Simplified Mathieu's equation with linear friction

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Abstract: *Consider a second order differential linear periodic equation. The friction coefficient is real positive constant. Some transformation of the solution and its first derivative allow writing two-order differential equations with void friction coefficients. The solutions of these equations are periodic functions or sum of periodic function and an oscillating function with monotone linear increasing amplitude. The second order equation with linear friction is recast as a first order system. The coefficients of the principal fundamental matrix solution of the system are explicit analytical functions.*

Key Words: *surface waves, Mathieu*'s *differential equation, parametric resonance, linear stability, characteristic exponents.*

1. INTRODUCTION

The approximate theory of infinitesimal standing waves is very fruitful in its application to problems with various special boundary configurations. The linear character of both the equations and boundary conditions allowed finding some explicit solutions. Let (x, y)-plane be at the undisturbed fluid free fixed simply connected surface Ω having a piecewise smooth boundary contour $\partial\Omega$. A special case of particular interest is the irrotational flow of perfect incompressible fluid. It is assumed that the bottom fluid is bounded by a rigid fixed surface $z = -h < 0$. The flow velocity $\mathbf{v}$ is related to the potential ϕ by the formula

$$\mathbf{v} = \nabla \phi$$

Differential operations ∇, div and Δ are performed with the variables x, y, z. The potential has the following expression

$$\phi(x, y, z, t) = H(x, y) \cosh (z + h) \cos \omega t, \qquad (x, y) \in \Omega, \quad z \in [0, h], \quad t \in \boldsymbol{R}$$

The continuity equation is fulfilled in the fluid domain.

$$\text{div}\, \mathbf{v} = \Delta \phi = 0$$

Consequently, the function H is a solution of *Hermann von Helhmoltz* equation.

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}=-k^2 H\,,\qquad (x,y)\in\Omega,\quad k=\text{const.}>0$$

Let $\mathbf{n}$ (n_1, n_2) be the normal vector to the contour $\partial\Omega$, it is necessary that the boundary flow condition be fulfilled

$$\frac{dH}{dn}=\frac{\partial H}{\partial x}n_1+\frac{\partial H}{\partial y}n_2=0\,,\qquad (x,y)\in\partial\Omega(x,y)$$

The form of the free surface is given by

$$\zeta(x,y,t)\sim H(x,y)\sin\omega t$$

If the surface Ω is an ellipse, then *Mathieu*'s equation can be obtained by expressing the above equation in elliptical coordinates and by the method of separation of variables [1].

2. PROBLEM FORMULATION

Consider the following two-order linear differential equation with respect to real dimensionless time t.

$$\frac{d^2 Z}{dt^2}+2\lambda\frac{dZ}{dt}+\left(\lambda^2+Q\right)Z=0 \tag{1}$$

We assume that the λ coefficient is a positive constant. The following Q function is a reasonable approximation of Mathieu's coefficient [1], [2], [3].

$$Q=1+\frac{8q\left(1-2\cos 2t\right)}{1+q-2q\cos 2t}\,,\qquad -1/9<q<1/9 \tag{2}$$

We recast the equation (1) as a first-order system. The following system is obtained.

$$\frac{dZ}{dt}=W,\qquad \frac{dW}{dt}=-\left(\lambda^2+Q\right)Z-2\lambda W \tag{3}$$

Let $(X, U)^{\mathrm{T}}$ and $(Y, V)^{\mathrm{T}}$ be the characteristic solutions of these system.

$$\frac{dX}{dt}=U,\qquad \frac{dU}{dt}=-\left(\lambda^2+Q\right)X-2\lambda U\,,\qquad X(0)=1,\quad U(0)=0 \tag{4}$$

$$\frac{dY}{dt}=V,\qquad \frac{dV}{dt}=-\left(\lambda^2+Q\right)Y-2\lambda V\,,\qquad Y(0)=0,\quad V(0)=1 \tag{5}$$

The problem is to give the analytical expressions of these solutions.

