Boundary Vorticity Dynamics at Very Large Reynolds Numbers

Horia DUMITRESCU[1], Vladimir CARDOS*[,1], Ion MALAEL[2]

*Corresponding author
[1]"Gheorghe Mihoc – Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy
Calea 13 Septembrie no. 13, 050711 Bucharest, Romania
horiadumitrescu@yahoo.com, v_cardos@yahoo.ca*
[2]National Research and Development Institute for Gas Turbine, COMOTI, Bucharest, Romania
ion.malael@comoti.ro

DOI: 10.13111/2066-8201.2015.7.3.8

Received: 15 June 2015 / Accepted: 13 august 2015

Abstract: *During the last century, the theoretical investigations in the boundary vorticity dynamics were based on the concept of a fluid of small-constant viscosity, i.e. the Newtonian fluid, described by the non-slip condition at the wall and the linear law for fluid friction. But the application of current ideas in non-linear hydrodynamic stability theory to the flow in shear layers showed the existence of a class of flows involving concentrations of vorticity, also visible both in experimental conditions and industrial environments. The role of concentrated vorticity in fluid dynamics phenomena, concerning both the vorticity creation at the boundary and the response/reaction to the flow field is not entirely understood. The main purpose of this paper is to bring about a better mechanism of vorticity creation at the wall beneath a flow using the concept of thixotropic fluid associated with an active vorticity governed by the vorticity transport equation that is able to react back on the fluid flow. Such a viscoelastic behavior can be easily forecasted by the relationship between the value of the critic Reynolds number,* Re_c, *and the equilibrium kinetic viscosity* $\nu_0, \mathrm{Re}_c = e^2\nu_0^{-1}$, *argued in the sequel by means of the thixotropic fluid concept.*

Key Words: *Boundary layer structure, Shear waves, Vortex dynamics*

1. INTRODUCTION

When starts from rest and interacts with the solid/fluid surface any flow introduced at a physical surface experiences a kind of thixotropic behavior and can memorize the initial conditions at the start up time, i.e. the impact conditions. A thixotropic viscous response shows an overshoot in the stress which makes its behavior quite complicated due to the unknown deformation prehistory on the fluid [1]. The impact is a process of momentum exchange between colliding bodies within a short time of contact when the vorticity is acquired at the physical solid/fluid surface. The impact problems of Newtonian fluid flows are drastically simplified by a vanishing contacting time assumption. This idealization loses the sudden jump in the velocity distribution, i.e. the onset of vorticity creation at a boundary.

Consequently, the class of flows of engineering interest, representing small or large periodic perturbations from a skewed shear layer must be approached by means of a thixotropic fluid able to describe better the impact process and, thus, the genesis of boundary vorticity that is the cornerstone of its dynamics. Hence, the fundamental relation for the Newtonian fluid is replaced by a more general one

$$\tau_w = \mu\omega_w \text{ on } \partial B, \tag{1}$$

where ω_w is the transverse vorticity over two-dimensional wall ∂B, which can take values from 1- uniform vorticity distribution, up to e^2 - highly non-uniform vorticity distribution [2], and μ accounts for its change during the start up flow.

Equation (1) for skewed shear layers represents, in general, a transverse shearing stress, that in contrast to the Newtonian classical frictional stress contains also elastic torsion Fourier components for both the small and concentrated vorticity, described in the more general form of the torsion pressure

$$p_{torsion} = \mu\omega,\ \omega \in [1, e^2] \text{ on } \partial \mathrm{B}. \tag{2}$$

The concept of torsion for vorticity wires is used for better understanding of various vorticity-boundary interactions and the role of concentrated vorticity on its transport in large Reynolds number flows.

2. SOLID/FLUID BOUNDARY-FLOW INTERACTION

Any flow experiences various interactions from physical (solid/fluid) surfaces and, in fact, the theoretical fluid dynamics is the (hi)story of the development of these interactions. But Fluid Dynamics has provided and still provides a number of problems mathematically unsolved and/or poorly understand from a physical point of view, as paradoxes. The early issue of paradigmatic nature was the D'Alembert's paradox of drag or "small" D-paradox, in contrast to the present nonsolved problem of turbulence, or "big" T-paradox. As D'Alembert's paradox is the result of the use of the concept of perfect fluid, it is rationally to consider that the problem of turbulence can be similarly the result of an improper fluid concept − the Newtonian fluid, too restrictive $\tau = \mu \frac{du}{dy}$ and $\mu = \mu_0$, to describe intricate phenomena as the turbulence [3]. In the paper, such a phenomenon is represented by the issue of vorticity-boundary interaction involving a complicated two-step process: the vorticity-creation at the solid-fluid/ fluid-fluid interface via non-constant viscosity and no slip condition followed by its relaxation/dispersion into the flow field depending on the Reynolds number as a control parameter governing the entire process.

