Strain resolving method of composite plane plates

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Abstract: *The paper deals with the extension of isotropic plates problem to the case of composite plates. In order to perform it, the Kirchhoff-Love hypotheses were "softened" by some additional ones. Considering the constitutive laws for composite materials the stress functions were eliminated by using Cauchy equations. As a result a partial derivative equation in displacements was obtained. Finally the boundary condition formulation was extended for the case of complex composite plates.*

Key Words*: composite materials, isotropic plates problem*

1. INTRODUCTION

This paper aims at finding an approachable form in displacements of equations governing the mechanical response of composite plates. Unlike the solutions proposed by S. G. Lekhnitskij [1,2], utilizing the facilities offered by several simplifying assumptions that will be presented below, intermediate equations requiring an integration (usually numerical one) can be eliminated.

Eventually, a partial differential equation of 4^{th} order (the analytical solution in displacements of the anisotropic problem of the composite plate) – which is a generalization for the anisotropic problem of the Sophie-Germain equation, can be determined.

Thus, an analytical alternative of the numerical solutions with stress solving that are proposed by most works dealing with the composite plates mechanics, is approached.

The starting point in this case is the Kirchhoff-Love hypothesis that allows a considerable simplification of the equations of the anisotropic elasticity theory, which in their turn are "weakened" by new hypotheses as will be shown below.

In addition, exploiting favorably the assumptions made, analytical expressions are proposed for all types of boundary conditions.

2. WORKING HYPOTESIS

1. During deformation, the normal vectors to the central plane of the plate remain straight and normal to the median deformed surface.

   This hypothesis implies that

$$\gamma_{xz} = \gamma_{yz} = 0 \tag{1}$$

From constitutive equation of orthotropic composites considering principal anisotropic directions, it results unequivocally.

$$\tau_{xz} = \tau_{yz} = 0 \tag{1'}$$

"Weakening" of this hypothesis consists in interpreting previous relation (1') as an indicator for the fact that unitary efforts τ_{xz}, τ_{yz} are very small with respect to the normal ones, but their gradient could be appreciable, namely

$$\frac{\partial \tau_{xz}}{\partial z} \neq 0 \tag{1''}$$

2. During deformation of the laminate its thickness is constant, which equates with the condition

$$\varepsilon_z = 0 \tag{2}$$

3. The small strain hypothesis is also considered thus resulting that the points located in median plane of the plate remain on the same vertical after deformation.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}_0 - z \cdot \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2 \cdot \dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}_0 \approx -z \cdot \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2 \cdot \dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}_0 \tag{3}$$

3. THE PLATE PROBLEM SOLVED IN DISPLACEMENTS

The constitutive equation can be written with respect to the main directions of anisotropy in the form

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{Bmatrix} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{4}$$

where

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{66} = G_{12} \tag{5}$$

In relation to any coordinate system equation (4) becomes

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ 2\varepsilon_{XY} \end{Bmatrix} \tag{6}$$

where

$$\begin{Bmatrix} \overline{Q}_{11} \\ \overline{Q}_{22} \\ \overline{Q}_{12} \\ \overline{Q}_{66} \\ \overline{Q}_{16} \\ \overline{Q}_{26} \end{Bmatrix} = \begin{bmatrix} m^4 & n^4 & 2m^2n^2 & 4m^2n^2 \\ n^4 & m^4 & 2m^2n^2 & 4m^2n^2 \\ m^2n^2 & m^2n^2 & m^4+n^4 & -4m^2n^2 \\ m^2n^2 & m^2n^2 & -2m^2n^2 & \left(m^2-n^2\right)^2 \\ m^3n & -mn^3 & mn^3-m^3n & 2\left(mn^3-m^3n\right) \\ mn^3 & -m^3n & m^3n-mn^3 & 2\left(m^3n-mn^3\right) \end{bmatrix} \cdot \begin{Bmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66} \end{Bmatrix} \tag{7}$$

with $m = \cos\theta$ and $n = \sin\theta$; θ is the angle made by the fibers direction (the principal anisotropic direction) and Ox axis direction.