3. USEFUL TRANSFORMATIONS

The functions X, U, Y and V fulfill the relations

$$\frac{d}{dt}\left(XV-UY\right)=-2\lambda\left(XV-UY\right),\qquad (XV-UY)(0)=1$$

The solution of this equation has the expression

$$X\,V - U\,Y = \exp(-2\lambda t) \tag{6}$$

Consider the following transformations in order to solve the above systems.

$$X = z\exp(-\lambda t),\quad X(0)=1 \quad\Rightarrow\quad z(0)=1,\quad w = \frac{dz}{dt} \tag{7}$$

$$U = \frac{dX}{dt} = (w - \lambda z)\exp(-\lambda t),\quad U(0)=0 \quad\Rightarrow\quad w(0)=\lambda \tag{8}$$

$$Y = y\exp(-\lambda t),\quad Y(0)=0 \quad\Rightarrow\quad y(0)=0,\quad v = \frac{dy}{dt} \tag{9}$$

$$V = \frac{dY}{dt} = (v - \lambda y)\exp(-\lambda t),\quad V(0)=1 \quad\Rightarrow\quad v(0)=1. \tag{10}$$

The vectors $(z, w)^T$, $(y, v)^T$ and $(x, u)^T$ are the solutions of the following systems

$$\frac{dz}{dt} = w,\qquad \frac{dw}{dt} = -Qz,\qquad z(0) = 1,\qquad w(0) = \lambda \tag{11}$$

$$\frac{dy}{dt} = v,\qquad \frac{dv}{dt} = -Qy,\qquad y(0) = 0,\qquad v(0) = 1 \tag{12}$$

$$\frac{dx}{dt} = u,\qquad \frac{du}{dt} = -Qx,\qquad x(0) = 1,\qquad u(0) = 0 \tag{13}$$

Consequently,

$$z = x + \lambda y,\qquad w = u + \lambda v \tag{14}$$

4. RESULTS

The expressions of solutions $(x, u)^T$ and $(y, v)^T$ have been obtained in [3].

$$x = \frac{\cos t - q\cos 3q}{1-q} = \frac{\cos t}{1-q}\left(1 + 3q - 4q\cos^2 t\right) \tag{15}$$

$$u = -\frac{\sin t - 3q\sin 3t}{1-q} = -\frac{\sin t}{1-q}\left(1 + 3q - 12q\cos^2 t\right) \tag{16}$$

Let us introduce the following notations

$$\alpha = \sqrt{\frac{1-q}{1+3q}},\qquad \beta = \frac{8\alpha q}{(1+3q)^2},\qquad \gamma_0 = \frac{4q(1-3q)}{(1+3q)^2} \tag{17}$$

Denote Γ_α the periodic function

$$\Gamma_\alpha(t) = \int_0^t \left(\frac{\alpha}{1 + (\alpha^2 - 1)\cos^2 t} - \operatorname{sign}\alpha\right)dt,\qquad \Gamma_\alpha(t+\pi) = \Gamma_\alpha(t) \tag{18}$$

Finally it results

$$y = (\alpha^2 - \gamma_0 \cos^2 t) \sin t + \beta\, x\, (\Gamma_\alpha + t) \tag{19}$$

$$v = (1 + 3\,\gamma_0 \sin^2 t) \cos t + \beta\, u\, (\Gamma_\alpha + t) \tag{20}$$

The ($y - \beta\ t\ x$) difference is a periodic function but ($\beta\ t\ x$) is an oscillating function with monotone linear increasing amplitude.