The perturbation caused by the motion of the solid/fluid surfaces into a flow is firstly an impact problem related to its start-up which is drastically simplified by engulfing the impact event into a confused initial condition when $t \to 0$. The unknown origin of this condition is the main drawback of previous works [4], [5], [6], [7]. Moreover, the improper initial condition leads to an ill-posed problem of the Navier-Stokes equations, which makes it inadequate to describe the turbulent motion of the Newtonian fluid governed by the linear relation $\tau = \mu \frac{du}{dy}$ and constant viscosity. Therefore, the vorticity-boundary interaction problem will be considered as an entire process composed by three events:

Events	Description
Impact, surface-flow colliding	boundary-flow momentum exchange; velocity distribution skews
Vorticity creation at boundaries	sudden increase of vorticity to the value of e^2 for stable flows [2]
Relaxation, the reaction of created vorticity to the flow field	- at large Reynolds number, $\mathrm{Re}_x < \nu_0^{-1}$, the response in Newtonian fluid: vorticity diffusion flux [4]; -at very large Reynolds, $\nu_0^{-1} \le \mathrm{Re}_x \le e^2\nu_0^{-1}$, the response in viscoelastic type fluid: vorticity transport by longitudinal-transverse waves

In the sequel it is account for thixotropic effects on the viscosity due to the flow-induced collision.

2.1 Impact-vorticity creation

First step of vorticity-boundary interaction process involves the perturbations in time generating the changes of vorticity at the boundary of a flow. Stuart's study, of a mixing layer of than *y* form reported on the existence of a class of hydro dynamically stable flows representing skewed shear layers, with strong vorticity concentrations. The change of the flow pattern in terms of the level of concentration is discussed with reference to the vorticity given by

$$\omega = e^{-2\Psi} = -\left[\cosh y + A\cos x\right]^2 , \tag{3}$$

where $\Psi = \ln\left(\cosh y + A\cos x\right)$ is the stream function, *x* is the longitudinal coordinate in the direction of mean flow and *y* is the transverse coordinate normal to that direction; the *x* and *y* velocity components are $u = \frac{\partial\Psi}{\partial y}$ and $v = -\frac{\partial\Psi}{\partial x}$, respectively.

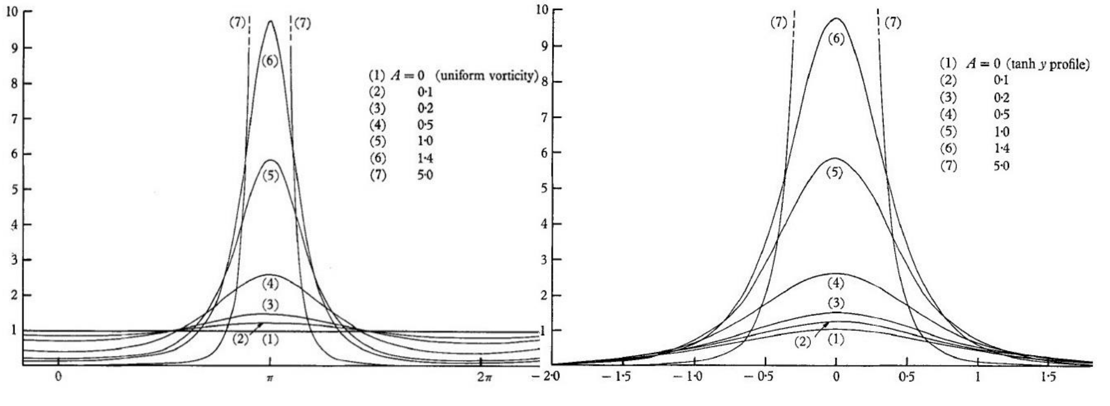

Figure 1. – Gauss like vorticity distribution

The Gauss function like solution named Stuart's e^{ω} –solution, represents a spatially periodic mixing layer, consisting of a single row of vortical structures, with constant circulation, $\Gamma = 4\pi$, around the contour: $y=\pm\infty, x=0,2\pi$, Fig. 1. e^{ω}-solution can be summarized as:

1) for the same amount of vorticity entering the flow field it can describe different distribution of ω with x and y changes and values from e^0 (uniform distribution) to e^2 (strongly skewed distribution);

2) the motion following impact is initiated with a set of point vortices on the boundary, Fig. 2;

3) the time-dependent solutions simulating flows generating small and large periodic perturbations in shear layers are stationary nondispersive waves traveling along with the flow in the longitudinal direction obtained by translation of axes, Fig. 3. As it is seen in Fig. 3, the shape of waves depends to a great extent on the concentration of vorticity [8].

