In Memoriam Elie Carafoli

Vortex theory of the ideal wind turbine

Horia DUMITRESCU, Vladimir CARDOŞ, Alexandru DUMITRACHE
and Florin FRUNZULICĂ

horiadumitrescu@yahoo.com, v_cardos@yahoo.ca,
alex_dumitrache@yahoo.com, ffrunzi@yahoo.com
"Gh. Mihoc, C. Iacob" Institute of Statistical Mathematics and
Applied Mathematics of the Romanian Academy, Bucharest

DOI: 10.13111/2066-8201.2009.1.2.14

Abstract

Based on the concepts outlined by Joukowsky nearly a century ago, an analytical aerodynamic optimization model is developed for rotors with a finite number of blades and constant circulation distribution. In the paper we show the basics of the new model and compare its efficiency with results for rotors designed using the optimization model of Betz.

1. Introduction

In the history of rotor aerodynamics two schools have dominated the conceptual interpretation of the optimum rotor. In Russia, Joukowsky [3] defined the optimum rotor as one having constant circulation along the blades, such that the vortex system for a N_b – bladed rotor consists of N_b helical tip vortices of strength Γ and an axial center vortex of strength $-N_b\Gamma$. The other school, which essentially was formed by Prandtl and Betz [1], assumed that optimum efficiency is obtained when the distribution of circulation along the blades produces a rigid helicoidally wake that moves in the direction of its axis with a constant velocity. This distribution, the so-called Goldstein circulation function, is rather complex and difficult to determine accurately [2]. Hence, in practice, the blades are normally modeled using Blade-Element-Momentum (BEM) theory, corrected by the tip correction of Prandtl [4]. In the following, we consider various classical theories for the optimum rotor. First, we repeat the Betz limit using axial momentum theory. Next, we consider a rotor with an infinite number of blades using the general momentum theory to include rotational velocities. Finally, we analyze a realistic rotor with a finite number of blades using the new solution of the Goldstein function based a recent mathematical approximation for the induced velocities [5]. The present modifications of the vortex theory for the ideal rotor enables for the first time to compare the theoretical maximum efficiency of rotors designed using Goldstein's circulation function (Betz rotor) with rotors designed using constant circulation (Joukowsky rotor) along the blades.

2. Theory

Dimensionless Parameters.

The aerodynamic operation of a wind turbine can be characterized by the following quantities: the rotor angular velocity Ω_0, the rotor torque Q or the power output $P = \Omega_0 Q$ and the rotor thrust T. These quantities are put into dimensionless form as follows:

$$\lambda_0 = \Omega_0 R_0 / V \quad \text{(rotor tip-speed ratio)}, \tag{1}$$

$$C_p = P / \left(1/2 \rho A_0 V^3\right) \quad \text{(power coefficient)}, \tag{2}$$

$$C_T = T / \left(1/2 \rho A_0 V^2\right) \quad \text{(thrust coefficient)}, \tag{3}$$

where R_0 is the radius of the rotor, V is the undisturbed wind speed, ρ is the density of the air and $A_0 = \pi R_0^2$ is the area of the rotor. The maximum power that can be extracted from a stream of air contained in the area equivalent to that swept out by the rotor corresponds to the maximum value of the power coefficient defined in equation (2).

Optimum Rotor: Rankine-Froude Theory. We first consider the simple axial momentum theory as it originated by Rankine [6], W. Froude [7] and R.E. Froude [8]. Here, we consider axial flow past an actuator disk representing the axial load on a rotor. Denoting by u_{z_0} the axial velocity in the rotor plane, the axial interference factor is defined as

$$a = \frac{V - u_{z_0}}{V} \tag{4}$$

From one-dimensional axial momentum theory, we get the following expressions for the axial load (thrust) and power extraction

$$T = 2\rho A_0 u_{z_0} \left(V - u_{z_0}\right) = 2\rho A_0 V^2 a(1-a) \tag{5}$$

$$P = u_{z_0} T = 2\rho A_0 V^3 a(1-a)^2 \tag{6}$$

Introducing the dimensionless power coefficient, equation (2), we get

$$C_P = 4 A_0 a(1-a)^2 \text{ with } C_{P_{\max}} = \frac{16}{27} = 0.593 \text{ for } a = \frac{1}{3} \tag{7}$$

This result is usually referred to as the Betz limit and states the upper maximum for power extraction: not more than 59% of the kinetic energy contained in a stream tube having the same cross section as the disk area can be converted to useful work by the disk. However, it does not include the losses caused by the rotation of the wake, and therefore, it represents a conservative upper maximum.

