Calculation of the zeros of some special functions

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DOI: 10.13111/2066-8201.2018.10.3.10

Received: 26 June 2018/ Accepted: 15 July 2018/ Published: September 2018

Abstract: *We consider solutions, with infinity of zeros, of real second-order linear differential equations. The graph of a solution locates the zeros with a certain precision, which means that zeros are between some successive numerical values of the solution. Once the first zero is calculated, the given equation is converted into a new first-order equation. The zeros of this new equation form a subset of numerical values of the solution of the first order equation.*

Key Words: *Bessel functions, roots, non-linear differential equations, orthogonal functions.*

1. TRANSFORMING BESSEL'S EQUATION

In the monograph "*A Treatise on the Theory of Bessel Functions*" [1], the properties of the function $J_\nu(x)$ with the real index ν and the real argument $x \geq 0$ are specified. In particular, this function is a solution of the linear differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\left(x-\frac{\nu^2}{x}\right)y=0 \tag{1}$$

We shall consider the transformation and the next derivatives

$$y = x^\nu Z, \qquad \frac{dy}{dx} = x^\nu\frac{dZ}{dx} + \nu x^{\nu-1}Z \tag{2}$$

$$\frac{d^2y}{dx^2} = x^\nu\frac{d^2Z}{dx^2} + 2\nu x^{\nu-1}\frac{dZ}{dx} + \nu(\nu-1)x^{\nu-2}Z \tag{3}$$

The zeros of the Z function are also zeros of the y function. The Z function is a solution of the linear equation

$$\frac{d^2Z}{dx^2}+Z = -\frac{2\nu+1}{x}\frac{dZ}{dx} \tag{4}$$

A system of two non-linear equations for the unknowns H and θ with the argument x, is associated with this equation, according to the following identities

$$Z \equiv H \sin\theta \tag{5}$$

$$\frac{dZ}{dx} = \frac{dH}{dx}\sin\theta + H\frac{d\theta}{dx}\cos\theta \equiv H\cos\theta \tag{6}$$

These identities result in successive formulas

$$\frac{d\theta}{dx} = 1 - \frac{1}{H}\frac{dH}{dx}\tan\theta \tag{7}$$

$$H^2 = Z^2 + \left(\frac{dZ}{dx}\right)^2 \tag{8}$$

$$H\frac{dH}{dx} = \left(Z + \frac{d^2Z}{d^2x}\right)\frac{dZ}{dx} = -\frac{2\nu+1}{x}\left(\frac{dZ}{dx}\right)^2 = -\frac{2\nu+1}{x}(H\cos\theta)^2 \tag{9}$$

Consequently, we have the non-liner differential system

$$\frac{dH}{dx} = -\frac{2\nu+1}{x}H\cos^2\theta \tag{10}$$

$$\frac{d\theta}{dx} = 1 + \frac{2\nu+1}{x}\sin\theta\cos\theta \tag{11}$$

The zeros of the y and Z functions correspond to the np values of the function $\theta(x)$. It is therefore useful to specify the equation system for the unknowns H and x in relation to the argument θ.

$$\frac{dx}{d\theta} = \frac{x}{x + (\nu + 0.5)\sin 2\theta} \tag{12}$$

$$\frac{dH}{d\theta} = -\frac{(\nu + 0.5)H[1 + \cos 2\theta]}{x + (\nu + 0.5)\sin 2\theta} \tag{13}$$

We admit that a positive root $j_{0,\nu}$ of the function $J_\nu(x)$ has been calculated. The initial useful condition of solution $x(.)$, is $x(-\pi) = j_{0,\nu,}$. For the integration interval $[-\pi, (N-1)\pi]$ with $(Np+1)$ division points, the $x(\theta_{sp})$ values of the numerical solution will be the roots of the specified function.

2. CALCULATING SUBSETS OF FIRST ZEROS

Knowing a root of the function $J_\nu(x)$, the following roots can be calculated by solving a single differential equation. For calculating the values of function $J_\nu(x)$, it would be necessary to specify the initial condition of the monotone decreasing function H.

