fluid – boundary system which has opposite tendencies: return force (wall pressure) and inertia (viscosity/shear of fluid). The “return force” tries to cancel out by imparting a suitable “velocity” to the moving part. The higher the flow rate U, the stronger the return force. For the oscillating fluid-boundary system, the “return force” is due to the boundary vorticity or wall pressure, which retards the fluid particles not to adhere to boundaries. The second property, “inertia” opposes any change of the flow state and is due to the viscosity of the fluid which must adjust itself continuously and respond the flow/stress. Therefore, the physical concept of the Newtonian fluid is an ill-defined concept, being too rigid to reliably describe the turbulent motions. As d’Alembert’s paradox was solved by a paradigm change from the concept of ideal fluid to the concept of Newtonian (linear) viscous fluid, it follows naturally that the turbulence crisis is a paradigmatic one due to the idealized Prandtl-hypothesis and an ill-defined medium/fluid. The issue with the Newtonian fluid concerns firstly, the impossibility to describe the intense/concentrated boundary vorticity, i.e. its creation as the reaction of the boundary, and secondly, the impossibility to reflect the change from any state of microstructure of fluid to another for different states of flow, i.e. thixotropic/nonlinear effects [6]. From the mechanical point of view the Newtonian fluid can describe only some simple/laminar flows (sparse/diluted vorticity/small angular velocity and constant viscosity/mass), in contrast to the turbulent flows involving variations of both vorticity concentration and viscosity.

The paper aims to remove the above drawbacks and to develop a vorticity wake mechanism for the turbulence production and its sustenance using the dual concept: torsion of vorticity filaments and thixotropic/nonlinear fluid. In contrast to the previous energy dissipation models, the proposed model is based on a self creation dispersion mechanism of the boundary vorticity which manifests an elastic behavior.

2. ON PHYSICS OF VORTICITY

The problem of vorticity is a more complex question than the simple definition of vorticity flux (Lighthill, 1966) [7] that concerns the vorticity creation and boundary vorticity dynamics, inducing the velocity field of both laminar and turbulent flow as a result of a collision process produced during the starting of motion. The fluids as deformable bodies without own shape, when start from rest, experience interactions between the flowing fluid and the physical surfaces enclosing the flow. These interactions are a kind of impact process where there is a momentum exchange between the flow and its boundary surfaces, with zero mass flux. Within a short time of contact a post-impact shear flow occurs where two main effects are triggered off by the flow-induced collision: dramatic redistribution of the momentum and the boundary vorticity creation followed by the shear stress/viscosity change in the microstructure of the fluid which at the beginning behaves as linear reactive medium (laminar flow) and latter as nonlinear dispersive medium (turbulent flow). The disturbances of the starting flow ($t \to 0$) cause the entanglement of the wall-bounded flow (stream function $\Psi = 0, t \to 0$) inducing a wall torsion pressure (suction) in the form of concentrated vorticity balls [8], as the result of boundary reaction. The concentrated vorticity balls are vorticity sources whence waves are emitted and propagated through the flow field [8], [9]. Such vorticity concentrations are observed both experimentally [10] and computationally [9] in Fig. 1, as well as in the natural case of tornados. The phenomenology of the impact process of any starting fluid motion shows that at the beginning the vorticity lines are subjected to a torsion pressure followed by their concentration at the boundaries. The vorticity conglomerations at sufficiently large Reynolds numbers are then dispersed in a shear layer by shear waves.



a) b)

Figure 1 - Two-dimensional turbulent flow: a) visualized in a soap film [10];
b) computed plate wall flow (LES model, $\mathrm{Re} = 2 \cdot 10^6$) [9].

In contrast with the common view that the origin of turbulence lies in the instability of some basic laminar flows, here the turbulence is the result of the boundary vorticity dynamics from the flow-induced collision and its consequences where shear waves are emitted and propagated in the basic flow field from the vibrating concentrated vorticity at the boundaries.