Replacing (3) in (6) we obtain the constitutive equation in relation with the strain function of medium plane; it results

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = -z \cdot \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \cdot \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\cdot\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}_0 \tag{8}$$

If the mass forces are neglected in equilibrium equations, we can write

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} = 0 \end{cases} \tag{9}$$

Because the ‘hard’ hypothesis of Kirchhoff does not involve the corresponding nullity of the unitary efforts, it can be ‘weakened’ assuming that these slides are small compared to the gradient of plate deformation– as in (10)

$$\frac{\partial v}{\partial z} = -\frac{\partial w}{\partial y} + \gamma_{yz} = 0 \tag{10}$$

NOTE. “Weakening” Kirchhoff's first assumption is legitimate in order to ensure the balance of the plate element under transverse loads

To estimate the transverse efforts in plate the equilibrium relations (9) are considered.

From the first relation in system (9) one deduces

$$\frac{\partial \tau_{zx}}{\partial z} = -\left\{ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right\} \tag{11}$$

or, by using relations from (8) it can be written

$$\frac{\partial \tau_{zx}}{\partial z} = z \cdot \left\{ \begin{array}{l} \frac{\partial}{\partial x}\left[\overline{Q}_{11}\frac{\partial^2 w}{\partial x^2} + \overline{Q}_{12}\frac{\partial^2 w}{\partial y^2} + 2\cdot\overline{Q}_{13}\frac{\partial^2 w}{\partial x \partial y}\right] + \\ \frac{\partial}{\partial y}\left[\overline{Q}_{13}\frac{\partial^2 w}{\partial x^2} + \overline{Q}_{23}\frac{\partial^2 w}{\partial y^2} + 2\cdot\overline{Q}_{33}\frac{\partial^2 w}{\partial x \partial y}\right] \end{array} \right\} \tag{12}$$

or

$$\frac{\partial \tau_{zx}}{\partial z} = z \cdot \left\{ \overline{Q}_{11}\frac{\partial^3 w}{\partial x^3} + \left(\overline{Q}_{12} + 2\cdot Q_{33}\right)\frac{\partial^3 w}{\partial x \partial y^2} + 3\cdot\overline{Q}_{13}\frac{\partial^3 w}{\partial x^2 \partial y} + \overline{Q}_{23}\frac{\partial^3 w}{\partial y^3} \right\} \tag{13}$$

If we note

$$A = \overline{Q}_{11}\frac{\partial^3 w}{\partial x^3} + \left(\overline{Q}_{12} + 2\cdot Q_{33}\right)\frac{\partial^3 w}{\partial x \partial y^2} + 3\cdot\overline{Q}_{13}\frac{\partial^3 w}{\partial x^2 \partial y} + \overline{Q}_{23}\frac{\partial^3 w}{\partial y^3} \tag{14}$$

and integrating the (13) equation with respect to *z* variable it is obtained

$$\tau_{zx} = \frac{z^2}{2}\cdot A + f_1(x, y) \tag{15}$$

Where $f_1(x,y)$ represents an arbitrary function which will be determined from outline conditions. Thus from the assumption that on the plate surfaces (at $z = \pm\frac{h}{2}$) there are no forces in plane($\tau_{zx} = 0$) one deduces

$$f_1(x, y) = -\frac{h^2}{8}\cdot A \tag{16}$$

namely in the end

$$\tau_{zx} = \left(\frac{z^2}{2} - \frac{h^2}{8}\right)\cdot A \tag{17}$$

Where *h* is the plate thickness.

Similarly, for the second component of the transverse effort we can successively deduce the following:

From the second relation of system (9) it can be deducted

$$\frac{\partial \tau_{zy}}{\partial z} = -\left\{\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y}\right\} \tag{18}$$

or, using the relations from (8)

$$\frac{\partial \tau_{zy}}{\partial z} = z \cdot \left\{ \begin{array}{l} \frac{\partial}{\partial x}\left[\overline{Q}_{13}\frac{\partial^2 w}{\partial x^2} + \overline{Q}_{23}\frac{\partial^2 w}{\partial y^2} + 2\cdot\overline{Q}_{33}\frac{\partial^2 w}{\partial x \partial y}\right] + \\ \frac{\partial}{\partial y}\left[\overline{Q}_{12}\frac{\partial^2 w}{\partial x^2} + \overline{Q}_{22}\frac{\partial^2 w}{\partial y^2} + 2\cdot\overline{Q}_{23}\frac{\partial^2 w}{\partial x \partial y}\right] \end{array} \right\} \tag{19}$$

or

$$\frac{\partial \tau_{zy}}{\partial z} = z \cdot \left\{ \overline{Q}_{13} \frac{\partial^3 w}{\partial x^3} + \left(\overline{Q}_{12} + 2 \cdot Q_{33} \right) \frac{\partial^3 w}{\partial x^2 \partial y} + 3 \cdot \overline{Q}_{23} \frac{\partial^3 w}{\partial x \partial y^2} + \overline{Q}_{22} \frac{\partial^3 w}{\partial y^3} \right\} \tag{20}$$