The first positive zeros of J_0, J_1, J_2 and J_3, according to the Mathcad 7 program, are determined with the following instructions

$x = 2.404826,\quad j_{0,0} = \mathrm{root}(\mathrm{J0}(x),x),\quad j_{0,0}= 2.404826,\quad \mathrm{J0}(j_{0,0}) = 1.150\cdot 10^{-10}$ (14)

$x = 3.831706,\quad j_{0,1} = \mathrm{root}(\mathrm{J1}(x),x),\quad j_{0,1}= 3.831706,\quad \mathrm{J1}(j_{0,1}) = 6.026\cdot 10^{-12}$ (15)

$x = 5.135622,\quad j_{0,2}= \mathrm{root}(\mathrm{Jn}(2, x), x),\quad j_{0,2}= 5.135622,\quad \mathrm{Jn}(2, j_{0,2}) = -5.136\cdot 10^{-11}$ (16)

$x = 6.380162,\quad j_{0,3}= \mathrm{root}(\mathrm{Jn}(3, x), x),\quad j_{0,3}= 6.380162,\quad \mathrm{Jn}(3, j_{0,3}) = 1.568\cdot 10^{-11}$ (17)

These zeros values are the initial conditions of equation (12) in the cases $\nu=0$, $\nu=1$, $\nu=2$ respectively $\nu=3$. The integration interval is $[-\pi, (N-1)\pi]$. Each subinterval $[k\pi, (k+1)\pi]$ has $p+1$ nodes. Let S be the function

$$S(\nu,\theta,x) = \frac{x}{x + (\nu + 0.5)\sin\theta} \tag{18}$$

The system of the four equations (12) is considered.

$$uo = \begin{bmatrix} j_{0,0} \\ j_{0,1} \\ j_{0,2} \\ j_{0,3} \end{bmatrix}, \qquad D(\theta,u) = \begin{bmatrix} S(0,\theta,u_0) \\ S(1,\theta,u_1) \\ S(2,\theta,u_2) \\ S(3,\theta,u_3) \end{bmatrix} \tag{19}$$

The solution is obtained with Carl David Runge and Wilhelm Kutta method [2].

$$N = 5 \qquad p = 32 \qquad \mathrm{R} := \mathrm{rkfixed}(uo,-\pi, N\pi-\pi, Np,D) \tag{20}$$

The vectors of numerical values of the solutions are

$$\mathrm{X0} :=\mathrm{R}^{<1>},\qquad \mathrm{X1} :=\mathrm{R}^{<2>},\qquad \mathrm{X2} :=\mathrm{R}^{<3>},\qquad \mathrm{X3} :=\mathrm{R}^{<4>} \tag{21}$$

The subset of zeros of the functions $J_0(x)$, $J_1(x)$, $J_2(x)$ and $J_3(x)$ have the components

$$\mathrm{s} := 0..N \qquad j_{\mathrm{s},0} := \mathrm{X0}_{\mathrm{sp}} \qquad j_{\mathrm{s},1} := \mathrm{X1}_{\mathrm{sp}} \qquad j_{\mathrm{s},2} := \mathrm{X2}_{\mathrm{sp}} \qquad j_{\mathrm{s},3} := \mathrm{X3}_{\mathrm{sp}} \tag{22}$$

For $\nu = 0, 1$ these zeros agrees with values in *Tafeln Höherer Funktionen* [3].

$$\mathrm{j}^{<0>} = \begin{bmatrix} 2.404826 \\ 5.520078 \\ 8.653728 \\ 11.791534 \\ 14.930918 \\ 18.071064 \end{bmatrix} \quad \mathrm{j}^{<1>} = \begin{bmatrix} 3.831706 \\ 7.015587 \\ 10.173468 \\ 13.323692 \\ 16.47063 \\ 19.615859 \end{bmatrix} \quad \mathrm{j}^{<2>} = \begin{bmatrix} 5.135622 \\ 8.417244 \\ 11.619841 \\ 14.795952 \\ 17.95982 \\ 21.116997 \end{bmatrix} \quad \mathrm{j}^{<3>} = \begin{bmatrix} 6.380162 \\ 9.761023 \\ 13.015201 \\ 16.223466 \\ 19.409415 \\ 22.58273 \end{bmatrix} \tag{23}$$