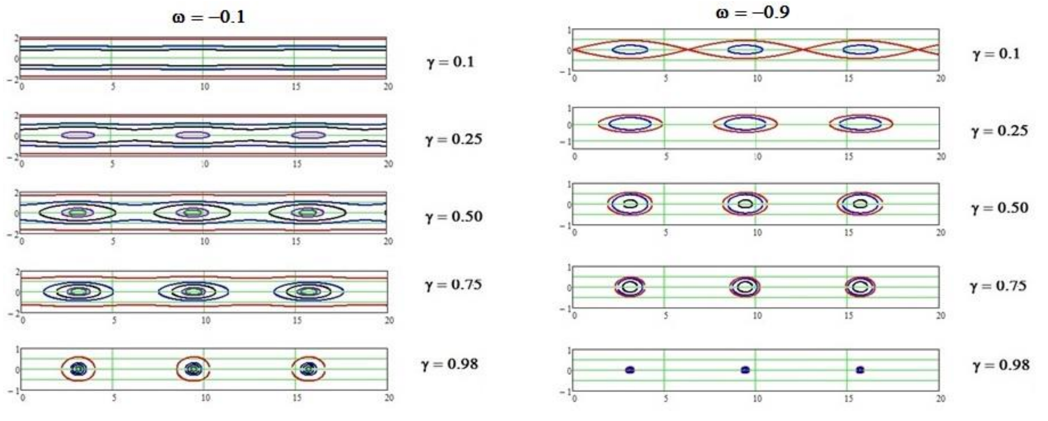

Figure 2. – Contours of constant vorticity for several ω

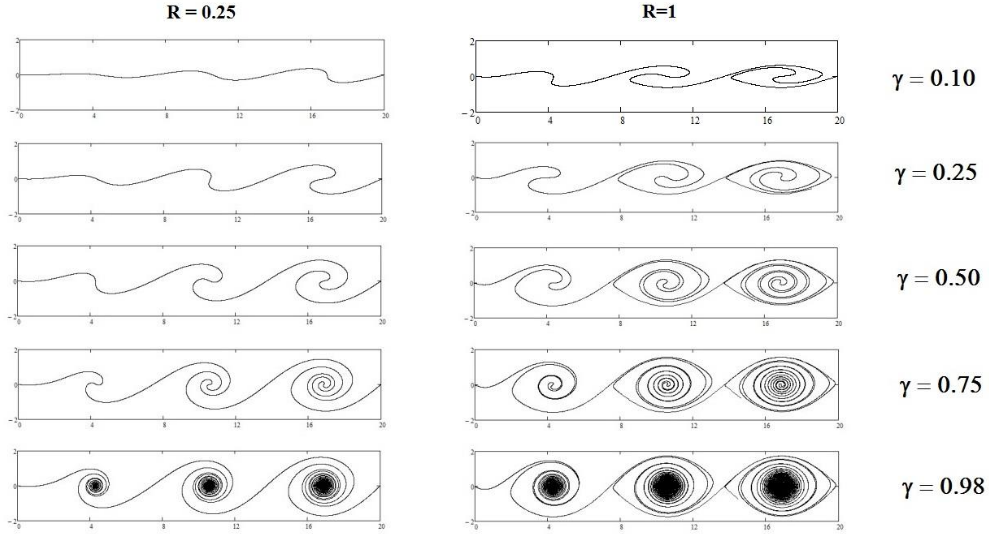

Figure 3. – Effect of γ on streakline patterns R=0.25 and R=1, any ω.

$$\int_{e^0}^{e^2} d(\ln\omega) = 2\,. \tag{4}$$

The relation (4) shows the consistence of the concept of torsion wire for the concentrated vorticity.

Two important parameters must be taken in consideration in order to describe the vorticity distribution entering the mixing layer:

$$\omega = -R\frac{1-\gamma^2}{\left[\cosh y+\cos\left(x-t\right)\right]^2}, \tag{5}$$

- the circulation parameter defined as the extreme velocity ratio $R=\left(U_1-U_2\right)/\left(U_1+U_2\right)$, and
- the vorticity concentration parameter γ calculated as the ratio of maximum total vorticity to maximum mean vorticity (at $y = 0$, $x\text{-}t = \pi$) [9].

$$\frac{\omega_{\max}}{\bar{\omega}_{\max}} = \left(1+\gamma\right)\sqrt{\frac{1+\gamma}{1-\gamma}}. \tag{6}$$

The concentration of vorticity at the boundary was simulated/provoked by successive increments $\delta\omega$ for several values of vorticity ω and vorticity concentration parameter γ according to (5).

This result evidently shows that the correct modeling of a vorticity-boundary interaction requires to associate the torsion concept of vorticity with a viscoelastic type fluid able to react back on the fluid flow.

2.2 Post-impact evolution

The post-impact relaxation/reaction of the concentrated boundary vorticity aims at describing the mechanism of vorticity change at the wall and also at estimating the amount of vorticity entering the flow.