This is understood in the sense that any flow starts from rest at some moment in time, and as long as the Reynolds number, or the reduced frequency of vorticity is small, the flow remains laminar (creeping motion of vorticity). As the Reynolds number/frequency of vorticity increases the boundary shear waves set off a wide instability range, which is followed by transition and then a fully developed turbulent state. This is the visible face or large scale of the flow field, i.e. the velocity fluctuations, whereas the poorly known origin of turbulence is the invisible face or small scale of the flow field, i.e. the vorticity fluctuations.

Therefore, the origin of turbulence is a problem of boundary vorticity dynamics that needs a holistic approach of the motion process containing the linked up events: the flow surface colliding and starting vorticity creation, the non-dispersive creeping motion of vorticity (laminar flow) and the dispersive vibrating motion of vorticity (turbulent flow).

Each perturbation, small or large, is in fact a kind of impact between the flow and its boundaries exchanging the momentum between bodies within a short time of contact, without mass flux. As a result, the initial velocity distribution is rapidly skewed/squeezed, the vorticity is created and organizes itself into more and more concentrated structures, thus at the boundary there is a set of point-vortices. The exact solutions of the equations of inviscid motion found by Stuart [8] (e^{τ} - solutions, $\tau \in [0,2]$) can describe strong vorticity concentrations developing in skewed shear layers. The concentration level of vorticity is estimated on a natural logarithmic scale e^{τ} from e^0 - sparse/weak vorticity, up to e^2 – concentrated vorticity, where the index τ is a measure of the concentration of vorticity. In contrast to the sparse vorticity transported by the ideal fluid flow according to the laws given by Helmholtz [11], the concentrated vorticities are transported by waves, their shape depending mostly on the concentration of vorticity, Fig. 2.

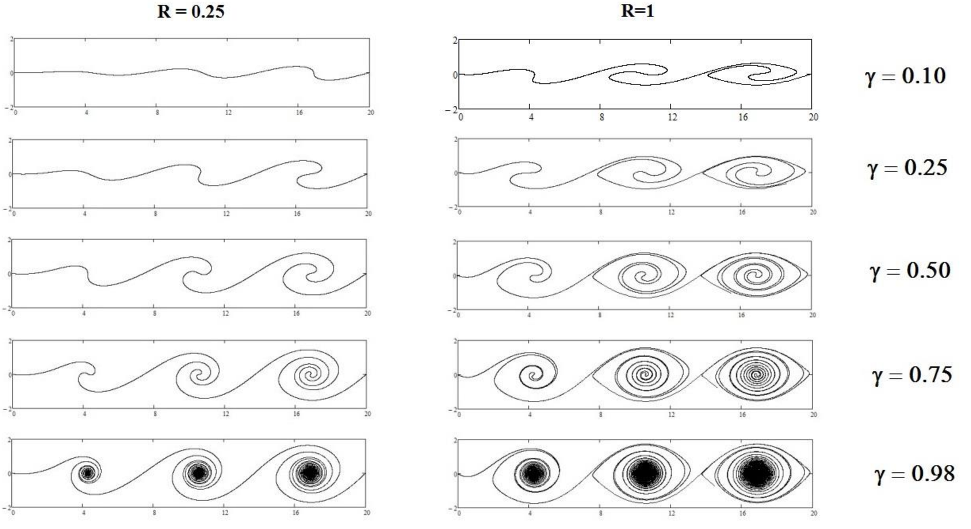


Figure 2 - Effect of γ (concentration), R (circulation) parameters on streakline patterns [9].

From the physical point of view, the concentration of vorticity at boundaries is a local compression of flow induced by the torsion of vorticity wires. The concept of torsion of the concentrated vorticity allows a better understanding of the boundary vorticity creation and its dynamics, which becomes an active one just after impact. Essentially, the boundary vorticity dynamics contains small amplitude vibrational motions generating vorticity weak waves that create the covered/hidden field of flow. The onset of self-sustained vorticity waves is the mechanical origin of turbulence.