Optimum Rotor: General Momentum Theory. Utilizing the general momentum theory, Glauert [9] developed a simple model for the optimum rotor that included rotational velocities. In this approach, Glauert treated the rotor as a rotating axisymmetric actuator disk, corresponding to a rotor with an infinite number of blades. Denoting the angular velocity of the rotor blade as Ω_0, and the azimuthal velocity in the rotor plane as u_{θ_0}, the azimuthal interference factor is defined as

$$a' = \frac{u_{\theta_0}}{\Omega_0 r}. \tag{8}$$

Employing Euler's turbine equation with some additional assumptions, in dimensionless form, the total power is found as

$$C_P = 8\lambda_0^2 \int_0^1 a'(1-a)x_0^3 dx_0 \tag{9}$$

where $x_0 = r / R_0$. By assuming that the different stream tube elements behave independently of each other, it is possible to optimize the integrand for each x_0 separately leading to the relationship

$$a' = \frac{1-3a}{4a-1}. \tag{10}$$

The analysis shows that the optimum axial interference factor is no longer a constant but will depend on the rotation of the wake, and that the operating range for an optimum rotor is $1/4 \le a \le 1/3$. The relations between a, a', $a' x_0^2 \lambda_0^2$ and $\lambda_0 x_0$ for an optimum rotor are given in Table 1, and the maximal power coefficient as a function of tip speed ratio is shown in Table 2. The optimal power coefficient approaches 0.593 at large tip speed ratios only.
It shall be mentioned that these results are valid only for a rotor with an infinite number of blades, and that the analysis is based on the assumption that the rotor can be optimized by considering each blade element independently of the remaining blade elements.

Table 1. Flow conditions for the optimum actuator disk

a	a'	$a' x_0^2 \lambda_0^2$	$\lambda_0 x_0$
0.25	∞	0	0
0.26	5.500	0.0296	0.073
0.27	2.375	0.0584	0.157
0.28	1.333	0.0864	0.255
0.29	0.812	0.1136	0.374
0.30	0.500	0.1400	0.529
0.31	0.292	0.1656	0.753
0.32	0.143	0.1904	1.150
0.33	0.031	0.2144	2.630
1/3	0	0.2222	∞

Table 2. Power coefficient as function of tip speed ratio for optimum actuator disk

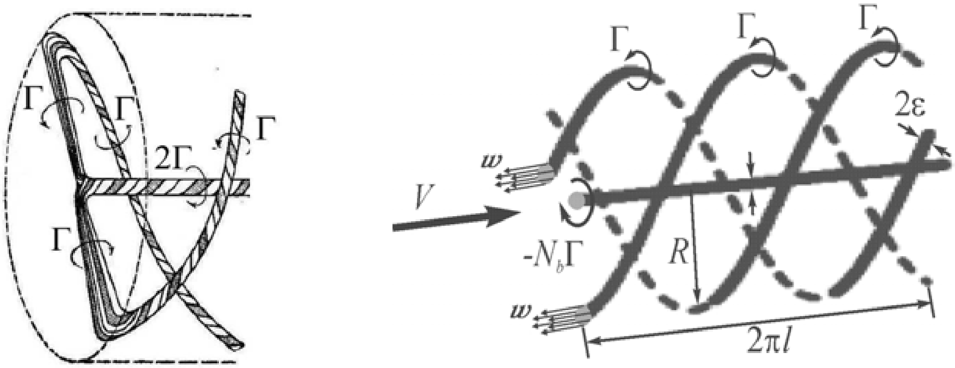
λ_0	C_{Pmax}
0.5	0.288
1.0	0.416
1.5	0.480
2.0	0.512
2.5	0.532
5.0	0.570
7.5	0.582
10.0	0.593

Optimum Rotor: Vortex Theory. The flow over a real rotor with a finite number of blades is very different from the properties of the flow models used previously to describe the optimum rotor. Indeed, important phenomena such as tip losses and azimuthally dependencies of the induced velocities are neglected in the momentum theory of the optimum rotor. An alternative model is the vortex theory in which each of the rotor blades is replaced by a lifting line about which the circulation is associated with the bound vorticity, and a vortex sheet is continuously shed from the trailing edge.