In the case of Newtonian fluids, Lighthill described the vorticity production at a solid boundary as a slow diffusion process of the vorticity similarly to Fourier's heat conduction

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\frac{\partial\omega}{\partial y} \text{ on } \partial B, \tag{7}$$

where diffusivity constant ν is the kinematic viscosity, as that for momentum, Fig. 4 [4]. However, Lighthill's mechanism is a qualitative one that can predict only the weak vorticities entering the flow.

Its restriction is in fact the constraint of the too stiff Newtonian fluid. The vorticity dynamics as a whole requires the use of a thixotropic fluid hypothesis [1] to more accurately simulate the torsion shearing stress of vorticity at impact and post-impact.

A typical behavior of the viscosity of a thixotropic fluid is shown in Fig. 5(a, b). At impact, the viscosity will initially be higher, but will decrease and end up at the same value or at a lower level depending on how vigorously the material was initially loaded.

Similarly, the post-impact thixotropic response is seen as an overshoot in the stress controlled flow.

The time of shearing is accounted by the Reynolds number based on the reference viscosity ν_0.

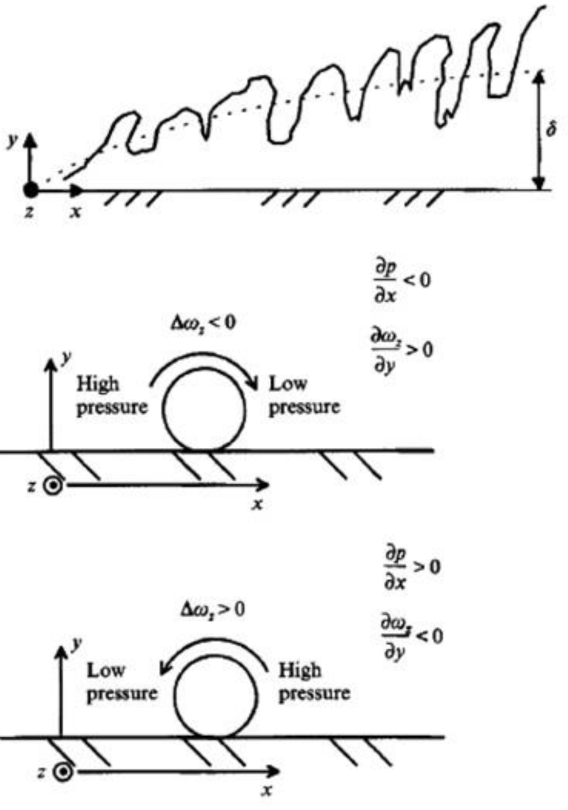

Figure 4. – Lighthill diffusion mechanism of vorticity change at the wall beneath a non-zero gradient pressure flow

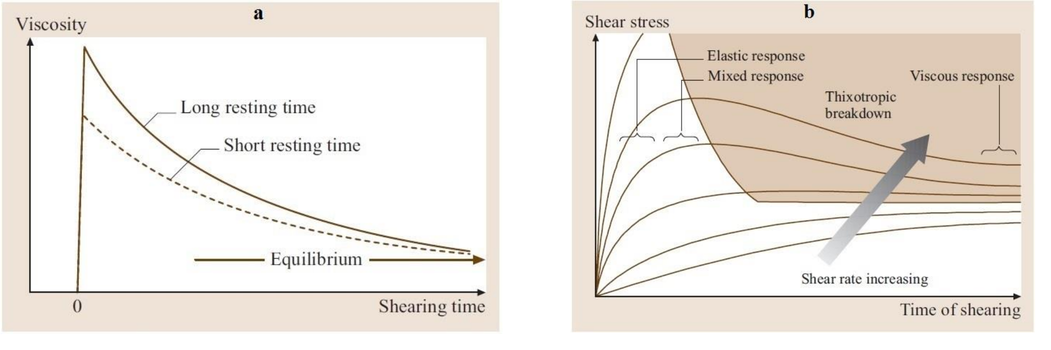

Figure 5. – Thixotropic effects of start-up flow a) shearing of a thixotropic fluid, b) various regions of start-up flow [1]

An assumption with respect to the deformation of the fluid at the time after impact is necessary. In this case, it is sufficient to assume that the admissible distribution of vorticity is affined to the one of the fluid, $\omega\nu$ = constant, which together with (2) defines the constitutive relation of viscoelastic type fluid. Furthermore, the vorticity-boundary interaction recasts into the problem of a vorticity source where waves are emitted and propagate in the flow field. Figure 6 illustrates the wave mechanism of vorticity transport near the wall beneath a flow at very large Reynolds numbers, $\mathrm{Re}_x = \frac{1}{\nu_0} - \frac{e^2}{\nu_0}(\mathrm{Re}_c)$ (see next section).

