Analytical solutions of a particular Hill's differential system

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Abstract: *Consider a second order differential linear periodic equation. This equation is recast as a first-order homogeneous Hill's system. For this system we obtain analytical solutions in explicit form. The first solution is a periodic function. The second solution is a sum of two functions; the first is a continuous periodic function, but the second is an oscillating function with monotone linear increasing amplitude. We give a formula to directly compute the slope of this increase, without knowing the second numerical solution. The periodic term of second solution may be computed directly. The coefficients of fundamental matrix of the system are analytical functions.*

Key Words: *linear differential equation, dynamic system, parametric resonance*

1. PROBLEM FORMULATION

Consider the following second order periodic linear differential equation with respect to real dimensionless time *t*.

$$\frac{d^2 z}{dt^2} + Q(t)\, z = 0 \tag{1}$$

In what follows we assume the coefficient *Q* to be a real positive bounded continuous periodic function with *t* real argument. The period of *Q*(t) equals π.

By the working hypothesis we admit the particular case.

$$Q = 1 + \frac{4k\,(5 - 6\cos^2 t)\cos^2 t}{1 - k\cos^4 t}, \qquad -0.25 < k \le 0.2 \tag{2}$$

The set of solutions to *Hill*'s equation is two dimensional real space [1], [2]. The function *x* is a periodic solution, [3], [4].

$$x = \frac{1 - k\cos^4 t}{1 - k}\cos t, \qquad \frac{d^2 x}{dt^2} + Q\, x = 0, \qquad x(0) = 1, \qquad \frac{dx}{dt}(0) = 0 \tag{3}$$

The second solution to Hill's equation is described by *Floquet* theory. Let *u* be the derivative of solution *x*.

The *x* and *u* periodic functions check the next *Hill*'s *system*, [5].

$$u=-\frac{1-5k\cos^4 t}{1-k}\sin t,\qquad \frac{dx}{dt}=u,\quad \frac{du}{dt}=-Qx,\qquad x(0)=1,\quad u(0)=0 \tag{4}$$

Each of functions *a*, *b* and σ depend on the parameter *k*.

$$a=\sqrt{1-\sqrt{k}},\quad b=\sqrt{1+\sqrt{k}},\quad \sigma=\frac{k}{4(a+b)}\left[1-k+(7-5k)\sqrt{1-k}\,\right] \tag{5}$$

The unknown y_{p} and v_{p} functions represent the periodic solution of the next non-homogeneous system.

$$\frac{dy_{\text{p}}}{dt}=v_{\text{p}},\qquad \frac{dv_{\text{p}}}{dt}=-Q\,y_{\text{p}}-2\sigma u\,,\qquad y_{\text{p}}(0)=0,\quad v_{\text{p}}(0)=1-\sigma \tag{6}$$

The function *y* and its derivative *v* have the following expressions:

$$\begin{aligned} &y=y_p+\sigma t x,\qquad \frac{dy}{dt}=v=v_p+\sigma x+\sigma t u\\ &\frac{dv}{dt}=\frac{dv_p}{dt}+2\sigma u+\sigma t\frac{du}{dt}=-Qy_p-\sigma tQx \end{aligned} \tag{7}$$

This function and its derivative represent a solution of the homogeneous *Hill*'s system.

$$\frac{dy}{dt}=v,\qquad \frac{dv}{dt}=-Q\,y\,,\qquad y(0)=0,\quad v(0)=1 \tag{8}$$

Since the derivative of the following expression is zero, the functions *x, u, y, v* have the integral property.

$$xv-u\,y=1 \tag{9}$$

Consequently, it results the *Floquet*'s expression of the fundamental matrix, [5], [6], [7].

$$\Phi(t)=\begin{bmatrix} x & y\\ u & v\end{bmatrix}=\begin{bmatrix} x & y_{\text{p}}\\ u & v_{\text{p}}+\sigma x\end{bmatrix}+\sigma t\begin{bmatrix} 0 & x\\ 0 & u\end{bmatrix}=\begin{bmatrix} x & y_{\text{p}}\\ u & v_{\text{p}}+\sigma x\end{bmatrix}\exp\left(\begin{bmatrix} 0 & \sigma\\ 0 & 0\end{bmatrix}t\right) \tag{10}$$

The problem is to prove the expression of the coefficient σ and to obtain analytically periodic solution y_{p}, v_{p}.