If denoting

$$B = \overline{Q}_{13} \frac{\partial^3 w}{\partial x^3} + \left(\overline{Q}_{12} + 2 \cdot Q_{33} \right) \frac{\partial^3 w}{\partial x^2 \partial y} + 3 \cdot \overline{Q}_{23} \frac{\partial^3 w}{\partial x \partial y^2} + \overline{Q}_{22} \frac{\partial^3 w}{\partial y^3} \tag{21}$$

And the (20) equation is integrated with respect to variable z it is obtained

$$\tau_{zy} = \frac{z^2}{2} \cdot B + f_2(x, y) \tag{22}$$

Where $f_2(x,y)$ represents an arbitrary function which will be determined from outline conditions.

Thus from the assumption that on the plate surfaces (at $z = \pm \frac{h}{2}$) there are no in plane forces in the plane($\tau_{zx} = 0$) one deduces

$$f_2(x, y) = -\frac{h^2}{8} \cdot B \tag{23}$$

and finally

$$\tau_{zy} = \left(\frac{z^2}{2} - \frac{h^2}{8} \right) \cdot B \tag{24}$$

In order to obtain an equation to be solved trough strain, in the third equation of (9) system, we determine

$$\frac{\partial \sigma_z}{\partial z} = -\left\{ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} \right\} \tag{25}$$

Or, using relations from (17) and (24) we write

$$\frac{\partial \sigma_z}{\partial z} = -\left\{ \frac{\partial}{\partial x} \left[\left(\frac{z^2}{2} - \frac{h^2}{8} \right) \cdot A \right] + \frac{\partial}{\partial y} \left[\left(\frac{z^2}{2} - \frac{h^2}{8} \right) \cdot B \right] \right\} \tag{26}$$

or

$$\frac{\partial \sigma_z}{\partial z} = \left(\frac{h^2}{8} - \frac{z^2}{2} \right) \cdot \left[\overline{Q}_{11} \frac{\partial^4 w}{\partial x^4} + 4 \cdot \overline{Q}_{13} \frac{\partial^4 w}{\partial x^3 \partial y} + 2 \cdot \left(\overline{Q}_{12} + 2 \cdot Q_{33} \right) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4 \cdot \overline{Q}_{23} \frac{\partial^4 w}{\partial x \partial y^3} + \overline{Q}_{22} \frac{\partial^4 w}{\partial y^4} \right] \tag{27}$$

If note

$$C = \overline{Q}_{11}\frac{\partial^4 w}{\partial x^4} + 4\cdot\overline{Q}_{13}\frac{\partial^4 w}{\partial x^3\partial y} + 2\cdot\left(\overline{Q}_{12} + 2\cdot Q_{33}\right)\frac{\partial^4 w}{\partial x^2\partial y^2} + 4\cdot\overline{Q}_{23}\frac{\partial^4 w}{\partial x\partial y^3} + \overline{Q}_{22}\frac{\partial^4 w}{\partial y^4} \tag{28}$$

and integrating the (27) equation with respect to z variable it results

$$\sigma_z = \left(\frac{h^2}{8}\cdot z - \frac{z^3}{6}\right)\cdot C + f_3(x, y) \tag{29}$$

Where $f_3(x,y)$ represents an arbitrary function which will be determined from outline conditions.