If the index ν does not have integer value, it is applied Poisson's formula [3].

$$\boldsymbol{J}(\nu,x) = \int_0^{\frac{\pi}{2}} \cos(x\cos(q))\cdot(\sin(q))^{2\cdot\nu}\,dq, \qquad 2\nu > -1. \tag{24}$$

$$J_\nu(x) = \boldsymbol{J}(\nu,x)\,\frac{2\left(x/2\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+1/2)} \tag{25}$$

$\Gamma(.)$ is the gamma function [4]. For example

$\nu = 0.5,\ x = \pi,\ \mathrm{j}\nu_0 = \mathrm{root}(\mathbf{J}(\nu, x), x),\ \mathbf{J}(\nu, \mathrm{j}\nu_0) = -9.63\cdot 10^{-12},\ \mathrm{j}\nu_0 - \pi = 3.024\cdot 10^{-8}$ (26)

The functions { $J_\nu(x\, j_{s,\nu})$, s = 0, 1, 2, ...} defined on the interval [0,1], are an orthogonal functions subset [5], [6].

3. EXAMPLES OF THE ORTHOGONAL FUNCTIONS

Let *a, b* be the positive real numbers and functions $A(x) = y(ax)$ and $B(x) = y(bx)$, where $y(x)$ function is a solution of equation (1). Hence

$$\left(a^2 x - \frac{\nu^2}{x}\right) A(x) + \frac{d}{dx}\left[x\frac{dA}{dx}\right] = 0, \qquad \left(b^2 x - \frac{\nu^2}{x}\right) B(x) + \frac{d}{dx}\left[x\frac{dB}{dx}\right] = 0 \tag{27}$$

$$(a^2 - b^2)\, xAB = A\left(x\frac{d^2B}{dx^2} + \frac{dB}{dx}\right) - B\left(x\frac{d^2A}{dx^2} + \frac{dA}{dx}\right) = \frac{d}{dx}\left[x\left(A\frac{dB}{dx} - B\frac{dA}{dx}\right)\right] \tag{28}$$

By integration, we have formula

$$(a^2 - b^2)\int_0^1 x\,y(ax)\,y(bx)\,dx = \left[x\left(A\frac{dB}{dx} - B\frac{dA}{dx}\right)\right]_0^1 = y(a)\,b\frac{dy}{dx}(b) - y(b)\,a\frac{dy}{dx}(a) \tag{29}$$

If *a* and *b* are zeros $A(a) = y(a) = 0$, $B(b) = y(b) = 0$ and $a \neq b$, then $y(ax)$ and $y(bx)$ are orthogonal functions.