2. EXPLICIT CHARACTERISTIC COEFFICIENT

Formulas (7) and (9) result in the non-homogeneous equation for y_{p}.

$$x\,(v_{\text{p}}+\sigma\,x+\sigma\,t\,u\,)\;-\;u\,(\,y_{\text{p}}+\sigma\,t\,x\,)\;=x\,(\,\frac{dy_{\text{p}}}{dt}+\sigma\,x\,)-u\,y_{\text{p}}=1 \tag{11}$$

Let *C* be the variable constant of integration.

$$y_{\text{p}}=C\,x\quad\Rightarrow\quad \frac{dC}{dt}=\frac{1}{x^2}-\sigma\qquad C(0)=0. \tag{12}$$

where

$$\frac{1}{x^2}=\frac{(1-k)^2}{\cos^2 t}\left[1+\frac{1}{(1-k\cos^4 t)^2}-1\right] \tag{13}$$

The derivative of function C_0 is a singularity of function x^{-2}. The product $C_0\,x$ will be a periodic bounded function.

$$C_0=(1-k)^2\tan t, \qquad C_0\,x=(1-k)\,(1-k\cos^4 t)\sin t, \qquad C_\sigma=C-C_0 \tag{14}$$

Consequently,

$$y_p=C_0\,x+C_\sigma\,x \quad \Rightarrow \quad \frac{dC_\sigma}{dt}=\left[\frac{1}{x^2}-\frac{(1-k)^2}{\cos^2 t}\right]-\sigma \qquad C_\sigma(0)=0. \tag{15}$$

Given the derivative of the geometric series we will have Taylor's series development.

$$\frac{1}{x^2}-\frac{(1-k)^2}{\cos^2 t}=\frac{(1-k)^2}{\cos^2 t}\left[\frac{1}{(1-k\cos^4 t)^2}-1\right]=k(1-k)^2\sum_{n\geq 0}(n+2)k^n\cos^{2+4n} t \tag{16}$$

The first expression of the characteristic coefficient will be

$$\sigma=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left[\frac{1}{x^2}-\frac{(1-k)^2}{\cos^2 t}\right]dt=\frac{2}{\pi}(1-k)^2\int_0^{\frac{\pi}{2}}\frac{2k\cos^2 t-k^2\cos^6 t}{(1-k\cos^4 t)^2}dt \tag{17}$$

By integrating term by term the above function, we obtain the second expression of σ:

$$\sigma=k(1-k)^2\sum_{n\geq 0}g_n\,k^n, \qquad g_n=\frac{2}{\pi}(n+2)\int_0^{\frac{\pi}{2}}\cos^{2+4n} t\;dt \tag{18}$$

The recurrence relation of constant coefficients g_n, has the following expression.

$$g_0=1, \quad g_{n+1}=\frac{(n+3)(4n+3)(4n+5)}{8(n+1)(n+2)(2n+3)}g_n \tag{19}$$

The convergence velocity of the series is acceptable for low values of parameter k.

3. SECOND EXPLICIT SOLUTION

The functions b and a depending on the k parameter, according to formulas (5), verify the following identities:

$$a^2 b^2=1-k, \qquad a^2+b^2=2, \qquad b^2-a^2=2\sqrt{k} \tag{20}$$

We make the change of the variable of integration.

$$s=\tan t, \qquad \frac{1}{x}=\frac{1-k}{\cos t}\frac{1}{1-k\cos^4 t}, \quad \cos^2 t=\frac{1}{s^2+1}, \quad \frac{1}{x}=\frac{1-k}{\cos t}\frac{(s^2+1)^2}{(s^2+1)^2-k} \tag{21}$$