In the vortex theory of Joukowsky [3] each of the blades is replaced by a lifting line about which the circulation associated with the bound vorticity is constant, resulting in a free vortex system consisting of helical vortices trailing from tips of the blades and a rectilinear hub vortex, as sketched in Fig. 1a). Using vortex theory, the bound vorticity serves to produce the local lift on the blades while the trailing vortices induce the velocity field in the rotor plane and in the wake. The fundamental expressions for the forces acting on a blade (Fig. 2) is most conveniently expressed by the Kutta– Joukowsky theorem, which in vector form reads

$$d\mathbf{L} = \rho \mathbf{U}_0 \times \mathbf{\Gamma}_0 dr\,. \tag{11}$$

where $d\boldsymbol{L}$ is the lift force on a blade element of radial dimension dr, $\boldsymbol{U}_0$ is the resultant relative velocity and $\mathbf{\Gamma}_0 = const$ is the bound circulation.



a b

Figure 1: (a) Vortex system corresponding to lifting line theory of the ideal propeller of Joukowsky [3]; (b) helical vortex structure representing the ideal far wake behind the Joukowsky rotor.

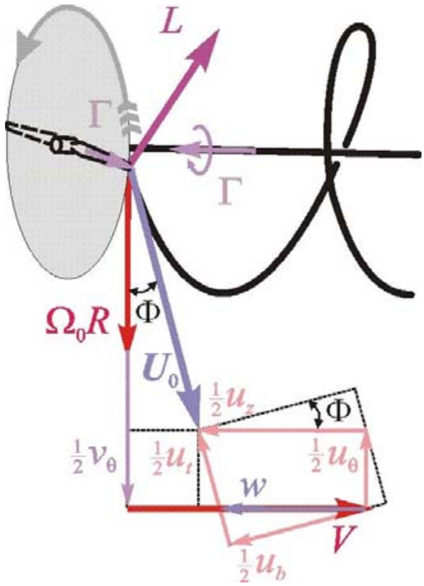


Figure 2: Velocity triangles in the rotor plane of an ideal wind turbine.

Let u_{z_0} and u_{θ_0}. be the axial and circumferential components of the velocity, respectively, induced at a blade element in the rotor plane by the free tip vortices, and v_{θ_0} the circumferential velocity induced by the hub vortex. Then, in accordance with Fig. 2, we can write the local torque dQ of the rotor as follows

$$dQ = \rho \Gamma_0 \left(V - u_{z_0}\right) r dr , \tag{12}$$

Integrating these quantities (12) along each blade and summing up, we get the following expression for the power

$$P = \rho N_b \Gamma_0 \Omega_0 \int_0^{R_0} \left(V - u_{z_0}\right) r dr , \tag{13}$$

where N_b is the number of blades.

For a rotor with a finite number of blades we can replace the free vortex system behind the rotor (Fig. 1*a*) by a vortex system, extended to infinity in both directions (Fig. 1*b*). The vortex system consists of a multiplet of helical tip vortices with constant pitch h and circulation Γ in which a finite vortex core moves backward (in the case of a propeller) or forward (in the case of a wind turbine) in the direction of its axis with a constant velocity $V - w$. The far wake includes an additional rectilinear hub vortex of strength $- N_b\Gamma$, resulting in a simple formula for the induced velocity, which only consists of the circumferential component,

$$v_\theta = \frac{\Gamma N_b}{2\pi R}, \tag{14}$$

Denoting the angle between the axis of the tip vortex and the rotor plane as Φ, the pitch is given as

$$h = 2\pi R \tan\Phi\,, \tag{15}$$

or, in alternative dimensional form,

$$l = h/2\pi = R\tan\Phi\,, \tag{16}$$

where R is the radial extent of the tip vortices. According to [5], in cylindrical coordinates (r, θ, z) the components of fluid velocity induced by N_b helical vortices outside the vortex cores are given as