Taking into account the initial real conditions

$$Z(0) = Z_0, \qquad W(0) = W_0 \tag{21}$$

we can write

$$\begin{pmatrix} Z(t) \\ W(t) \end{pmatrix} = \begin{vmatrix} X(t) & Y(t) \\ U(t) & V(t) \end{vmatrix} \begin{pmatrix} Z(0) \\ W(0) \end{pmatrix} = \begin{vmatrix} z & y \\ w - \lambda z & v - \lambda y \end{vmatrix} \begin{pmatrix} Z_0 \\ W_0 \end{pmatrix} \exp(-\lambda t) \tag{22}$$

Consequently,

$$Z(t) = \{\, [\, x(t) + \lambda\, y(t)\,]\, Z_0 + y(t)\, W_0 \,\} \exp(-\lambda\, t) \tag{23}$$

$$W(t) = -\lambda\, Z(t) + \{\, [\, u(t) + \lambda\, v(t)\,]\, Z_0 + v(t)\, W_0 \,\} \exp(-\lambda\, t) \tag{24}$$

The set of these solutions is two-dimensional real space [4], [5], [6]. For directly calculation of $(X, U, Y, V)^T$ solution it is useful to consider the fourth-order differential system (4) and (5), [7], [8], [9]. The constant solution $(0, 0)^T$ of the first system (3) is asymptotically stable [10], [11].

$$\lambda > 0 \quad \Rightarrow \quad \lim_{t\to\infty} Z(t) = 0, \quad \lim_{t\to\infty} W(t) = 0 \tag{25}$$

Let $f(t)$ and $g(t)$ be the periodic functions

$$\lambda > 0 \quad \Rightarrow \quad \lim_{t\to\infty} Z(t) = 0, \quad \lim_{t\to\infty} W(t) = 0 \tag{26}$$

Let $\mathbf{P}(t)$ be the following 2π-periodic matrix

$$\mathbf{P}(t) = \begin{bmatrix} x(t) & f(t) + \beta x(t)\Gamma_\alpha(t) \\ u(t) & g(t) + \beta u(t)\Gamma_\alpha(t) \end{bmatrix} \tag{27}$$

This matrix is invertible unimodular matrix

$$|\,\mathbf{P}(t)\,| = x(t)\, g(t) - u(t)\, f(t) = 1 \tag{28}$$

Consequently, in the case $\lambda = 0$, the principal fundamental matrix $\Phi_0(t)$ of the system (3) has the following expression

$$\Phi_0(t) = \begin{bmatrix} x(t) & y(t) \\ u(t) & v(t) \end{bmatrix} = \mathbf{P}(t) + \beta t \begin{bmatrix} 0 & x(t) \\ 0 & u(t) \end{bmatrix} \tag{29}$$

Taking into account formulae (17), let $\mathbf{B}_0$ be the parametric nilpotent matrix

$$\mathbf{B}_0 = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{B}_0^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{30}$$

Therefore

$$e^{\mathbf{B}_0 t} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!}(\mathbf{B}_0 t)^n = \mathbf{I} + \mathbf{B}_0 t \tag{31}$$

Consequently,

$$\Phi_0(0) = \mathbf{I}, \qquad \Phi_0(2\pi) = \mathbf{I} + 2\pi\mathbf{B}_0$$

The appropriate monodromy matrix has the expression

$$\mathbf{C}_0 = \mathbf{I} + 2\pi\mathbf{B}_0, \qquad \Phi_0(t + 2\pi) = \Phi_0(t)\mathbf{C}_0 \tag{32}$$

Taking into account the formulae (14) and (22), for $\lambda \geq 0$, the principal fundamental matrix solution of the system (3) has the expression

$$\Phi(t) = \begin{bmatrix} z & y \\ w - \lambda z & v - \lambda y \end{bmatrix} e^{-\lambda t} = \begin{bmatrix} x + \lambda y & y \\ u + \lambda v - \lambda(x + \lambda y) & v - \lambda y \end{bmatrix} e^{-\lambda t} \tag{33}$$

Therefore,

$$|\,\Phi(t)\,|\, e^{\lambda t} = \cdots = (\, x\,v - u\,y\,) = (\, x\,g - u\,f\,) = 1 \tag{34}$$