Hence

$$\frac{1}{x^2}=\frac{(1-k)^2}{\cos^2 t}\left\{1+\frac{k}{(s^2+1)^2-k}\right\}^2=\frac{(1-k)^2}{\cos^2 t}(1+E) \tag{22}$$

where

$$E=\frac{2k}{(s^2+1)^2-k}+\frac{k^2}{\left[(s^2+1)^2-k\right]^2} \tag{23}$$

Taking into account equation (12), function $h(s)$, is the solution of the following equation:

$$C(t)=h(s),\quad s=\tan t\,,\qquad \frac{dC}{dt}=\frac{dh}{ds}\frac{1}{\cos^2 t}=(1+s^2)\frac{dh}{ds}=\frac{1}{x^2}-\sigma \tag{24}$$

The function E is a rational function.

$$\frac{dh}{ds}=(1-k)^2(1+E)-\frac{\sigma}{1+s^2},\qquad h(0)=0 \tag{25}$$

First of all, we check the following identities:

$$(s^2+1)^2-k=s^4+2s^2+1-k=s^4+(a^2+b^2)\,s^2+a^2b^2=(s^2+a^2)\,(s^2+b^2) \tag{26}$$

Analogously, we will have:

$$\frac{2k}{(s^2+1)^2-k}=\sqrt{k}\left(\frac{1}{s^2+a^2}-\frac{1}{s^2+b^2}\right) \tag{27}$$

Thus,

$$\frac{k^2}{\left[(s^2+1)^2-k\right]^2}=\frac{k}{4}\left[\frac{1}{(s^2+a^2)^2}+\frac{1}{(s^2+b^2)^2}\right]-\frac{\sqrt{k}}{4}\left(\frac{1}{s^2+a^2}-\frac{1}{s^2+b^2}\right) \tag{28}$$

Finally, we rewrite (23) in the formula:

$$E=\frac{k}{4}\left[\frac{1}{(s^2+a^2)^2}+\frac{1}{(s^2+b^2)^2}\right]+\frac{3\sqrt{k}}{4}\left(\frac{1}{s^2+a^2}-\frac{1}{s^2+b^2}\right) \tag{29}$$

On the other hand, we have identities:

$$\frac{d}{dt}\left(\frac{s}{s^2+a^2}\right)=\frac{1}{s^2+a^2}-\frac{2s^2}{(s^2+a^2)^2}=\frac{a^2-s^2}{(s^2+a^2)^2}=\frac{2a^2}{(s^2+a^2)^2}-\frac{1}{s^2+a^2} \tag{30}$$

The function E is the sum of two terms:

$$E=E_1+E_2$$

$$E_1=\frac{k}{4}\frac{d}{ds}\left[\frac{s}{2a^2(s^2+a^2)}+\frac{s}{2b^2(s^2+b^2)}\right] \tag{31}$$

The second term has the following expression:

$$E_2=\left(\frac{k}{8a^2}+\frac{3\sqrt{k}}{4}\right)\frac{1}{s^2+a^2}+\left(\frac{k}{8b^2}-\frac{3\sqrt{k}}{4}\right)\frac{1}{s^2+b^2} \tag{32}$$

The solution of the differential equation (25) has two terms that check equations (33).

$$h = h_1 + h_2$$

$$h = h_1 + h_2, \quad \frac{dh_1}{ds} = (1-k)^2(1+E_1), \quad \frac{dh_2}{ds} = (1-k)^2 E_2 - \frac{\sigma}{s^2+1} \tag{33}$$

The solution of the first equation is:

$$h_1(s) = (1-k)^2\left\{s + \frac{k}{8}\left[\frac{s}{a^2(s^2+a^2)} + \frac{s}{b^2(s^2+b^2)}\right]\right\} \tag{34}$$

The argument of the second equation is the function s depending on $t \in [0, \pi/2)$.