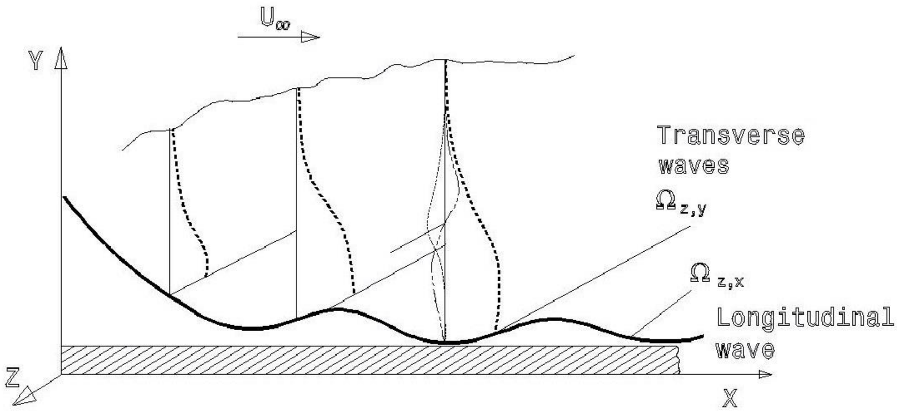

Figure 6. – The wave mechanism of vorticity transport of the wall beneath a flow (Re_x – control parameter)

3. VORTICITY WAVES IN PLATE BOUNDARY LAYER

Generally, the Cauchy motion equations can be written as [7]

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = \vec{f} - \frac{1}{\rho}\left(\nabla\varphi + \nabla\times\vec{A}\right), \tag{8}$$

where $\vec{f}$ is an external force and the irrotational potential φ and the vector potential $\vec{A}$ characterize the two basic dynamic processes: the longitudinal compressing/expanding process, and the transverse torsion shearing process, respectively. $\nabla\varphi$ is the velocity field outside a shear layer and $\nabla\times\vec{A}$ represents the velocity field as the rotation of a vector potential $\vec{A}$ $\left(u = \frac{\partial\Psi}{\partial y}, v = -\frac{\partial\Psi}{\partial x}\right)$, but both paradigms of the incompressible flow satisfy the continuity equation. The Stokes-Helmholtz decomposition of the stress tensor $\overline{\overline{\mathrm{T}}}$

$$\nabla\overline{\overline{\mathrm{T}}} = -\nabla\varphi - \nabla\times\vec{A}, \nabla\cdot\vec{A} = 0, \tag{9}$$

allows, for thixotropic fluid, a distinction between two systems of fluid waves with different propagation speeds: the fast longitudinal L-waves and the slower transverse T-waves.

We shall illustrate the simplest example of application of the boundary–layer equation (flow along a very thin flat plate) where the vorticity dynamics is the result of coupling the longitudinal-transverse wave system. In this case, velocity of the potential flow is constant, that is $\varphi = U_\infty x, \frac{dp}{dx} = 0$, and $\vec{A} = \Psi(x, y)\vec{k}$ with Ψ as a stream function defined by

$$u = \frac{\partial\Psi}{\partial y}, v = -\frac{\partial\Psi}{\partial x}. \tag{10}$$

With

$$\Psi = \sqrt{2\nu x U_\infty}\, f(\eta), \delta(x) \approx \left(\frac{x\nu}{U_\infty}\right)^{1/2}, \eta = \frac{y}{\delta(x)}, \tag{11}$$

where $f(\eta)$ is the dimensionless stream function, $\delta(x)$ is a scaled measure of the boundary layer thickness (up to the approximation u=0.99U_∞) and $u/U_\infty = f'(\eta)$ is the similarity law of the velocity profile, the boundary layer equations and their boundary equations become one ordinary differential equation for the stream equation [10]

$$kf''' + f\,f'' = 0\,, \tag{12}$$

$$\begin{aligned} &\eta = 0: \quad f = 0,\ f' = 0, \\ &\eta \to \infty: \quad f' = 1 \end{aligned}. \tag{13}$$

This nonlinear third order equation and the three boundary conditions completely determine its solution. In the case of a Newtonian fluid, the wall value $f''(0) = f_w''$ is the local wall frictional shear stress

$$\tau_w(x) = \mu\left(\frac{\partial u}{\partial y}\right)_w = 0.332\mu U_\infty\sqrt{\frac{U_\infty}{\nu x}}\,, \tag{14}$$

which is misinterpreted as the dissipation of the boundary vorticity since the flow-induced collision process is completely reversible [1].