$$u_z(r,\theta) = \frac{N_b\Gamma}{2\pi l}\begin{Bmatrix}1\\0\end{Bmatrix} + \frac{\Gamma R}{\pi l^2}\sum_{n=1}^{N_b}\sum_{m=1}^{\infty} m\begin{Bmatrix}I_m(mr/l)K_m'(mR/l)\\ I_m'(mR/l)K_m(mr/l)\end{Bmatrix}\cos(m\chi_n)\,, \tag{17}$$

$$u_\theta(r,\theta) = \frac{N_b\Gamma}{2\pi r}\begin{Bmatrix}0\\1\end{Bmatrix} + \frac{\Gamma R}{\pi r l}\sum_{n=1}^{N_b}\sum_{m=1}^{\infty} m\begin{Bmatrix}I_m(mr/l)K_m'(mR/l)\\ I_m'(mR/l)K_m(mr/l)\end{Bmatrix}\cos(m\chi_n)\,, \tag{18}$$

where $I_m(x)$ and $K_m(x)$ are modified Bessel functions and $\chi_n = \theta + \frac{2\pi(n-1)}{N_b} - \frac{z}{l}$

When the dominant first two singularity terms are extracted, (17) and (18) are reduced to the following rough-and-ready formulas [9]

$$u_z(r,\theta) = \frac{N_b\Gamma}{2\pi l}\begin{Bmatrix}1\\0\end{Bmatrix} + \frac{\Gamma}{2\pi l}\frac{\sqrt[4]{l^2+R^2}}{\sqrt[4]{l^2+r^2}}\sum_{n=1}^{N_b}\mathrm{Re}\left[\frac{\pm e^{i\chi_n}}{e^{\mp\xi}-e^{i\chi_n}} + \frac{l}{24}\left(\frac{3r^2-2l^2}{\left(l^2+r^2\right)^{3/2}} + \frac{3R^2-2l^2}{\left(l^2+R^2\right)^{3/2}}\right)\ln\left(1-e^{\xi+i\chi_n}\right)\right]$$

,

$$u_\theta(r,\theta) = \frac{N_b\Gamma}{2\pi r}\begin{Bmatrix}0\\1\end{Bmatrix} + \frac{\Gamma}{2\pi r}\frac{\sqrt[4]{l^2+R^2}}{\sqrt[4]{l^2+r^2}}\sum_{n=1}^{N_b}\mathrm{Re}\left[\frac{\pm e^{i\chi_n}}{e^{\mp\xi}-e^{i\chi_n}} + \frac{l}{24}\left(\frac{3r^2-2l^2}{\left(l^2+r^2\right)^{3/2}} + \frac{3R^2-2l^2}{\left(l^2+R^2\right)^{3/2}}\right)\ln\left(1-e^{\xi+i\chi_n}\right)\right]$$

,

where $e^{\xi} = \frac{r\left(l+\sqrt{l^2+R^2}\right)\exp\left(\sqrt{l^2+r^2}\right)}{R\left(l+\sqrt{l^2+r^2}\right)\exp\left(\sqrt{l^2+R^2}\right)}$. Here we use the notations "$\pm$" or "$\mp$" or "$\begin{Bmatrix}x\\x\end{Bmatrix}$", where the upper sign or symbol corresponds to $r < R$, and the lower to $r \geq R$. It follows immediately from Eqs. (17) and (18) that the velocity of the resulting motion of the helical vortex in the axial direction remains constant and obeys the following relationship

$$u_\tau \equiv u_z + \frac{r}{l}u_\theta = \frac{N_b\Gamma}{2\pi l}\,. \tag{19}$$

Moreover, if the vorticity field in the helix cores is collinear to the center of the helical lines, then condition (19) holds true for all points of the flow field, including the vortex cores [5]. The velocity component orthogonal to u_r and u_τ is given as

$$u_\chi \equiv u_\theta - \frac{r}{l} u_z . \tag{20}$$

Introducing azimuthally averaged induced velocities as $a \equiv \langle u_z \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} u_z d\theta$, from Eq. (17) we get

$$a = \frac{N_b \Gamma}{2\pi l} . \tag{21}$$

It should be mentioned that the averaged induced velocity is identical to the wake interference factor a and that it takes the same constant value as the velocity in Eq. (19). In accordance with (14) and (21), it is possible to derive a simple relation between the circumferential component velocity induced by the hub vortex and the interference factor a induced by the helical multiplet

$$v_\theta = \frac{al}{R} . \tag{22}$$

Assuming the relative wake motion with constant axial speed, w, to correspond to half the averaged induced velocity, $1/2a$, the helical vortices included in the finite vortex cores of radius, ε, are translated in the bi-normal direction with the velocity

$$u_b \equiv w \cos\Phi = \frac{1}{2} a \cos\Phi \equiv \frac{1}{2} a \frac{l}{\sqrt{R^2 + l^2}} . \tag{23}$$