Consequently,

$$\Phi(t) = \left\{ \begin{bmatrix} x & 0 \\ u + \lambda(v - x) & v \end{bmatrix} + y \begin{bmatrix} \lambda & 1 \\ -\lambda^2 & -\lambda \end{bmatrix} \right\} e^{-\lambda t}$$

so that

$$\Phi^{-1}(t) = \left\{ \begin{bmatrix} v & 0 \\ -u - \lambda(v - x) & x \end{bmatrix} + y \begin{bmatrix} -\lambda & -1 \\ \lambda^2 & \lambda \end{bmatrix} \right\} e^{\lambda t}$$

We recall the formulae (15) - (20) and we deduce successively

$$x(2\pi) = x(0) = 1, \quad u(2\pi) = u(0) = 0, \quad y(0) = 0, \quad v(0) = 1, \quad \Phi(0) = \mathbf{I} \tag{35}$$

$$v(t + 2\pi) = v(t) + 2\pi\beta\, u(t), \qquad v(2\pi) = 1 \tag{36}$$

$$y(t + 2\pi) = y(t) + 2\pi\beta\, x(t), \qquad y(2\pi) = 2\pi\beta \tag{37}$$

$$\Phi(2\pi) = \begin{bmatrix} 1 + 2\pi\beta\lambda & 2\pi\beta \\ \lambda - \lambda(1 + 2\pi\beta\lambda) & 1 - 2\pi\beta\lambda \end{bmatrix} e^{-2\pi\lambda} \tag{38}$$

But, $\Phi(t + 2\pi)$ is also a fundamental matrix solution.

$$\Phi(t + 2\pi) = e^{-2\pi\lambda}\,\Phi(t) \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\pi\beta e^{-\lambda t}\, \Phi^{-1}(t) \left(u \begin{bmatrix} 0 & 0 \\ \lambda & 1 \end{bmatrix} + x \begin{bmatrix} \lambda & 1 \\ -\lambda^2 & -\lambda \end{bmatrix} \right) \right\}$$

$$\Phi(t + 2\pi) = e^{-2\pi\lambda}\, \Phi(t) \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\pi\beta(xv - uy) \begin{bmatrix} \lambda & 1 \\ -\lambda^2 & -\lambda \end{bmatrix} \right\}$$

Therefore, using the formula (34), there exists an invertible constant matrix **C,** so that

$$\Phi(t+2\pi)=\Phi(t)\,\mathbf{C},\quad \mathbf{C}=\Phi(2\pi)$$

The appropriate monodromy matrix for the system (3) or (41) has the expression

$$\mathbf{C}=\mathbf{C}(\lambda,q)=\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}+2\pi\beta(q)\begin{bmatrix}\lambda&1\\-\lambda^2&-\lambda\end{bmatrix}\right\}e^{-2\pi\lambda},\quad \lambda\ge 0 \tag{39}$$

$$\beta(q)=\frac{8q}{(1+3q)^2}\sqrt{\frac{1-q}{1+3q}},\qquad |q|<0.11 \tag{40}$$

$$\begin{bmatrix}\dfrac{dZ}{dt}\\ \dfrac{dW}{dt}\end{bmatrix}=\mathbf{A}(t)\begin{bmatrix}Z\\ W\end{bmatrix},\qquad \mathbf{A}(t)=\begin{bmatrix}0&1\\ -\lambda^2-\dfrac{8q(1-2\cos 2t)}{1+q-2q\cos 2t}-1&-2\lambda\end{bmatrix} \tag{41}$$

5. CONCLUSIONS

The problem of determining the Floquet multipliers of linear differential periodic systems is often difficult. For the special case of simplified *Mathieu*'s equation but with linear friction (1), the characteristic coefficients of the appropriate system (3) or (41) have explicit analytical expressions and also the monodromy matrix **C**.

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