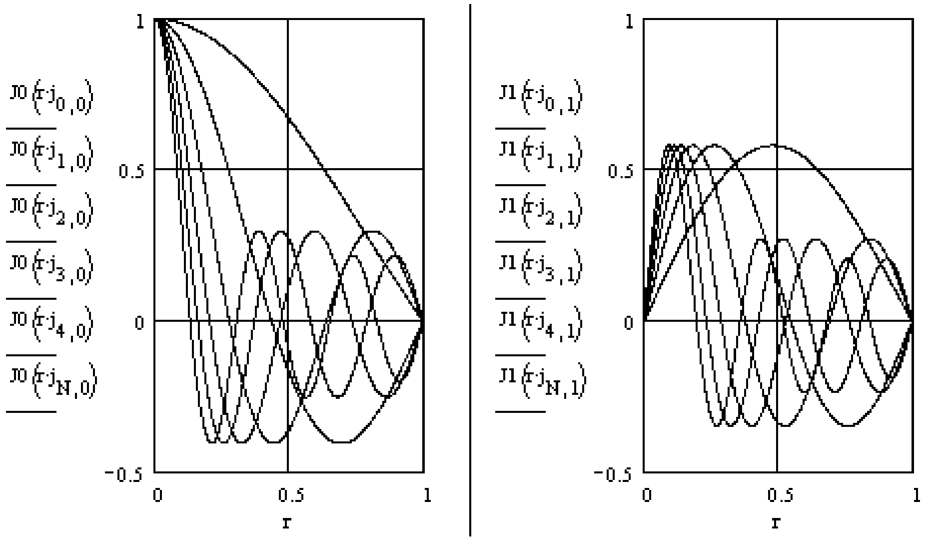

Figure 1. The graphs of orthogonal Bessel's functions on interval [0, 1]

$$s \neq m \quad \Rightarrow \quad \int_0^1 x\, J_\nu(j_{s,\nu}x)\, J_\nu(j_{m,\nu}x)\, dx = 0 \tag{30}$$

The square of the norm has the expression

$$J_\nu(j_{m,\nu}) = 0 \quad \Rightarrow \quad \int_0^1 x\,[\,J_\nu(j_{m,\nu}x)\,]^2\, dx = \tfrac{1}{2}\,[\,J_{\nu+1}(j_{m,\nu})\,]^2 \tag{31}$$

From (29) if $a(dy/dx)(a) + h\, y(a) = 0$ and $b(dy/dx)(b) + h\, y(b) = 0$ then functions $y(ax)$ and $y(bx)$ are also orthogonal functions.

4. THE DIFFERENTIAL EQUATION OF THE FUNCTION *f*

Let h be a real positive number [5]. We consider solution $y(x)$ of equation (1) and function $f(x)$.

$$f = x\frac{dy}{dx} + h\,y, \qquad x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \left(\frac{\nu^2}{x} - x\right)y \tag{32}$$

So the derivative also depends linearly of the solution y and its derivative.

$$\frac{df}{dx} = x\frac{d^2y}{dx^2} + \frac{dy}{dx} + h\frac{dy}{dx} = h\frac{dy}{dx} + \left(\frac{\nu^2}{x} - x\right)y \tag{33}$$

Inverting the algebraic linear system we have the representation

$$(h^2 - \nu^2 + x^2)\frac{dy}{dx} = h\frac{df}{dx} - \left(\frac{\nu^2}{x} - x\right)f, \qquad (h^2 - \nu^2 + x^2)y = -x\frac{df}{dx} + hf \tag{34}$$

The second derivative of function f has the expression

$$\frac{d^2f}{dx^2} = h\frac{d^2y}{dx^2} + \left(\frac{\nu^2}{x} - x\right)\frac{dy}{dx} - \left(\frac{\nu^2}{x^2} + 1\right)y \tag{35}$$

From Bessel's equation (32) we could write

$$\frac{d^2f}{dx^2} = -\frac{h}{x}\frac{dy}{dx} + h\left(\frac{\nu^2}{x^2} - 1\right)y + \left(\frac{\nu^2}{x} - x\right)\frac{dy}{dx} - \left(\frac{\nu^2}{x^2} + 1\right)y \tag{36}$$

Taking into account formulas (34), we will consider the differential equation of function $f(x)$.

$$(h^2 - \nu^2 + x^2)\frac{d^2f}{dx^2} = M(x)\frac{df}{dx} + N(x)\,f =$$
$$= \left(-\frac{h}{x} + \frac{\nu^2}{x} - x\right)\left[h\frac{df}{dx} - \left(\frac{\nu^2}{x} - x\right)f\right] + \left\{h\left(\frac{\nu^2}{x^2} - 1\right) - \left(\frac{\nu^2}{x^2} + 1\right)\right\}\left[-x\frac{df}{dx} + hf\right] \tag{37}$$

The coefficients of the equation have the following expressions

$$M = \frac{\nu^2 - h^2}{x} + x, \qquad N = (h^2 - \nu^2 + x^2)\left(\frac{\nu^2}{x^2} - 1\right) - 2h \tag{38}$$