$$h_2(s) = \\ = \frac{(1-k)^2}{8}\left\{\left(\frac{k}{a^2} + 6\sqrt{k}\right)\frac{1}{a}\arctan\frac{s}{a} + \left(\frac{k}{b^2} - 6\sqrt{k}\right)\frac{1}{b}\arctan\frac{s}{b} + \right\} - \sigma\arctan s \tag{35}$$

The rational following function has the equivalent expression:

$$\frac{1}{a^2(s^2+a^2)} + \frac{1}{b^2(s^2+b^2)} = \frac{(a^2+b^2)s^2 + a^4 + b^4}{(s^2+a^2)(s^2+b^2)} = \frac{2(s^2+1+k)}{(s^2+1)^2 - k} \tag{36}$$

Consequently,

$$h_1(s) = (1-k)^2 s\left[1 + \frac{k}{4}\frac{(s^2+1+k)}{(s^2+1)^2 - k}\right] \tag{37}$$

The first term C_1 of the function C have the expression:

$$C_1(t) = h_1(\tan t) = (1-k)^2\left[1 + \frac{k\cos^2 t}{4}\frac{1+k\cos^2 t}{1-k\cos^4 t}\right]\tan t \tag{38}$$

The y_p periodic solution of the equation (12) has a first explicit term y_{p1}.

$$y_p(t) = x(t)\,[\,C_1(t) + C_2(t)\,] = y_{p1}(t) + y_{p2}(t)$$

$$y_{p1}(t) = x(t\,)\,C_1\,t) = (1-k)\,[\,1 + 0.25\,k\cos^2 t + (0.25\,k^2 - k)\cos^4 t\,]\,\sin t \tag{39}$$

Derivation results in the explicit expression of the first term of the v_{p1} solution.

$$v_{p1}(t) = \frac{dy_{p1}}{dt}, \quad v_1(t) = (1-k)\{\,1 - 0.25k\,[2+(4k-19)\cos^2 t + 5(4-k)\cos^4 t\,]\}\cos t \tag{40}$$

The first coefficients of the expression in formula (35) define the functions of parameter k.

$$\beta_1 = \frac{(1-k)^2}{8a}\left(\frac{k}{a^2} + 6\sqrt{k}\right), \qquad \beta_2 = \frac{(1-k)^2}{8b}\left(\frac{k}{b^2} - 6\sqrt{k}\right) \tag{41}$$

It is necessary to cancel the limit value of the function h_2 (s) to infinity.

$$h_2(s) = \beta_1\arctan\frac{s}{a} + \beta_2\arctan\frac{s}{b} - \sigma\arctan s,\ \lim_{s\to\infty} h_2(s) = 0,\ \sigma = \beta_1 + \beta_2 \tag{42}$$

We find the value (5) of the coefficient σ.

$$\sigma = \beta_1 + \beta_2 \equiv \frac{k}{4(a+b)}\left[1 - k + (7-5k)\sqrt{1-k}\right], \quad a = \sqrt{1-\sqrt{k}}, \quad b = \sqrt{1+\sqrt{k}} \tag{43}$$

The solution h_2 has the expression that will ensure the periodicity of the C_2 function.

$$h_2(s) = \beta_1(\arctan\frac{s}{a} - \arctan s) + \beta_2(\arctan\frac{s}{b} - \arctan s) \tag{44}$$

The second term of the solution C has the expression

$$C_2(t) = h_2(\tan t) = \beta_1\left[\arctan\left(\frac{\tan t}{a}\right) - t\right] + \beta_2\left[\arctan\left(\frac{\tan t}{b}\right) - t\right] \tag{45}$$

This expression is correct for t in the range [0, π/2) and is the restriction of the next periodic integral function

$$C_2(t) = \int_0^t \left(\frac{\beta_1 a}{1+(a^2-1)\cos^2 t} + \frac{\beta_2 b}{1+(b^2-1)\cos^2 t} - \sigma\right)dt, \tag{46}$$