This drawback can be avoided using the concept of torsional vorticity-thixotropic fluid able to describe better the vorticity dynamics. Thus, (12) and (13) are completed with the vorticity transport equation and its initial/end boundary conditions:

$$\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = \nu(x)\nabla^2\omega\,, \tag{15}$$

$$\begin{aligned} &\left(y = 0, t \to 0\right): \quad \omega = e^2 \\ &\left(y = 0, t \to \infty\right): \quad \omega = e^0\,, \\ &y \ge \frac{\delta}{2}: \quad \omega = e^0 \end{aligned} \tag{16}$$

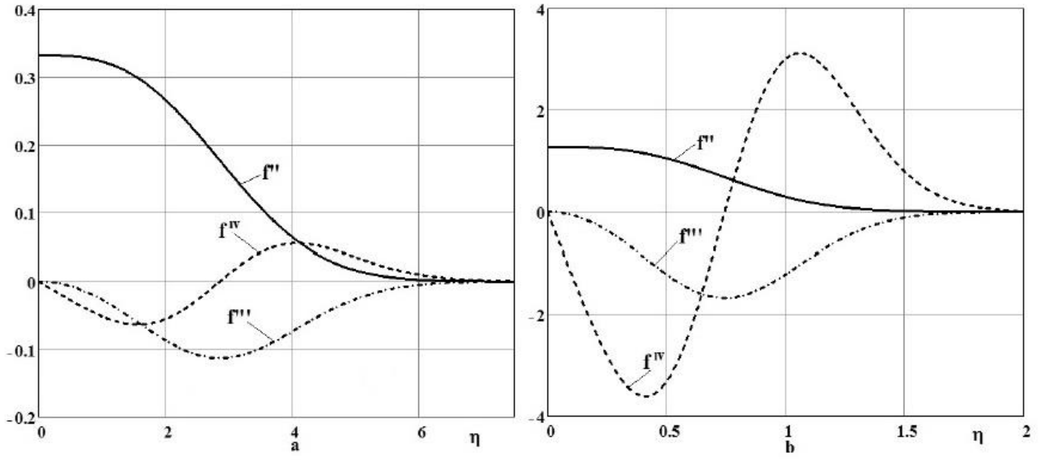

Figure 7. – Full Blasius type solution for T-waves, f'', elastic component, f''', dispersion component, f^{iv}: a) T-wave in laminar flow, k = 2; b) T-wave in turbulent flow, $k = e^{-2}$

The thin viscous-elastic sublayer $\left(y<\frac{\delta}{2}\right)$ is governed by the Blasius type solution $f'''(\eta)$ (Fig. 7), ignored before, and the constitutive equation

$$\sqrt{\omega\nu}=v_d\,, \tag{17}$$

where v_d is a dispersion velocity which is just the flow velocity U_e. Since during impact the whole process is reversible, the conservation of angular momentum leads to $\mathrm{Re}_c=\frac{e^2}{\nu_0}$ except a dimensional constant, which is the location where the vorticity amount created at the boundary completely enters the flow.

The Reynolds number is a measure of the frequency of the wave system and the constant k from (12) depend on its amplitude decreasing for large amplitudes.

The motion following impact is analyzed by using the vorticity transport equation near the wall, $y\to 0$, an implicit constitutive relation $\sqrt{\omega\nu}=U_e$ and the initial/end values of vorticity: $t\equiv\frac{x}{U_e}=0:\omega=e^2,\, t\equiv\frac{x_c}{U_e}=\mathrm{Re}_c\frac{\nu_0}{U_e^2}:\omega=e^0$.

We set $g=\frac{\omega}{e^2},\, \log\mathrm{Re}_x=\xi$ and then obtain the following ordinary equation for the wall vorticity distribution

$$\begin{aligned} &g''-gg'=0\\ &\xi=0:g=1 \qquad ,\\ &\xi=\log\frac{e^2}{\nu_0}:g=e^{-2} \end{aligned} \tag{18}$$

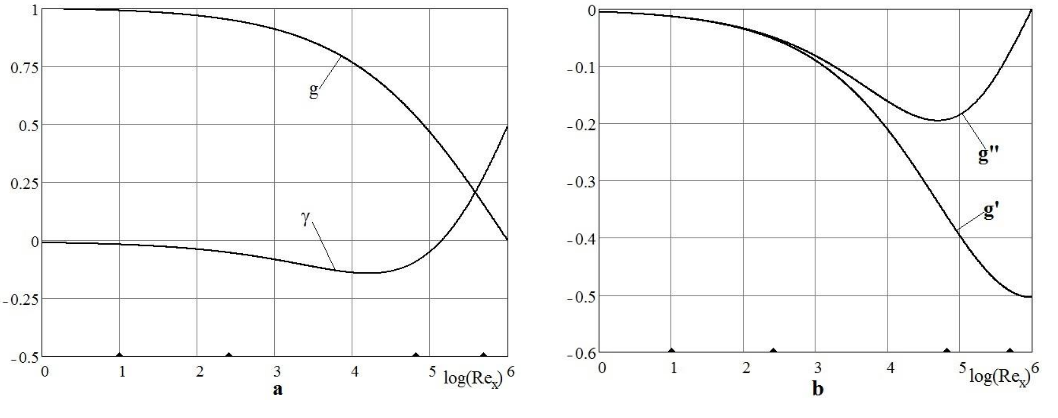

Figure 8. – Vorticity change after a distribution of the state of equilibrium: a) vorticity transported by L-wave *g* and concentration degree of vorticity γ; b) vorticity transported by T-wave *g'* and vorticity dispersed by flow momentum *g''*

In this case, the partial differential equation (15) has been transformed into one nonlinear ordinary differential equation which simulates a nonlinear longitudinal wave steepening in flow field.