The problem of finding the induced equilibrium motion of multiple of helical vortices in an unbounded domain was solved in [5]. Employing the velocity component u_χ, defined by

$$u_b = -u_\chi \frac{l}{\sqrt{R^2 + l^2}} . \tag{24}$$

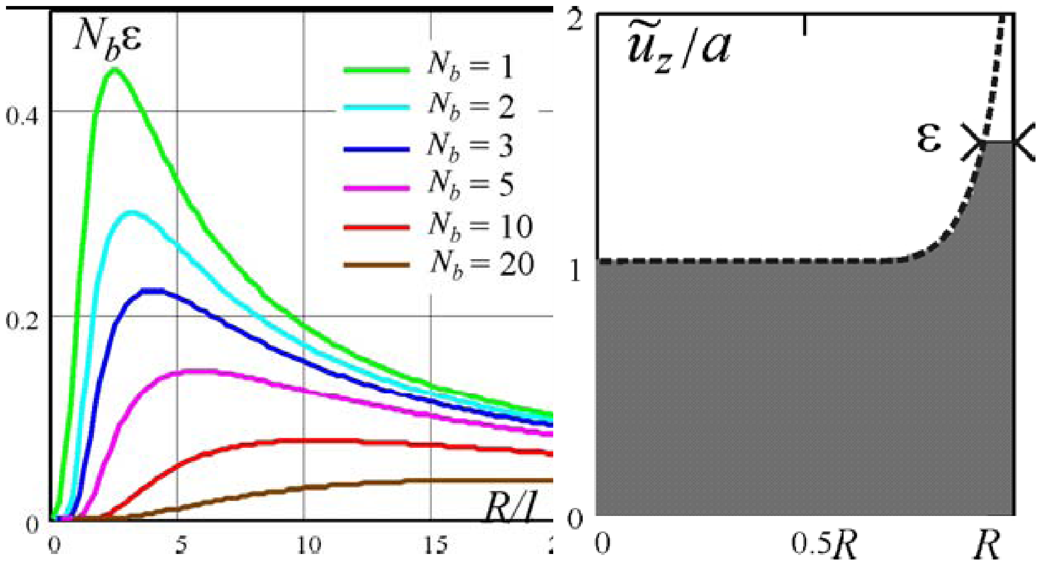
the total induced and self-induced motion of the tip vortices can be found as

$$\begin{aligned} \frac{4\pi R}{\Gamma} u_{\chi_{ind}}(\varepsilon) \equiv \bar{u}_{\chi_{ind}}(\varepsilon) = N_b - \frac{\sqrt{1+\tau^2}}{\tau} - \frac{N_b}{\tau^2} + \\ + \frac{1}{\tau\left(1+\tau^2\right)^{1/2}} \left(\ln \frac{\varepsilon N_b \left(1+\tau^2\right)^{3/2}}{\tau} + \frac{1}{4} \right) - \frac{\tau}{\left(1+\tau^2\right)^{7/2}} \left(\tau^4 - 3\tau^2 + \frac{3}{8} \right) \frac{\varsigma(3)}{N_b^2} . \end{aligned} \tag{25}$$

where $\tau = l/R$ is the non-dimensional pitch and $\zeta(3)$ = 1.20206… is the Riemann zeta function.

Finally, to determine the conditions for equilibrium motion of the far wake we must define a radius ε of the tip vortex core by solving the equation

$$\bar{u}_{\chi_{ind}}(\varepsilon) = -\frac{N_b}{\tau^2} . \tag{26}$$



a) *b*)

Figure 3 (*a*) The vortex core radius for equilibrium motion of tip vortex multiplet as function of helical pitch for different numbers of blades;
(*b*) Singularity elimination in axial velocity profile along blade.

Figure 3*a* shows the vortex core radius as function of R/l for different numbers of blades. It may be noted that for a rotor with infinitely many blades the vortex radius tends to zero when the tip vortex system tends to a cylindrical surface.