Concomitantly with the vorticity creation the impact process induces microstructure changes of the flow properties resulting in a time dependent shear stress, $\nu = \nu(t)$, known as the thixotropic behavior of the flowing fluid. It is experimentally shown that the transient viscosities follow the line of the complex viscosity versus angular frequency [6]. This behavior can be described by a Klein-Gordon like wave equation [12]

$$\frac{d^2\nu(x)}{dx^2} = \frac{1}{U_e^2}\left(\omega_0^2 - \omega^2\right)\nu(x), \tag{1}$$

where ν_0 is steady shear viscosity and $\omega_0 = \nu_0^{-1}$ is the natural angular frequency of fluid, defined as the first zero of the Fourier coefficient **B** (ω) of a square pulse/impact. Thus, the solutions of Eq. 1, describe well enough the dual behavior of the thixotropic fluid, as a reactive medium, $\omega < \omega_0$, at impact inducing exponential waves (without energy dissipation), and as a dispersive medium which can support sinusoidal waves for ω above the natural frequency ν_0^{-1}. Equation (1) also shows that the microstructure takes time to respond to the flow/stress and its elastic response at low frequency is faster as the flow velocity U_e increases.

The disturbed post-impact flow is a boundary-layer type flow which is relaxed through a complicated wave system, which transports concentrated vorticity from boundaries to the flow field and rebuilds the flow microstructure. There is a non-dispersive transport of vorticity performed by exponential waves in the form of the laminar flows dominated by the frictional shear stress and a dispersive one which involves lightly damped sinusoidal waves by dry friction in turbulent flows. Hence, it is evident that the analysis of the impact-relaxation process requests another constitutive relation to describe the intricate behavior of viscous fluid. For the thixotropic fluid, such a relationship is a shear compliance defined as

$$\frac{1}{\rho} p_{torsion,w} \equiv \omega_w \nu = U_e^2 \text{ on } \partial \mathrm{B}, \tag{2}$$

where $p_{torsion,w}$ is the torsion pressure at wall, $\omega_w = e^{\tau}$ is the vorticity at a two-dimensional wall ($\partial \mathrm{B}$), τ is a torsion/concentration index $\tau \in [0,2]$, and ν (t) denotes the change of viscosity during the post-impact flow which is able to adjust itself continuously. The shear compliance expresses the law of equal action (U_e^2 - dynamic pressure) and reaction ($\omega_w \nu$ - torsion pressure) in a manner more exact than Newton's law of friction.

A non-steady fluid system involves an oscillating behavior of its opposite, intrinsic properties (vorticity and viscosity) and suddenly excited it decays as a big damped harmonic oscillator. The evolution is slow and a plotting on suited scales is necessary to visualize its extremely complicated route. Using the exponential scales and measure units e and ν_0^{-1}, the shear compliance, Eq. 2, can be written as

$$\begin{aligned} &e^{\tau}\left(\nu_0^{-1}\right)^{1/\tau} = \mathrm{Re}_x \text{ for } \mathrm{Re}_x < \mathrm{Re}_c \text{ (laminar flow)} \\ &e^{1/\tau}\left(\nu_0^{-1}\right)^{\tau} = \mathrm{Re}_x \text{ for } \mathrm{Re}_x \geq \mathrm{Re}_c \text{ (turbulent flow)} \end{aligned}, \tag{3}$$

where the critical Reynolds number $\mathrm{Re}_c = e^2 \nu_0^{-1}$ is the non-rolling condition for concentrated vorticity, which separates the non-periodic creeping motion of vorticity inducing laminar flow from the torsional vibration motion of vorticity generating turbulent flow, and ν_0^{-1} is the natural frequency of the thixotropic fluid. Equations (3) show the conservation of boundary vorticity in laminar flow, its dispersion in turbulent flow, respectively.