The final expression of the equation is

$$.\frac{d^2f}{dx^2} + f = \underline{B}(x)\frac{df}{dx} + C(x)f, \qquad \underline{B} = \frac{\nu^2 - h^2 + x^2}{x\left(h^2 - \nu^2 + x^2\right)}, \qquad C = \frac{\nu^2}{x^2} - \frac{2h}{h^2 - \nu^2 + x^2} \tag{39}$$

For calculating the values of function f given in relation (32), account is taken of a recurrence relation [1], [4].

$$f(x) = 0.5\,x\,[\,J_{\nu-1}(x) - J_{\nu+1}(x)\,] + h\,J_\nu(x) \tag{40}$$

5. THE EQUATION FOR ZEROS OF THE FUNCTION *f*

It is suitable to use the functions K and w defined by following relations

$$f = K\sin w, \qquad \frac{df}{dx} = \frac{dK}{dx}\sin w + K\frac{dw}{dx}\cos w \equiv K\cos w \tag{41}$$

These functions are solutions of the non-linear differential system (42) and (43).

$$\frac{dw}{dx} = 1 - \frac{1}{K}\frac{dK}{dx}\tan w, \qquad K^2 = f^2 + \left(\frac{df}{dx}\right)^2 \tag{42}$$

From equation (39) of function *f* and definition (41) we could write

$$K\frac{dK}{dx} = \left(f + \frac{d^2 f}{d^2 x}\right)\frac{df}{dx} = \left(\underline{B}\frac{df}{dx} + Cf\right)\frac{df}{dx} = (\underline{B}K\cos w + CK\sin w)K\cos w \tag{43}$$

Therefore the differential system for *K* and *w* has the expressions

$$\frac{1}{K\cos w}\frac{dK}{dx} = \underline{B}\cos w + C\sin w, \qquad \frac{dw}{dx} = 1 - (\underline{B}\cos w + C\sin w)\sin w \tag{44}$$

Hence

$$\frac{dx}{dw} = \frac{1}{1 - [\underline{B}(x)\cos w + C(x)\sin w]\sin w} \tag{45}$$

The first positive zeros of function *f*, according to the Mathcad program, are determined with the following instructions.

$$h := 0.5, \quad \nu := 2, \qquad f(z) := 0.5\cdot z\cdot(Jn(\nu-1,z) - Jn(\nu+1,z)) + h\cdot Jn(\nu,z) \tag{46}$$

$$z1 := 3.31075389 \quad x1 := root(f(z1),z1) \quad x1 = 3.31075389 \quad f(x1) = 2.21\cdot 10^{-12} \tag{47}$$

If the initial value z1 is different from the root x1, then the z1 value is changed so that z1 = x1. From (39) it results the expressions of the coefficients

$$\underline{B}(z) := \frac{\nu^2 - h^2 + z^2}{z(h^2 - \nu^2 + z^2)}, \qquad C(z) := \frac{\nu^2}{z^2} - \frac{2h}{h^2 - \nu^2 + z^2} \tag{48}$$

Let *F* be the function

$$F(x,w) := \frac{1}{1 - [\underline{B}(x)\cos w + C(x)\sin w]\sin w} \tag{49}$$

The $[w_0, w_1]$ interval of integration have $N > 4$ components of 2π length.

$$w0 := -\pi \qquad N := 5 \qquad w1 := N\cdot\pi - \pi \qquad p := 32 \tag{50}$$

The solver parameters and solution matrix of equation (45).

$$ic_0 := x1 \qquad D(w,u) := F(u_0,w) \qquad R := rkfixed(ic, w0, w1, N\cdot p, D) \tag{51}$$

Solution x_{mp} values and the zeros of function *f* defined in (32), (40), (46) and (54).

$$x := R^{\langle 1\rangle} \qquad m := 0..N \qquad k_m := x_{m.p} \qquad err_m := f(k_m) \tag{52}$$

$$k^T = [\ 3.310754 \quad 6.787223 \quad 10.021532 \quad 13.209152 \quad 16.378533 \quad 19.538786\ ]$$
$$10^7\, err^T = [\ 21.22\cdot 10^{-5} \quad 1.11 \quad -1.55 \quad 1.89 \quad -2.17 \quad 2.42\ ] \tag{53}$$