Let γ be the function:

$$\gamma(t) = \frac{\beta_1 a}{1+(a^2-1)\cos^2 t} + \frac{\beta_2 b}{1+(b^2-1)\cos^2 t} = \frac{k(1-k)}{4}\frac{1+(6-5k)\cos^2 t}{1-k\cos^4 t} \tag{47}$$

Consequently,

$$C_2(t) = \int_0^t \gamma(t)dt - \sigma t, \qquad y_{p2}(t) = x(t)\, C_2(t) \tag{48}$$

Thus,

$$v_{p2}(t) = \frac{dy_{p2}}{dt} = u(t)C_2(t) + x(t)\left[\gamma(t) - \sigma\right] \tag{49}$$

The periodic solution (y_p, v_p) of the non-homogeneous system (6) is calculated according to formulas (38), (39), (47), (48) and (49).

The solution (y, v) of the system (8) will be

$$y(t) = y_{p1}(t) + y_{p2}(t) + \sigma\, t\, x(t), \quad v(t) = v_{p1}(t) + v_{p2}(t) + \sigma\, x(t) + \sigma\, t\, u(t) \tag{50}$$

4. RESULTS

The analytical solutions of the differential system of formulas (4), (6) and (8) have the following expressions specified in the MATCAD program [8], [9].

The graphs in Figure 1 show the y_p and v_p periodic solution of the non-homogeneous system (6) and also the two additive components y_{p1}, y_{p2}, and v_{p1}, v_{p2}.

The numerical solutions of the system (4), (6) and (8) result from the second part of the program below.

$$k := 0.2$$

$$Q(t) := 1 + 4\cdot k\cdot\frac{5-6\cdot\cos(t)^2}{1-k\cdot\cos(t)^4}\cdot\cos(t)^2 \qquad x(t) := \frac{1-k\cdot\cos(t)^4}{1-k}\cdot\cos(t) \qquad u(t) := -\frac{1-5\cdot k\cdot\cos(t)^4}{1-k}\cdot\sin(t)$$

$$yp1(t) := (1-k)\cdot\left[1-k\cdot\cos(t)^4+\frac{k}{4}\cdot\cos(t)^2\cdot\left(1+k\cdot\cos(t)^2\right)\right]\cdot\sin(t)$$

$$vp1(t) := (1-k)\cdot\left[1-\frac{k}{4}\cdot\left[(2+(4\cdot k-19))\ \cos(t)^2+5\cdot(4-k)\cdot\cos(t)^4\right]\right]\cdot\cos(t)$$

$$a := \sqrt{1-\sqrt{k}} \qquad b := \sqrt{1+\sqrt{k}} \qquad \sigma := \frac{k}{4\cdot(a+b)}\left[1-k+(7-5\cdot k)\cdot\sqrt{1-k}\right] \qquad \sigma = 0.15840148$$

$$\gamma(t) := \frac{k\cdot(1-k)}{4}\cdot\frac{1+(6-5\cdot k)\cdot\cos(t)^2}{1-k\cdot\cos(t)^4} \qquad C2(t) := \int_0^t \gamma(t)\,dt-\sigma\cdot t \qquad C2\left(\frac{\pi}{2}\right) = 6.8\cdot 10^{-10}$$

$$yp2(t) := x(t)\cdot C2(t) \qquad yp(t) := yp1(t)+yp2(t) \qquad y(t) := yp(t)+\sigma\cdot t\cdot x(t)$$

$$vp2(t) := u(t)\cdot C2(t)+x(t)\cdot(\gamma(t)-\sigma) \qquad vp(t) := vp1(t)+vp2(t) \qquad v(t) := vp(t)+\sigma\cdot x(t)+\sigma\cdot t\cdot u(t)$$