In the present work we represent the vortex system behind the rotor by a set of helical vortices with finite core to eliminate the singularity of the induced velocity field in vicinity of the each filament described by solution (16) and (17). The regularization was achieved by restricting the velocity inside vortex cores Λ (also see Fig. 3*b*):

$$\tilde{u}_z(r,\theta)=\begin{cases}\dfrac{u_z(r,\theta)}{a} & if \quad (r,\theta)\notin\Lambda \\ \dfrac{u_z(R-\varepsilon,\theta)}{a} & if \quad (r,\theta)\in\Lambda\end{cases}. \tag{27}$$

Thus, for any given value of the wake pitch l and number of rotor blades b N_b , we compute the radius of the tip vortex core and determine the finite velocity values induced by the vortices in all points of the unbounded space.

The above introduced vortex system, extended to infinity in both directions, can be described by parameters defining the far wake properties in the so-called Trefftz plane, which per definition is the plane normal to the relative wind far downstream of the rotor.

It now remains to establish the characteristics in the rotor plane in order to utilize the Kutta–Joukowsky theorem to determine the power (13). In accordance with Helmholtz' vortex theorem the bound circulation Γ_0 about a blade element is uniquely related to the circulation Γ of a corresponding tip vortex at the Trefftz plane. It is here seen that there is a simple relation between the bound circulation and the interference factor,

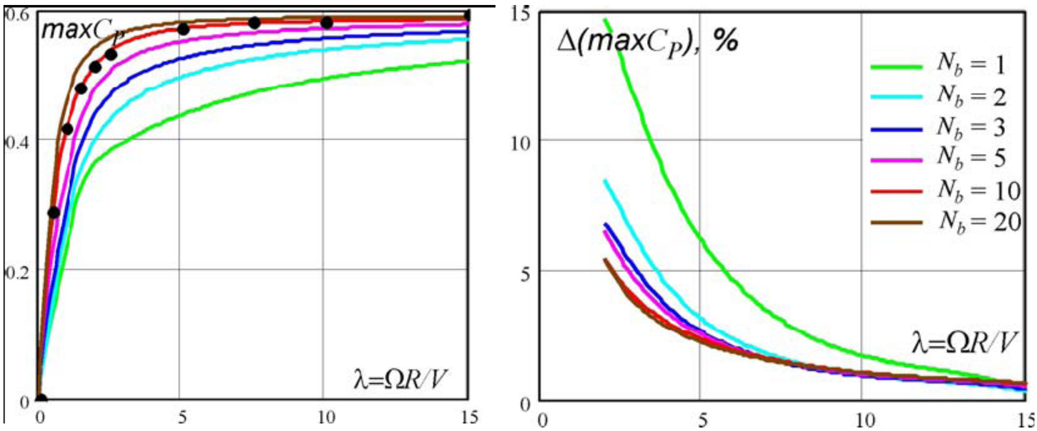
$$N_b\Gamma_0 = 2\pi la \,. \tag{28}$$

If the expansion of the wake is neglected ($R_0 = R$), it is readily seen that the induced velocities by the vortex system in the rotor plane tend to be half the induced velocity at a corresponding point in the Trefftz plane (see e.g. [3]). Thus, as a first order approximation we assume that

$$v_{\theta_0} = \frac{1}{2}v_\theta; \quad u_{\theta_0} = \frac{1}{2}u_\theta \text{ and } u_{z_0} = \frac{1}{2}u_z \,. \tag{29}$$

From simple geometric considerations in the rotor plane (Figure 2*b*), using Eqs. (22) and (23), the angular pitch is given as

$$\mathrm{tg}\Phi = \frac{V - w}{\Omega_0 R + \frac{1}{2}v_\theta} = \frac{V - \frac{1}{2}a}{\Omega_0 R + \frac{1}{2}a\tau^2} = \frac{l}{R} \,. \tag{30}$$



a) *b*)

Figure 4 (*a*) Power coefficient as function of tip speed ratio for different number of blades of an optimum rotor. Points: General momentum theory [4]; lines: Present theory; (b) Difference between the optimum coefficients of the "Joukowsky rotor" with constant circulation along the blades and a "Betz rotor" with circulation given by Goldstein's function [5].