Figure 8a shows the formation of a bore penetrating into the flow and the concentration degree of vorticity calculated from the vorticity balance at a boundary

$$\gamma = 1 - g + g' + g'', \tag{19}$$

where:

1 is the torsional vorticity/pressure at impact;

g is wall vorticity transported by longitudinal wave (L);

g' is vorticity transferred to transverse wave (T), Fig. 8b;

g'' is vorticity dispersed by flow momentum, Fig. 8b.

The values marked in Fig. 8 follow the change of the vorticity/viscosity in the post-impact flow:

- $\mathrm{Re}_x = \nu_0^{-1/2}$ is the onset of the vorticity transport by weak waves (Tollmien-Schlichting waves) in the range ($\nu_0^{-1/2} - \nu_0^{-1}$) of Re_x the approximation of Lighthill's vorticity flux diffusion is applied (Fig. 4b);
- $\mathrm{Re}_x = \nu_0^{-1}$ is the onset of the concentrated vorticity transported by strong transverse waves (lambda-vortical structures, turbulent spots) for $\mathrm{Re}_x \geq \nu_0^{-1}$ the proposed wave mechanism works (Fig. 6);
- $\mathrm{Re}_c = e^2 \nu^{-1}$ is the change of the creeping motion of vorticity to the vibrational one, i.e. the classic laminar-turbulent transition.

The mechanism of longitudinal-transverse wave coupling involves an exciting longitudinal wave (L) depending on the Reynolds number, followed by a number of dispersive transverse waves (T).

The transverse wave is a superimposing of three components with different roles: the torsion shear component, *f''*, excited by the longitudinal wave, $f_w'' = g'$, in terms of the Reynolds number, the elastic intrinsic component, f''', retarding the fast component, f_w'', up to $f_{\max}'''$ at $y = \delta/2$ where $\omega = e^0$, and the dispersion component of the Newtonian fluid, f^{iv}, is transported along with the flow.

The vorticity waves in laminar flows are dispersive non dissipative waves, which transfers to the flow field about half amount of vorticity created at the boundary.

This is a general result showing that the angular momentum induced by a boundary in the form of the vorticity (as a specific angular momentum) in laminar flows, can be recovered at the most of its half amount.

The other half locked at the boundary in a thin elastic sublayer is the active vorticity of wave system.

In the range of the large Reynolds numbers ($\nu_0^{-1/2} - \nu_0^{-1}$) the momentum-vorticity coupling is rather a diffusive one and its effect on the drag increase can be estimated by interpolation of the end values.

Figure 9 illustrates the increasing effect of the viscous drag caused by the torsional loading of the thixotropic fluid within the short time of impact [10].

Moreover, the numerical results presented in Fig. 10, show that the unsteady RANS (URANS) technique was unable to account for the fluctuations of flow field induced by the concentrated vorticity.

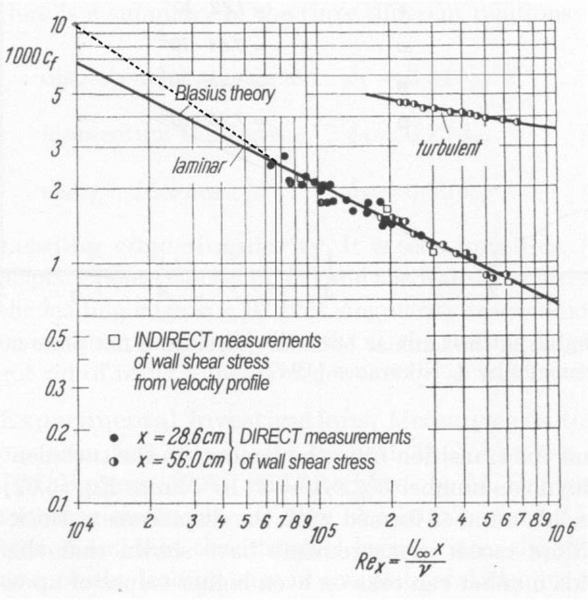

Figure 9. – Local skin-friction coefficient of a flat plate at zero incidence: ─ Blasius theory; --- thixotropic effect

LES technique [11] reproduces the solutions most accurately and consistently because it resolves the unsteady fluctuations that capture the wave process within the shear layer.