Thus from the assumption that on inner plate surface (at $z = +\frac{h}{2}$) there are no normal forces to the plane ($\sigma_z = 0$) and also on upper surface (at $z = -\frac{h}{2}$) the p pressure is acting, it is successively obtained

$$f_3(x, y) = -\frac{h^3}{24}\cdot C \tag{30}$$

or

$$\sigma_z = -\left(\frac{h^3}{24} - \frac{h^2}{8}\cdot z + \frac{z^3}{6}\right)\cdot C \tag{31}$$

and, for $z = -\frac{h}{2}$ we consider $\sigma_z = -p$ .I.e.,

$$\frac{h^3}{12}\cdot C = p \tag{32}$$

Or, explicitly, in the final form

$$\begin{aligned}&\overline{Q}_{11}\frac{\partial^4 w}{\partial x^4} + 4\cdot\overline{Q}_{13}\frac{\partial^4 w}{\partial x^3\partial y} + 2\cdot\left(\overline{Q}_{12} + 2\cdot Q_{33}\right)\frac{\partial^4 w}{\partial x^2\partial y^2} +\\ &4\cdot\overline{Q}_{23}\frac{\partial^4 w}{\partial x\partial y^3} + \overline{Q}_{22}\frac{\partial^4 w}{\partial y^4} = \frac{12\cdot p}{h^3}\end{aligned} \tag{33}$$

The (33) equation represents the displacement solution of composite plane plate problem transversally loaded which can be numerically integrated.

As already stated it is a generalization of Sophie Germain equation for the composite plane plate.

4. BOUNDARY CONDITIONS

4.1 Generalities

To achieve an optimization code of the plate composite structure in relation to its dynamic response to interaction with the fluid field, it is interesting to define each possible case of boundary conditions, especially in strain for the plate defined by the equation (31).

In a first step it is useful to define the sectional efforts inside the plate.

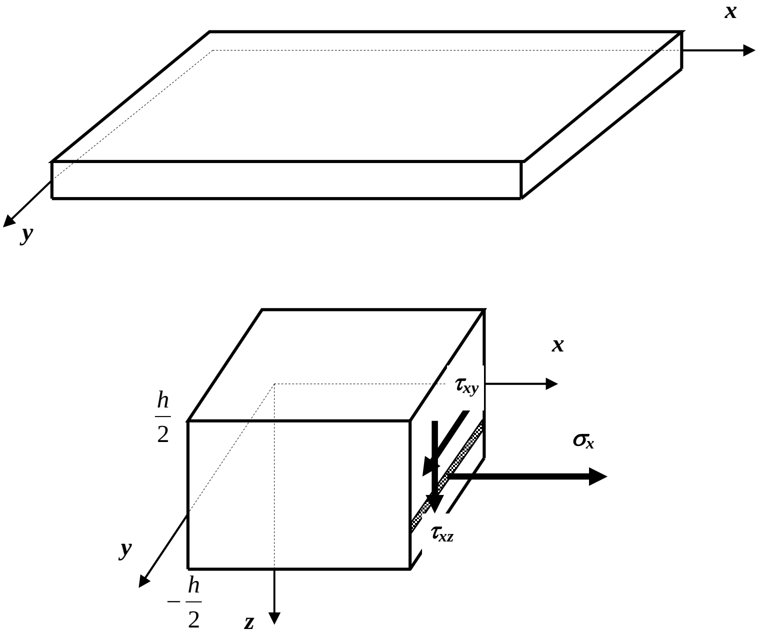

Fig. 1 Geometry of the plate with unitar length and width

Thus, as shown in picture it can be noted

$$M_x = 1 \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y \cdot z \cdot dz \text{ – bending moment} \tag{34}$$

$$M_y = 1 \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \cdot z \cdot dz \text{ – bending moment} \tag{35}$$

$$M_{xy} = 1 \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \cdot z \cdot dz \text{ – torque moment} \tag{36}$$

$$T_x = 1 \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \cdot dz \text{ – shear force on the face parallel to } Oy \tag{37}$$

$$T_y = 1 \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} \cdot dz \text{ – shear force on the face parallel to } Ox \tag{38}$$

4.2. Fixed edge

For the fixed edge the boundary conditions will block the movement of the points situated on the respective side along the vertical direction (*Oz*) as well as prevent the rotation of the plaque surface around its axis.