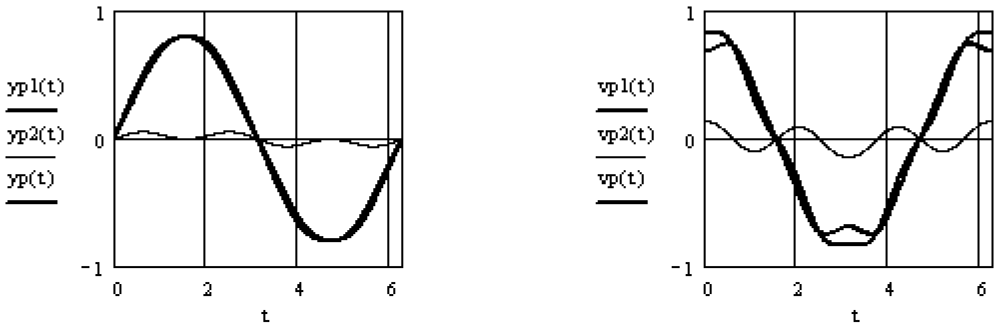

Figure 1. The graphs of the periodic solution y_p, v_p of the system (6)

The Φ fundamental matrix of the system (4) or (8) has the expression (10) of the matrix product of the periodic component $P\ (t)$ and the matrix (I_2+Bt) =exp (Bt). B is nilpotent matrix.

$$\Phi(t) = P(t)\exp(Bt), \quad B = \begin{bmatrix} 0 & \sigma \\ 0 & 0 \end{bmatrix}, \quad P(t) = \begin{bmatrix} x(t) & y_p(t) \\ u(t) & v_p(t)+\sigma x(t) \end{bmatrix} \tag{51}$$

A linear differential equation of the second order has two solutions in general. There are equations without periodic solutions, equations with a single periodic solution as in the treated example or equations with both periodic solutions.

Knowing a periodic solution depending on the parameter k, the integral σ of the formula (17) was specified.

Depending on the explicit expression (5), if σ is different from zero, the second solution is not periodic.

Numerical x, u, y_p, v_p, y, v solutions are denominated in capital letters.

$$uo := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1-\sigma \\ 0 \\ 1 \end{bmatrix} \quad D(t,u) := \begin{bmatrix} u_1 \\ -Q(t)\cdot u_0 \\ u_3 \\ -Q(t)\cdot u_2 - 2\cdot\sigma\cdot u_1 \\ u_5 \\ -Q(t)\cdot u_4 \end{bmatrix}$$

$$N := 1024$$

$$S := \mathrm{rkfixed}(uo, 0, 4\cdot\pi, N, D) \qquad t := S^{<0>}$$

$$X := S^{<1>} \quad U := S^{<2>} \quad Yp := S^{<3>}$$

$$Vp := S^{<4>} \quad Y := S^{<5>} \quad V := S^{<6>}$$

$$n := 0..N \qquad er_n := \left| Y_n - \left(Yp_n + \sigma\cdot t_n\cdot X_n\right) \right| \qquad err_n := \left| V_n - \left(Vp_n + \sigma\cdot X_n + \sigma\cdot t_n\cdot U_n\right) \right|$$

$$\max(er) = 3.4\cdot 10^{-9}$$

$$\max(err) = 3.7\cdot 10^{-9}$$

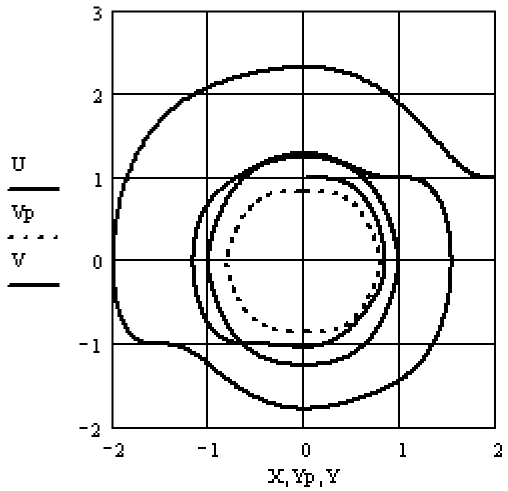

Figure 2. The graphs (X, U) and (Yp, Vp) show periodicity

The amplitude of the oscillations of the solution y increases proportionally to time t. It is important to integrate on the interval $[0, 2\pi]$, since $y(t + 2n\pi) = y(t) + 2n\pi\ \sigma\ x(t)$.