Eq. (30) can be written as

$$\Omega_0 l = V - \frac{1}{2}a - \frac{1}{2}a\tau^2 \,. \tag{31}$$

Introducing dimensionless variables

$$\bar{a} = \frac{a}{V}, \tag{32}$$

and inserting Eqs. (27), (28) and (31) into Eq. (13), the power can be determined from the following integral

$$P = \rho\pi R_0^2 V^3 \bar{a}\left(1-\frac{\bar{a}}{2}\left(1+\tau^2\right)\right)\left(1-\frac{\bar{a}}{2}\int_0^1 \tilde{u}_z(x,0)x dx\right). \tag{33}$$

Performing the integration and introducing the dimensionless power coefficient (Eq. 2), we get

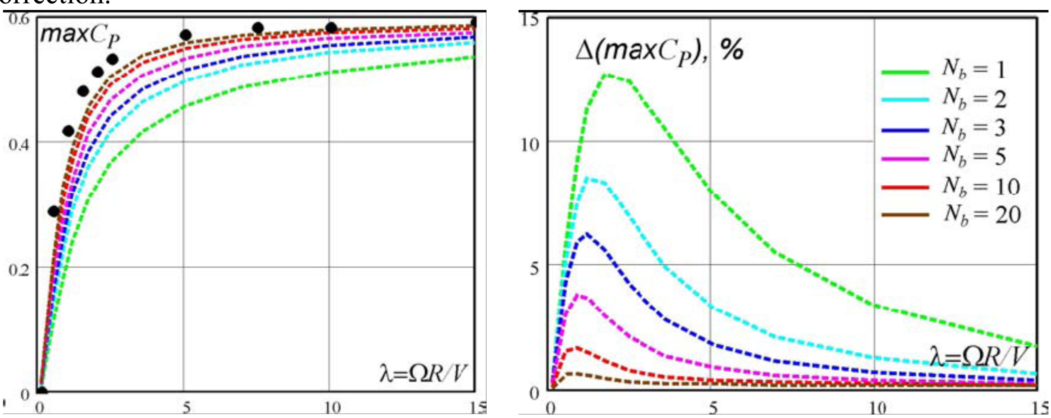
$$C_P = 2\bar{a}\left(1-\frac{1}{2}\bar{a}J_1\right)\left(1-\frac{1}{2}\bar{a}J_3\right), \tag{34}$$

where $J_1 = 1+\tau^2$ and $J_3 = \int_0^1 \tilde{u}_z(x,0)x dx$. For a given helicoidally wake structure, the power coefficient is seen to be uniquely determined, except for the parameter $\bar{a}$. Differentiation of C_P with respect to $\bar{a}$ yields the maximum value of C_{Pmax}, resulting in

$$\bar{a}\left(C_P = C_{P\max}\right) = \frac{2}{3J_1J_3}\left(J_1 + J_3 - \sqrt{J_1^2 - J_1J_3 + J_3^2}\right). \tag{35}$$

3. Results and Discussion

In the following we compare results from the new analytical models for wind turbine rotors with finite number of blades for the case of a rotor with constant circulation (Joukowsky rotor), described above, and a rotor designed using the Goldstein distribution (Betz rotor), recently developed in [5]. In addition we evaluate the error committed when approximating the aerodynamics of a rotor with finite number of blades by Prandtl's tip correction.



a) *b)*

Figure 5 (*a*) Approximation of the optimum power coefficient by Prandtl's tip correction as function of tip speed ratio. Points: General momentum theory [4]; lines: Prandtl's approximation; (b) Difference between Prandtl's approximation of the optimum coefficients and the solution for the "Betz rotor" with circulation given by Goldstein's function [5].

Fig. 4*a* presents the optimum power coefficient of the present model, Eqs. (33) and (34), as function of tip speed ratio for different number of blades. In Fig. 4*b* the corresponding difference between the optimum coefficients of the Joukowsky and the Betz rotor is depicted. From the figures it is evident that the optimum power coefficient of the Joukowsky rotor for all number of blades is higher than that for the Betz rotor.