Considering the system of axis from figure 1 the following conditions can be written

a) on the edge parallel to the *Oy* axis

$$\begin{cases} w = 0 \\ \dfrac{\partial w}{\partial x} = 0 \end{cases} \tag{39}$$

b) on the edge parallel to the *Ox* axis

$$\begin{cases} w = 0 \\ \dfrac{\partial w}{\partial y} = 0 \end{cases} \tag{40}$$

4.3. Simply supported edge

For simply supported edge the boundary conditions will block the movement of the points situated on the respective side along the vertical direction (*Oz*). On the other hand, the bending moment on this surface must be null because the simply support does not introduce reaction moment. Making reference to the system of axes in figure 1 the following conditions can be written:

a) on the edge parallel to the *Ox* axis

$$\begin{cases} w = 0 \\ M_x = 0 \end{cases} \tag{41}$$

The second relation from (41), taking (34) into account leads to the condition

$$M_x = 1 \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y \cdot z \cdot dz = 0 \tag{42}$$

Namely, if relation (8) is used it can be written

$$-\int_{-\frac{h}{2}}^{\frac{h}{2}} z \cdot \left(\overline{Q}_{21} \frac{\partial^2 w}{\partial x^2} + \overline{Q}_{22} \frac{\partial^2 w}{\partial y^2} + 2 \cdot \overline{Q}_{23} \frac{\partial^2 w}{\partial x \partial y} \right) \cdot z \cdot dz = 0 \tag{43}$$

Or, noting

$$D = \overline{Q}_{21} \frac{\partial^2 w}{\partial x^2} + \overline{Q}_{22} \frac{\partial^2 w}{\partial y^2} + 2 \cdot \overline{Q}_{23} \frac{\partial^2 w}{\partial x \partial y} \tag{44}$$

It results

$$D \cdot \left. \frac{z^3}{3} \right|_{-\frac{h}{2}}^{\frac{h}{2}} = 0 \quad \Rightarrow \quad D \cdot \frac{h^3}{12} = 0 \quad \Leftrightarrow \quad D = 0 \tag{45}$$

So, finally, the relation (41) becomes

$$\begin{cases} w=0 \\ \overline{Q}_{21}\dfrac{\partial^2 w}{\partial x^2}+\overline{Q}_{22}\dfrac{\partial^2 w}{\partial y^2}+2\cdot\overline{Q}_{23}\dfrac{\partial^2 w}{\partial x\partial y}=0 \end{cases} \tag{46}$$

b) on the edge parallel to the Oy axis

Similarly for the other side is written

$$\begin{cases} w=0 \\ M_y=0 \end{cases} \tag{47}$$

The second relation from (47), taking (35) into account leads to the condition

$$M_y=1\cdot\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_x\cdot z\cdot dz=0 \tag{48}$$

Namely, if relation (8) is used it can be written

$$-\int_{-\frac{h}{2}}^{\frac{h}{2}} z\cdot\left(\overline{Q}_{11}\frac{\partial^2 w}{\partial x^2}+\overline{Q}_{12}\frac{\partial^2 w}{\partial y^2}+2\cdot\overline{Q}_{13}\frac{\partial^2 w}{\partial x\partial y}\right)\cdot z\cdot dz=0 \tag{49}$$

Or, by noting

$$D1=\overline{Q}_{11}\frac{\partial^2 w}{\partial x^2}+\overline{Q}_{12}\frac{\partial^2 w}{\partial y^2}+2\cdot\overline{Q}_{13}\frac{\partial^2 w}{\partial x\partial y} \tag{50}$$

it results

$$D1\cdot\left.\frac{z^3}{3}\right|_{-\frac{h}{2}}^{\frac{h}{2}}=0 \quad\Rightarrow\quad D1\cdot\frac{h^3}{12}=0 \quad\Leftrightarrow\quad D1=0 \tag{51}$$

So, finally, the relation (51) becomes

$$\begin{cases} w=0 \\ \overline{Q}_{11}\dfrac{\partial^2 w}{\partial x^2}+\overline{Q}_{12}\dfrac{\partial^2 w}{\partial y^2}+2\cdot\overline{Q}_{13}\dfrac{\partial^2 w}{\partial x\partial y}=0 \end{cases} \tag{52}$$

4.4. Free edge

For the free edge the boundary conditions stem in the condition that all efforts in that section are null.