For the above $\mathrm{Re}_x = \nu_0^{-1}$ the transient flow in the neighborhood of the wall vibrates as a continuous and homogeneous string carrying transverse vorticity waves which permanently disperse vorticity. Figure 3 illustrates the wave system induced by the flow-boundary impact and the dispersion mechanism of concentrated vorticity that displays wide frequency/ Reynolds number spectrum from the low indifference Reynolds number $\mathrm{Re}_{lind} = e^2 \nu_0^{-1/2}$ - the onset of the weakest waves (TSW-Tolmien-Schlichting waves) up to the high indifference Reynolds number $\mathrm{Re}_{hind} = e^{1/2}\nu_0^{-2}$. For the last Re the wave system becomes a slightly damped one with the resonance close to the natural frequency ν_0^{-1}, emitting acoustic waves. The transitional flow displays a strong beat phenomenon with varicose aspect of vorticity for above $\mathrm{Re}_x = e^2\nu_0^{-1}$ where its frequency $\mathrm{Re}_x^{1/2}$ is far from the resonance frequency (Fig. 1). For above $\mathrm{Re}_x = e^{2/3}\nu_0^{-3/2}$, the vorticity is broken down in contra-rotating fragments/flocks and its frequency approaches the resonance frequency where the flow is full turbulent and it can be statistically described. The Reynolds number controls the wave system playing a role of tuning button that switches the frequency band from concentrated vorticity and small Re/low frequency and long wavelength to dispersed vorticity and high Re/high frequency and short wavelength.

The essential difference between the laminar flow and the turbulent flow is given by the difference between the behaviors of fluid as linear viscoelastic-reactive medium and nonlinear thixotropic-dispersive medium. That is, while both are time effects, the former is in the linear region, where the microstructure responds but remains unchanged and the latter takes place in the nonlinear region where the microstructure is broken down by deformation as well as responding to it.

At the smallest scales of the flow there is a vorticity field like an elastic coverlet/diaphragm over the boundary having a wide frequency spectrum. The fundamental difference between these two types of flows is given by the vibration frequency of the elastic coverlet. The nature and the location where the turbulence is generated justify the term of wall/elastic turbulence.

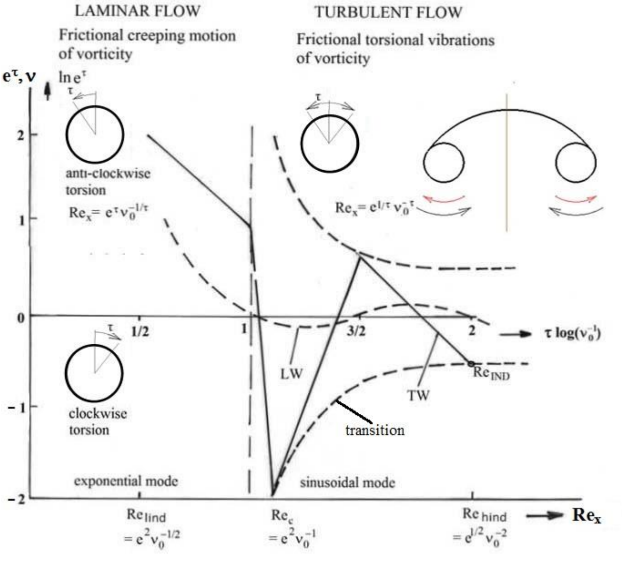


Figure 3 - Transverse waves (TW) modulated in amplitude on a longitudinal carrying wave (LW) and dispersion mechanism of concentrated vorticity by shear waves [9].

3. PLATE BOUNDARY-LAYER FLOW

The main reason for considering the Prandtl boundary-layer equations is to expose clearly ideas on the origin and extremely intricate mechanism of turbulence. A holistic approach of turbulence dynamics is used where the spectacular fluctuating motion/visible noise, and so hopelessly complex to describe, is disregarded and only the main motion is considered, but without the above remarked drawbacks.

Thus, the main (laminar) flow field is governed by the Navier-Stokes equations in the boundary-layer approximation and their similar solutions [1], where the law of equal action and reaction obeys a shear compliance relation, Eq. 3.