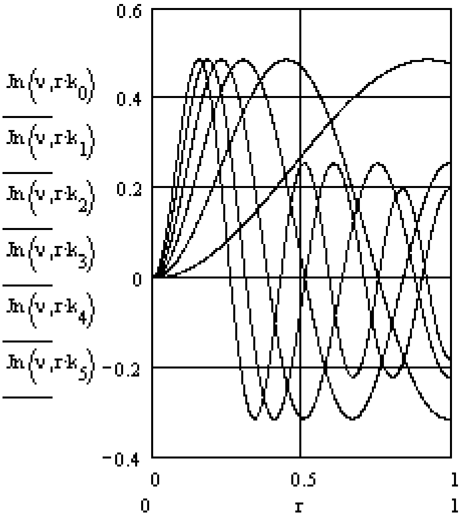

Figure 2. The graphs of orthogonal Dini's functions on interval [0, 1], for ν = 2

$$m = 0,1,2,\dots,N \qquad f(k_m) \equiv \left(x\frac{dJ_\nu(x)}{dx} + h\,J_\nu(x) \right)(k_m) = 0 \tag{54}$$

$$s \neq m \quad \Rightarrow \quad \int_0^1 x\,J_\nu(k_s x)\,J_\nu(k_m x)\,dx = 0 \tag{55}$$

$$\nu = 2,\; f(k_m) = 0 \quad \Rightarrow \quad \int_0^1 x\,[\,J_\nu(k_m x)\,]^2\,\mathrm{d}x = \frac{1}{2}\left(\frac{h^2 - \nu^2}{k_m^2} + 1 \right)[\,J_\nu(k_m)\,]^2 \tag{56}$$

From formula (29) for case $b = a$, inner product $< y(ax), y(bx) >$ would have an undetermined value.

$$(a^2 - b^2)\int_0^1 x\,y(ax)\,y(bx)\,dx = y(a)\,b\frac{dy}{db} - y(b)\,a\frac{dy}{da} \tag{57}$$

The l'Hôspital rule applies.

$$\frac{\partial}{\partial b}\left[y(a)\,b\frac{dy}{db} - y(b)\,a\frac{dy}{da} \right] = y(a)\left(b\frac{d^2y}{db^2} + \frac{dy}{db} \right) - \frac{dy}{db}\,a\frac{dy}{da} \tag{58}$$

Therefore

$$\int_0^1 x\,y(ax)\,y(ax)\,dx = -\lim_{b\to a}\frac{1}{2b}\left\{ y(a)\left(-b + \frac{\nu^2}{b} \right) y(b) - \frac{dy}{db}\,a\frac{dy}{da} \right\} \tag{59}$$

$$2\int_0^1 x\,y(ax)\,y(ax)\,dx = \left(1 - \frac{\nu^2}{a^2} \right) y(a)\,y(a) + \left(\frac{dy}{da} \right)^2 \tag{60}$$

If $y(a) = J_\nu(a) = 0$ then according to the recurrence relations we will have $\mathrm{d}y/\mathrm{d}a = J_{\nu+1}(a)$ and formula (31). If $a\,\mathrm{d}y/\mathrm{d}a + h\,y(a) = 0$, then will have formula (56).

6. RESULTS

In the theory of surface waves problems are often formulated which lead to the determination of roots of certain functions [7], [8], [9] and [10]. Problems with eigenvalues have been recently formulated [6], [11], [12]. In the above, the zeros of the functions of Bessel J_0, J_1, J_2, J_3 formulas (22), (23) and the series of orthogonal functions (30), (31) are specified. Also the zeros of function f are given in formulas (52), (53) and the orthogonal functions are given in formulas (55), (56). Only in the calculation of the first zero it is necessary to know the values of the function. The non-linear differential equation for determining the other zeros have coefficients with known rational expressions.

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