Making reference to the system of axes mentioned in figure 1 the following conditions can be written

a) on the edge parallel to the Ox axis

$$M_y=0,\quad M_{xy}=0,\quad T_x=0 \tag{53}$$

Because for determining the solution two independent relations in the form of boundary conditions, are needed, by using Saint-Venant's principle and considering that the shearing force T_x and the components of the torque moment M_{xy} (as distributed moment) are steady, from the last two equations from (53) it results the relation:

$$T_x + \frac{\partial M_{xy}}{\partial y} = 0 \tag{54}$$

Meaning, from (53) the boundary conditions become

$$\begin{cases} M_y = 0 \\ T_x + \dfrac{\partial M_{xy}}{\partial y} = 0 \end{cases} \tag{55}$$

The first relation from (55) is equivalent with the second relation from (47) i.e.

$$\overline{Q}_{11}\frac{\partial^2 w}{\partial x^2} + \overline{Q}_{12}\frac{\partial^2 w}{\partial y^2} + 2\cdot\overline{Q}_{13}\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{56}$$

After processing the second relation from (55) it results

$$-\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} dz + \frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy}\cdot z dz = 0 \tag{57}$$

or,

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{z^2}{2} - \frac{h^2}{8}\right)\cdot A\cdot dz - \frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}} z\cdot\left(\overline{Q}_{13}\frac{\partial^2 w}{\partial x^2} + \overline{Q}_{23}\frac{\partial^2 w}{\partial y^2} + 2\cdot\overline{Q}_{33}\frac{\partial^2 w}{\partial x \partial y}\right)\cdot z\cdot dz = 0 \tag{58}$$

where B is the notation from (21). If, denoting

$$E = \overline{Q}_{13}\frac{\partial^2 w}{\partial x^2} + \overline{Q}_{23}\frac{\partial^2 w}{\partial y^2} + 2\cdot\overline{Q}_{33}\frac{\partial^2 w}{\partial x \partial y} \tag{59}$$

Then (58) becomes

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{z^2}{2} - \frac{h^2}{8}\right)\cdot A\cdot dz - \frac{\partial}{\partial y}\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2\cdot dz = 0 \tag{60}$$

or

$$-A\cdot\frac{h^3}{12} - \frac{\partial}{\partial y}\left(E\cdot\frac{h^3}{12}\right) = 0 \quad \Leftrightarrow \quad A + \frac{\partial E}{\partial y} = 0 \tag{61}$$

And, replacing

$$\left[\overline{Q}_{11}\frac{\partial^3 w}{\partial x^3} + \left(\overline{Q}_{12} + 2\cdot\overline{Q}_{33}\right)\frac{\partial^3 w}{\partial x \partial y^2} + 3\cdot\overline{Q}_{13}\frac{\partial^3 w}{\partial x^2 \partial y} + \overline{Q}_{23}\frac{\partial^3 w}{\partial y^3}\right] + \frac{\partial}{\partial y}\left[\overline{Q}_{13}\frac{\partial^2 w}{\partial x^2} + \overline{Q}_{23}\frac{\partial^2 w}{\partial y^2} + 2\cdot\overline{Q}_{33}\frac{\partial^2 w}{\partial x \partial y}\right] = 0 \tag{62}$$

Finally, after processing, the two boundary conditions are written

$$\begin{cases} \overline{Q}_{11}\dfrac{\partial^2 w}{\partial x^2}+\overline{Q}_{12}\dfrac{\partial^2 w}{\partial y^2}+2\cdot\overline{Q}_{13}\dfrac{\partial^2 w}{\partial x\partial y}=0 \\ \overline{Q}_{11}\dfrac{\partial^3 w}{\partial x^3}+4\cdot\overline{Q}_{13}\dfrac{\partial^3 w}{\partial x^2\partial y}+\left(\overline{Q}_{12}+4\cdot\overline{Q}_{33}\right)\dfrac{\partial^3 w}{\partial x\partial y^2}+2\cdot\overline{Q}_{23}\dfrac{\partial^3 w}{\partial y^3}=0 \end{cases} \tag{63}$$

b) on the edge parallel to the O*y* axis

$$M_x = 0, \quad M_{xy} = 0, \quad T_y = 0 \tag{64}$$

Similar to the previous case it results the relation:

$$T_y + \frac{\partial M_{xy}}{\partial x} = 0 \tag{65}$$

Or, from (64) the boundary conditions become

$$\begin{cases} M_x = 0 \\ T_y + \dfrac{\partial M_{xy}}{\partial x} = 0 \end{cases} \tag{66}$$