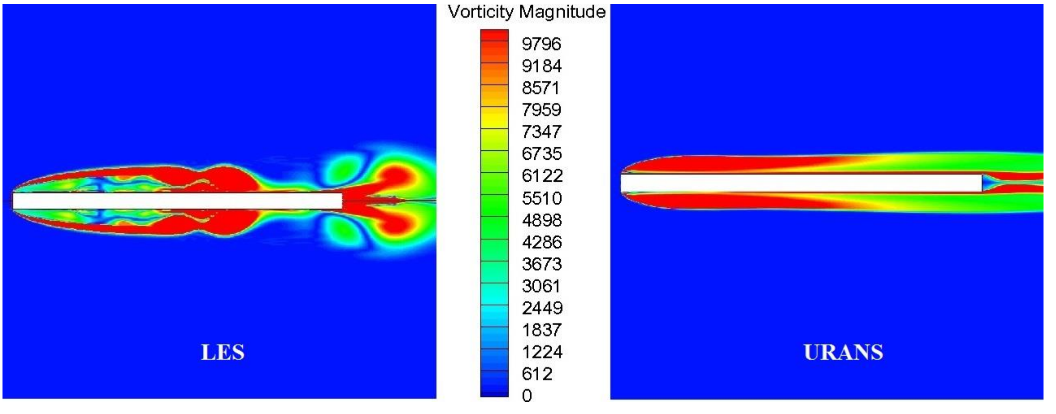

Figure 10. – Numerical results on the plate boundary layer Re=2/3 10^5 — 5 10^5.

4. CONCLUSIONS

The main aim of this paper is an extensive treatment of the boundary vorticity issue with in emphasis on aspects of a conceptual nature. Since the Lighthill's mechanism of boundary-vorticity based a diffusion process, analogous to the Fourier heat conduction, is not applied to very large Reynolds flows involving concentrated vorticities, a wave mechanism of vorticity transport at wall is proposed. Using the concept of torsional vorticity associated with a thixotropic fluid hypothesis, a wave system is devised for the boundary-vorticity dynamics as an intrinsic dispersive phenomenon conserving both angular momentum and energy. The visco-elastic type fluid is used to describe the onset of vorticity at physical surfaces caused by the surface-fluid impact. Then, the Stuart's vorticity model and some hidden solutions of Blasius equation can simulate the impact process by means of a dispersive longitudinal-transverse wave system: the longitudinal waves excite the torsional vorticity created at wall, and the vorticity is further transported by transversal waves into the flow field. On the other hand, the wave mechanism can be extended to turbulent flow. It is shown the importance of the torsional vorticity- thixotropic fluid concept to described the vorticity creation from a boundary and its transfer to the flow field including the changes of the creeping motion of vorticity to the vibration one at $\mathrm{Re}_c = e^2 / \nu_0$, i.e. the turbulent flow field.

ACKNOVLEDGMENTS

This work was realized through the Partnership programme in priority domains – PN II, developed through support provided from ANCS CNDI – UEFISCDI, project no. PN-II-PT-PCCA-2011-32-1670.

REFERENCES

[1] A. H. Barnes, N. Klasste, *Thixotropy, Rheopexy, Yield Stress*, Part C 9.2. in Handbook of Experimental Fluid Mechanics, ed. by C. Tropea, L. A. Yarin, F. J. Foss, Springer-Verlag Berlin Heidelberg, pp. 661-679, 2007.

[2] T. J. Stuart, On finite amplitude oscillations in laminar mixing layers, *Journal of Fluid Mechanics*, Vol. **29**, part 3, pp. 417-440, 1967.

[3] A. Tsinober, *The Essence of Turbulence as a Physical Phenomenon*, Springer Dordrecht Heidelberg, New York London 2014.

[4] M. J. Lighthill, *Introduction: Boundary layer theory*, in Laminar Boundary Theory, ed. by L. Rosenhead, Oxford University Press, Oxford, pp. 46-113, 1963.

[5] R. L. Panton, *Incompressible Flows*, Wiley Interscience, 1984.

[6] M. J. Lighthill, *An Informed Introduction to the Theoretical Fluid Dynamics*, Oxford University Press, Oxford, 1986.

[7] Z. J. Wu, M. J. Wu, Boundary vorticity dynamics since Lighthill's 1963 article. Review and development, *Theoretical and Computational Fluid Dynamics*, Vol. **10**, Springer-Verlag, pp. 459-474, 1998.

[8] H. Dumitrescu, V. Cardos, Vorticity waves in shear flows, *INCAS BULLETIN*, (online) ISSN 2247–4528, (print) ISSN 2066–8201, ISSN–L 2066–8201, Vol. **7**, Issue 1, DOI: 10.13111/2066-8201.2015.7.1.5, pp. 51-56, 2015.

[9] I. Gursul, D. Rockwell, Effect of concentration of vortices on streakline patterns, *Experimental in Fluids*, Vol. **10**, pp. 294-296, 1991.

[10] H. Schlichting, *Boundary-Layer Theory*, McGraw-Hill Book Company, New York, 1968.

[11] * * * ANSIS Fluent-Commercially available CFD software package based on the finite volume method, www.ansys.com, 2011.