The similar solutions provide a smooth passing from the wall-bounded flow dominated by the boundary vorticity dynamics (an autonomous motion) to the non-autonomous basic flow (here outer flow $U_e\left(x\right)=const.$), describing a kind of diffeomorphism in flow field [13].

In this context, the motion equations contain an irrotational potential $\nabla\varphi=U_\infty x$ (outside a shear layer) and a vector potential $\mathbf{A}=\psi\left(x,y\right)\mathbf{k}$ (inside and normal to shear layer) where ψ is the stream function defined by

$$u=\frac{\partial\psi}{\partial y},v=-\frac{\partial\psi}{\partial x}, \tag{4}$$

Using the approximations and notations of the boundary-layer flow [1]

$$\psi=\sqrt{2\nu xU_\infty}\,f\left(\eta\right),\delta\left(x\right)\approx\left(\frac{x\nu}{U_\infty}\right)^{1/2},\eta=\frac{y}{\delta\left(x\right)}, \tag{5}$$

where $f\left(\eta\right)$ is the dimensionless stream function, $\delta\left(x\right)$ is a scaled measure of the boundary-layer thickness (up to the approximation $u=0.99U_\infty$) and $\frac{u}{U_\infty}=f'\left(\eta\right)$ is the similarity law of the velocity profile, the boundary layer equations and their boundary equations become one ODE for the stream function

$$\begin{cases} k_w f'''+ff''=0 \\ \eta=0: f=0, f'=0 \\ \eta\to\infty: f'=1 \end{cases}. \tag{6}$$

This nonlinear third order equation and the three boundary conditions completely determine its solution, if the mute constant k_w can be known a priori.

Blasius found a solution for $k_w=2$, known as Blasius equation, but, Eq. 6 describes a more general phenomenon, that of the transverse standing vorticity/shear waves, called solitons, which retain their identity upon a collision.

The vorticity soliton identified here as Blasius soliton, depends on the function k_w directly related to Re_x via the shear compliance, Eq. 3 , which is responsible for coupling between the autonomous fast motion of vorticity and the velocity field of the non-autonomous slower flow,

$$\begin{aligned} &k_w=e^{\tau} \text{ for } \mathrm{Re}_x<\mathrm{Re}_c \text{ (weak coupling)} \\ &k_w=\left(\log\mathrm{Re}_x\right)^{-1} \text{ for } \mathrm{Re}_x\geq\mathrm{Re}_c \text{ (strong coupling)} \end{aligned}. \tag{7}$$

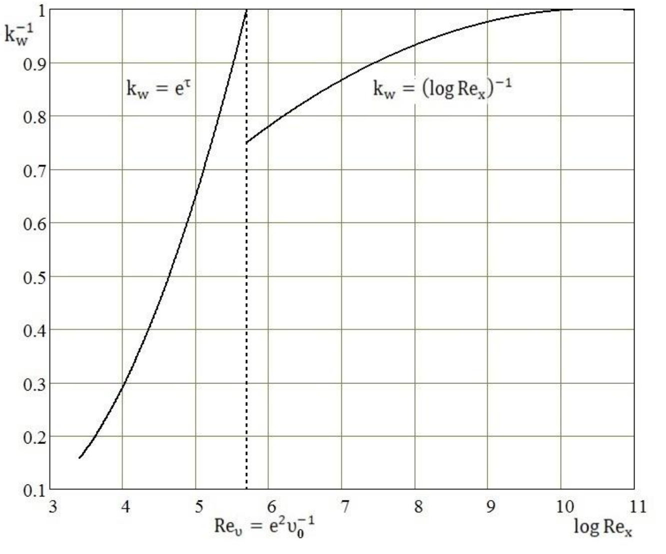


Figure 4 - Coupling function $k_w(\mathrm{Re}_x)$

Figure 4 shows the variation of the coupling function $k_w(\mathrm{Re}_x)$, that as a matter of fact is a vorticity creation boundary condition, with a physical support, for any wall-bounded flow. Now, justly Re_x known the successive solutions of Blasius soliton can be easy computed by a standard shooting technique.