The first relation from (66) is equivalent with the second relation from (41) i.e.

$$\overline{Q}_{12}\frac{\partial^2 w}{\partial x^2}+\overline{Q}_{22}\frac{\partial^2 w}{\partial y^2}+2\cdot\overline{Q}_{23}\frac{\partial^2 w}{\partial x\partial y}=0 \tag{67}$$

After processing the second relation from (66) it results

$$-\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{yz}dz+\frac{\partial}{\partial x}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy}\cdot zdz=0 \tag{68}$$

or,

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{z^2}{2}-\frac{h^2}{8}\right)\cdot B\cdot dz-\frac{\partial}{\partial x}\int_{-\frac{h}{2}}^{\frac{h}{2}}z\cdot\left(\overline{Q}_{13}\frac{\partial^2 w}{\partial x^2}+\overline{Q}_{23}\frac{\partial^2 w}{\partial y^2}+2\cdot\overline{Q}_{33}\frac{\partial^2 w}{\partial x\partial y}\right)\cdot z\cdot dz=0 \tag{69}$$

where A is the notation of (14). If the notation from (59) is considered, then (69) becomes

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{z^2}{2}-\frac{h^2}{8}\right)\cdot B\cdot dz-\frac{\partial}{\partial x}\int_{-\frac{h}{2}}^{\frac{h}{2}}z\cdot\left(\overline{Q}_{13}\frac{\partial^2 w}{\partial x^2}+\overline{Q}_{23}\frac{\partial^2 w}{\partial y^2}+2\cdot\overline{Q}_{33}\frac{\partial^2 w}{\partial x\partial y}\right)\cdot z\cdot dz=0 \tag{70}$$

or

$$-B\cdot\frac{h^3}{12}-\frac{\partial}{\partial x}\left(E\cdot\frac{h^3}{12}\right)=0 \quad\Leftrightarrow\quad A+\frac{\partial E}{\partial x}=0 \tag{71}$$

And, replacing

$$\left[\overline{Q}_{13}\frac{\partial^3 w}{\partial x^3}+\left(\overline{Q}_{12}+2\cdot\overline{Q}_{33}\right)\frac{\partial^3 w}{\partial x^2\partial y}+3\cdot\overline{Q}_{23}\frac{\partial^3 w}{\partial x\partial y^2}+\overline{Q}_{22}\frac{\partial^3 w}{\partial y^3}\right]+\frac{\partial}{\partial x}\left[\overline{Q}_{13}\frac{\partial^2 w}{\partial x^2}+\overline{Q}_{23}\frac{\partial^2 w}{\partial y^2}+2\cdot\overline{Q}_{33}\frac{\partial^2 w}{\partial x\partial y}\right]=0 \tag{72}$$

Finally, after processing, the two boundary conditions are written

$$\begin{cases} \overline{Q}_{21}\dfrac{\partial^2 w}{\partial x^2}+\overline{Q}_{22}\dfrac{\partial^2 w}{\partial y^2}+2\cdot\overline{Q}_{23}\dfrac{\partial^2 w}{\partial x\partial y}=0 \\ 2\cdot\overline{Q}_{13}\dfrac{\partial^3 w}{\partial x^3}+\left(\overline{Q}_{12}+4\cdot\overline{Q}_{33}\right)\cdot\dfrac{\partial^3 w}{\partial x^2\partial y}+4\cdot\overline{Q}_{23}\cdot\dfrac{\partial^3 w}{\partial x\partial y^2}+2\cdot\overline{Q}_{22}\dfrac{\partial^3 w}{\partial y^3}=0 \end{cases} \tag{73}$$

5. CONCLUSIONS

- In the case of composite plane plates subjected to transverse loads and if small displacements assumption is considered, the equation (33) is a relatively simple and easy solution to integrate, which eliminates the considerable calculus effort and additional errors introduced by classical numerical methods.
- The boundary conditions are easily applicable and allow a simple interpretation of the phenomena which have generated the mathematical conditions.
- By dropping the assumption of anisotropy the Sophie-Germain equation is re-found.
- The proposed method requires a comparison between classical and numerical method in terms of solving time and accuracy of the calculus.

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