5. CONCLUSIONS

The enunciation and theoretical demonstration of following propositions is the main contribution of the work. If $Q(t, k)$ is a set of functions according to formula (2), then differential equation (1) has a periodic solution $x(t)$ given by formula (3).

If the function $\sigma(k)$ has the algebraic expression (5), then the differential system (6) has a periodic solution $y_p(t)$, $v_p(t)$. The second fundamental solution $y(t)$ of equation (1) has the structural expression $y(t) = y_p(t) + t\ \sigma\ (k)\ x(t)$, and the fundamental matrix of the general system (8) has the expression (10). It also shows that, after relatively laborious calculations, the algebraic function $\sigma(k)$ is the value of an integral with the real k parameter.

The numerical calculation program shown in Figures 1 and 2 tests the truth of these propositions for the interval [0,2], in particular $k = 0.2$.

The second section begins the theoretical and constructive demonstration. The wronskian determinant of solutions x and y does not explicitly depend on Q. Given $u = \mathrm{d}x / \mathrm{d}t$ and $v = \mathrm{d}y / \mathrm{d}t$ we will obtain $xv - uy = 1$ for $y\ (t)$ or linear equation (11) for $y_p\ (t)$. The homogeneous equation has the solution $Cx(t)$. The method of variation of the integration constant requires $y_p(t) = C(t)x(t)$. The unknown function $C(t)$ checks a singular equation (12) in the sense that $x(\pi/2) = 0$. The singularity is isolated, resulting in the expression $C(t) = C_0(t) + C_\sigma(t)$. Although $C_0(t)$ is periodic, unbounded according to (14), product $x(t)C_0(t)$ is a periodic bounded function.

The term $C_\sigma(t)$ must be periodic, otherwise $y_p(t)$ would not be the periodic component of $y(t)$. Derivative $\mathrm{d}C_\sigma/\mathrm{d}t$ is a periodic function bounded according to (15). But it is known that the integral of a periodic function, is periodic, if and only if the mean value of the function is zero. Imputing this condition results in expression (17) of the coefficient $\sigma(k)$. This is an integral with parameter k.

Formulas (18) and (19) allow the calculation of $\sigma(k)$ as the limit of a set of functions. In the third section the expression of the periodic component $y_p(t)$ is determined.

With the change of the integration variable $C(t) = h(s)$, $s = \tan t$, the equation (25) results. The function $h(s)$ is the primitive of a rational function. By a decomposition in certain simple fractions, the need for expressing the $h(s) = h_1(s)+ h_2(s)$ function.

The $h_1(s)$ function has explicit expression (37). It results $C_1(t)$ and a first periodic $y_{p1}(t)$ term of the $y_p(t)$ component.

The restriction of the $h_2(s)$ function has the expression (35) in which the coefficient $\sigma(k)$ appears. For $h_2(s)$ to be bounded at the infinite point, from (35), (41), (42) the algebraic expression (43) or (5) of the coefficient $\sigma(k)$ results.

The $h_2(s)$ function will correspond to the restriction (45) of the function $C_2(t)$. The extension of this function has the expression (46) as integral to the function $\gamma(t)$ specified in (47). Knowing $C_2(t)$ determines the second component $y_{p2}(t)$ of the $y_p(t)= y_{p1}(t)+ y_{p2}(t)$ function.

In the fourth section there is a numerical program for testing the formulas given above.

For the exemplified case $k = 0.2$ the theoretical formulas are consistent with the numerical results. In the case of negative k values, the coefficients a and b are complex conjugates. The $\sigma(k)$ coefficient is however real as identity has been demonstrated (43). The function $\gamma(t)$ has also real values according to the identity (47). Finally, $C_2(t)$ and $y_p(t)$ functions has real values.

It is important to note that the graphs in Figure 2 are plotted by knowing only $Q(t)$ and the algebraic expression of the σ coefficient.

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