Figure 5 a, b shows the solution for the vorticity waves close to the ends of the Reynolds spectrum, i.e. flow field at small scale for Prandtl flow. In contrast to the strong shock/pressure waves, the weaker vorticity waves propagate under the form of the three wave packet: shear stress wave f'', elastic wave f''' and dispersion wave f^{iv}. The vorticity wave packet is a superimposing of three waves, each wave having different roles depending on the Reynolds number (the flow type), that is while the laminar flows (Fig 5a) are dominated by the shear stress waves induced by the creeping motion of vorticity (without energy dissipation), in turbulent flows (Fig. 5b) the elastic dispersive waves, induced by torsional vibrations of vorticity, are the key mechanism of the turbulence phenomenon.

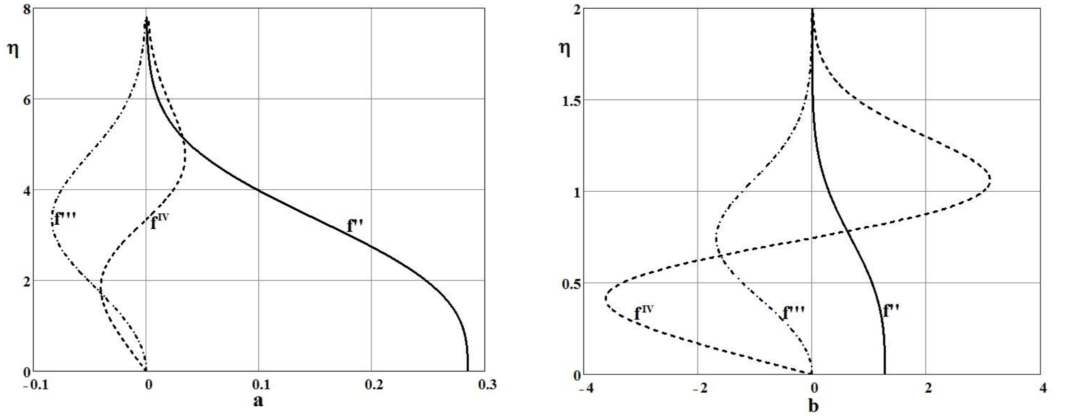


Figure 5 - Blasius soliton for T – waves, (Prandtl flow): f'' - shear wave, f''' - elastic wave, f^{Iv} –dispersion wave for a) $k_w = e$ and b) $k_w = e^{-2}$

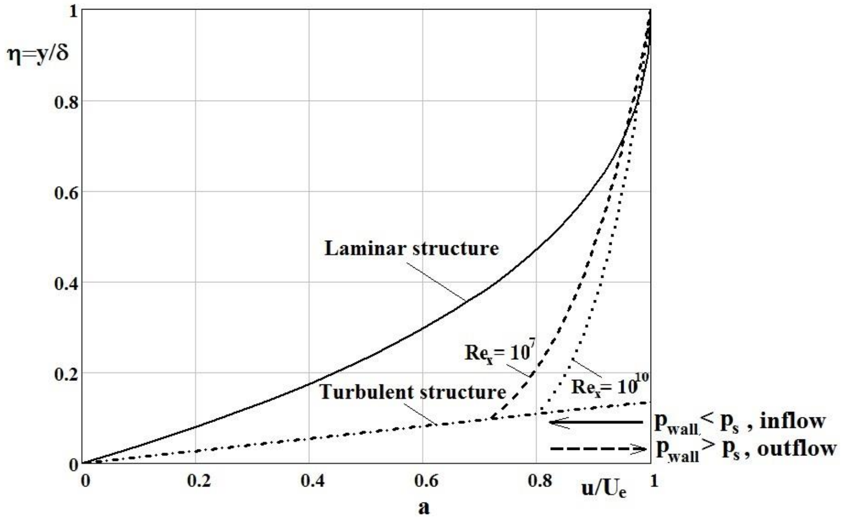


Figure 6 - Average velocity profiles and dynamic features of a wall-bounded flow:
——— laminar, – – – – – turbulent two layer model

Figure 6 shows the flow field at large scale marking also the vibration tendencies of flow near wall when the Reynolds number exceeds its critic value.