Fig. 5*a* shows an approximation of the optimum power coefficient as a function of tip speed ratio calculated for different number of blades, based on the dimensionless power coefficient from [5]

$$C_P = 2\overline{w}\left(1-\frac{1}{2}\overline{w}\right)\left(\widetilde{I}_1 - \frac{1}{2}\overline{w}\widetilde{I}_3\right), \tag{36}$$

where the mass coefficient $\widetilde{I}_1 = 2\int_0^1 \frac{F(x,l)x^3}{l^2+x^2}dx$ and the axial energy factor $\widetilde{I}_3 = 2\int_0^1 \frac{F(x,l)x^5}{\left(l^2+x^2\right)^2}dx$ were estimated with help of the reduction factor in Prandtl's tip correction approximation [4],

$$F(x,l) = \frac{2}{\pi}\cos^{-1} e^{-f(x,\tau)} \text{ with}$$
$$f(x,\tau) = \frac{N_b(1-x)\sqrt{1+\tau^2}}{2l} \tag{37}$$

Differentiation of Eq. (36) with respect to *w* determines the maximum value of the power coefficient, resulting in the value

$$\overline{w}\left(C_P = C_{P\max}\right) = \frac{2}{3\widetilde{I}_3}\left(\widetilde{I}_1 + \widetilde{I}_3 - \sqrt{\widetilde{I}_1^2 - \widetilde{I}_1\widetilde{I}_3 + \widetilde{I}_3^2}\right). \tag{38}$$

In Fig. 5 the C_P -values obtained with the original analytical solution using Goldstein's circulation [5] are subtracted from those obtained using the Prandtl approximation. The difference shows that the error committed by using Prandtl's tip correction formula, Eq. (37) depends on the number of blades, but in all cases results in a higher performance than obtained from the exact solution [5]. The difference, however, vanishes for $N_b \to \infty$.

4. Conclusions

An analytical optimization model has been developed for a rotor with a finite number of blades and constant circulation ("Joukowsky rotor"). The method is based on an analytical solution to the problem of the equilibrium motion of a helical vortex multiplet in a far wake. The main achievement of the model is that it eliminates the singularity of the solution at all operating conditions. In contrast to earlier models, the new model enables for the first time to determine the theoretical maximum efficiency of rotors with constant circulation and an arbitrary number of blades.

Optimum conditions for finite number of blades as function of tip speed ratio has been compared for two models: (a) "Joukowsky rotor" with constant circulation along blade (b) "Betz rotor" with circulation given by Goldstein's function [5]. For all tip speed ratios the "Joukowsky rotor" achieves a higher efficiency than the "Betz rotor".

REFERENCES

[1] *A. BETZ*, Schraubenpropeller mit Geringstem Energieverlust, ***Dissertation**,* Göttingen Nachrichten, Göttingen (1919).

[2] *S. GOLDSTEIN*, ***On the vortex theory of screw propellers***. Proc R Soc London A (1929); 123: 440–65.

[3] *N.E. JOUKOWSKY*, ***Vortex theory of screw propeller***, I-IV, I-III in Trudy Otdeleniya Fizicheskikh Nauk Obshchestva Lubitelei Estestvoznaniya. I in (1912) **16**(1); II in (1914) **17**(1); III in (1915) **17**(2); IV in Trudy Avia Raschetno-Ispytatelnogo Byuro (1918) no. 3 (in Russian). French translation in Théorie tourbillonnaire de l'hélice propulsive, Gauthier-Villars, Paris, (1929). 198p.

[4] *H. GLAUERT,* ***Airplane propellers Division L*** in Aerodynamic Theory, vol. IV, W.F. Durand (ed.). Springer: Berlin, (1935); 169–360.

[5] *V. OKULOV*, ***On the Stability of Multiple Helical Vortices***. J. Fluid Mech.(2004), 521: 319-342.

[6] *W.J.M. RANKINE*, ***On the mechanical principles of the action of propellers***. Transactions of the Institution of Naval Architects (1865); **6**: 13-30.

[7] *W. FROUDE,* ***On the elementary relation between pitch, slip and propulsive efficiency.*** Transactions of the Institution of Naval Architects (1878); **19**: 47.

[8] *R.E. FROUDE,* ***On the part played in propulsion by differences of fluid pressure***. Transactions of the Institution of Naval Architects (1889); **30**: 390-405.

[9] *Y. FUKUMOTO, V. OKULOV,* ***The velocity field induced by a helical vortex tube***, Phys. Fluids, (2005), **17**(10): 107101(1-19).