The analysis of the wall-bounded flow both at small scale (vorticity field) and large scale (velocity field) points out the self-sustained wave mechanism of turbulence, that is similar to synthetic jets generated by pulsed jets with zero net-mass flux and a small flow-momentum consumption [14], Fig. 7.

The thixotropic fluid in the turbulent wall-bounded flows operates as a diaphragm that alternatively sucks inflow and ejects outflow in a periodic manner, creating discrete/concentrated vortical structures followed by their dispersion by wall friction and transport by flow.

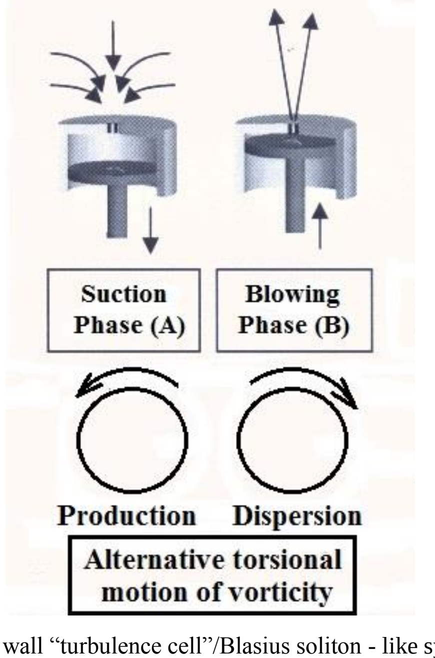


Figure 7: Principle of a wall “turbulence cell”/Blasius soliton - like synthetic jets generator

4. CONCLUSIONS

The objective of this paper has been to correctly pose the turbulence problem and make a contribution towards a better understanding of the phenomenon of turbulence in order to assist in the knowledge about the basic physical processes of turbulence, its generation and origin. There is no consensus on what is physics and what is mathematics of turbulence due to the lack of a theory itself to guide researches. In this context we have proposed a holistic approach of turbulence problem involving its origin and successive development from initial conditions: $\mathrm{Re}_c = e^2 \nu_0^{-1}$ - non-rolling wall condition for concentrated vorticity, at some boundary conditions-successive Blasius solitons, to a terminal (statistical) state: $\mathrm{Re}_{hinf} = e^{1/2} \nu_0^{-2}$ and last Blasius soliton.

Moreover, the holistic approach can elucidate some controversial questions:

- <u>The turbulence</u> is a flow/motion state of any slight/thin viscous fluid when a control parameter-Reynolds number exceeds its critical value ($\mathrm{Re}_c = e^2 \nu_0^{-1}$).
- <u>The origin of turbulence</u> is the result of the intrinsic property of shear-thinning fluids/materials that vigorously loaded when starting from rest are relaxed and at one time manifest a thixotropic response/behavior. That is, at impact short times the fluid structures cannot respond quickly, remain unchanged, and have a linear viscoelastic/laminar response, while further the response becomes a nonlinear thixotropic/turbulent one where the structure is broken down by deformation (here torsion of boundary vorticity) and the motion/velocity field is regularized.
- <u>The turbulence phenomena</u> contain the ensemble of flow manifestations observed/perceived/non-understood from diverse turbulent flow categories/types.
- <u>The problem of turbulence</u> is one of paradigmatic nature depending on the solution of the key issue of “medium” and flow-boundary reaction.

We hope that by means of a more correctly defined concept of the fluid (dual concept of torsional concentrated vorticity-thixotropic fluid, introduced in the paper), it is possible to improve the paradigms in research of turbulence thus stopping its crisis.

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