About zeros of some oscillations with dynamic friction

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Abstract: *Consider a second order differential non-linear equation having free boundary value conditions. Let be a solution having infinity of unknown zeros. The integral of energy gives the implicit correlation between the successive modules of the extreme values of oscillation. The method of successive approximations transforms this correlation into an algorithmic correlation. The decreasing sequence of the modules or local amplitudes converges to zero. For the local amplitude of oscillation inside the interval of two successive zeros, the length of the interval is a sum of two improper integrals. In order to obtain the values of these integrals, it is necessary to use series expansions. If the coefficient of dynamic friction is small and the amplitude reached a low enough value, then the polynomial functions are given for the numerical calculus of distances between zeros of the oscillation.*

Key Words: *non-linear differential equation.*

1. PROBLEM FORMULATION

Consider the following second order non-linear differential equation with respect to time t.

$$\frac{d^2u}{dt^2}+\frac{1}{2}\varepsilon\left|\frac{du}{dt}\right|\frac{du}{dt}+\omega^2\sin u=0 \tag{1}$$

We assume that the coefficients ε and ω are positive constants. If u is a solution, then $-u$ is also a solution. Therefore let us consider the oscillatory solution u: $[\,t_0, \infty\,]\to(-\pi,\pi)$ having the following boundary conditions

$$u(t_0)=0, \qquad \frac{du}{dt}(t_0)>0, \qquad u(\infty)=0. \tag{2}$$

This solution has infinity of zeros. Let be $(t_n\,,\, n\in \boldsymbol{N}\,)$ the increasing sequence of zeros of the solution or its derivative and $(a_n,\, n\in \boldsymbol{N}\,)$ the decreasing sequence of local amplitudes.

$$u(t_{2n})=0, \qquad \frac{du}{dt}(t_{2n+1})=0, \qquad a_n=\left|\,u(t_{2n+1})\,\right|. \tag{3}$$

We assume that the maximum value $a_0\in(0, \pi)$ is given.

2. ANALITICAL SOLUTION

In order to obtain the terms of these sequences it is suitable to use the following integral of equation

$$\frac{1}{2}\left(\frac{du}{dt}\right)^2 \exp\left(\varepsilon u\, sign\left(\frac{du}{dt}\right)\right)+\omega^2 V\left(u,\varepsilon\, sign\left(\frac{du}{dt}\right)\right)=c(t) \tag{4}$$

The function V has the expression

$$V(u,\varepsilon)=\frac{1}{1+\varepsilon^2}\left[1-(\cos u-\varepsilon\sin u)\exp(\varepsilon u)\right]\equiv V(-u,-\varepsilon) \tag{5}$$

The function $c(t)$ is piecewise function. Let c_0 be the constant of integration on interval (t_0,t_1). The missing initial value for the given equation results from the following formula:

$$\frac{1}{2}\left[\left(\frac{du}{dt}\right)(t_0)\right]^2=\omega^2 V(a_0,\varepsilon)=c_0 \tag{6}$$

The restriction of $c(t)$ function on the interval of decrease of solution from a_{2n} to $-a_{2n+1}$ is a constant of integration

$$\frac{du}{dt}<0,\quad c(t)=c_{2n+1}=V(a_{2n},-\varepsilon)=V(-a_{2n+1},-\varepsilon)=V(a_{2n+1},\varepsilon),\quad t_{4n+1}<t<t_{4n+3}. \tag{7a}$$

On the interval of the solution increase from $-a_{2n+1}$ to a_{2n+2} we can write

$$\frac{du}{dt}>0,\quad c(t)=c_{2n+2}=V(-a_{2n+1},\varepsilon)=V(a_{2n+2},\varepsilon)=V(a_{2n+1},-\varepsilon),\ t_{4n+3}<t<t_{4n+5}. \tag{7b}$$

Consequently, it results the implicit relation between local successive amplitudes.

$$0<a_0<\pi,\qquad V(a_{n+1},\varepsilon)=V(-a_n,\varepsilon),\qquad n=0,1,\cdots \tag{7}$$

Let x be a_n and let y be a_{n+1}. Substituting (5) in (7) we find the algebraic equation

$$(\cos y-\varepsilon\sin y)\exp(\varepsilon y)=(\cos x+\varepsilon\sin x)\exp(-\varepsilon x) \tag{8}$$

If we suppose that the dynamic coefficient ε and the given amplitude a_0 have small values, than we can use the following appropriate expressions for the local amplitude y.

$$y\cong Q(x)=x\left\{1-\frac{2}{3}\varepsilon x\left[1+\frac{1}{45}\left(3+22\varepsilon^2\right)x^2\right]+\frac{4}{9}\varepsilon^2x^2\left[1+\frac{1}{45}\left(9+26\varepsilon^2\right)x^2\right]\right\} \tag{9}$$

Otherwise, we introduce the constant δ, the function M and the sequence (y_m), namely

$$\delta=\operatorname{atan}\varepsilon,\qquad M(x,y)=\operatorname{acos}\left[\cos(\delta-x)\exp(-\varepsilon(x+y))\right]-\delta \tag{10}$$

$$y_0=x,\qquad y_{m+1}=M(x,y_m),\qquad m\in\mathbf{N}. \tag{11}$$

The solution y is the limiting value of this sequence.

If the number of iterations is J, then we use the following algorithm

$$q_0=a_0\qquad n=0..N\qquad m=0..J-1$$

$$q_{nJ+m+1}=M(q_{nJ},\ q_{nJ}\times \mathrm{if}(m=\mathbf{0},1,0)+q_{nJ+m}\times \mathrm{if}(m>\mathbf{0},1,0)) \tag{12}$$

$$n=0..N\qquad a_n=q_{nJ}\qquad c_n=V(a_n,\varepsilon).$$

The restrictions of equation (4) on intervals of successive zeros can be integrated using the method of separated variables.

It results the length of these intervals.

$$t_{2n+1} - t_{2n} = I(a_n, \varepsilon)/\omega, \qquad t_{2n+2} - t_{2n+1} = I(-a_n, \varepsilon)/\omega. \tag{13}$$

In this expression $I(x, \varepsilon)$ is an improper integral.

$$I(x,\varepsilon) \equiv \int_0^x \sqrt{\frac{\exp(\varepsilon u)}{2V(x,\varepsilon) - 2V(u,\varepsilon)}}\, du \tag{14}$$

3. RESULTS

In order to obtain the value of this integral it is first necessary to use series expansions. The V function is developed in series.

$$(1+\varepsilon^2)V(u,\varepsilon) = 1+\mathrm{Im}\{(\varepsilon - \mathrm{i})\exp[(\varepsilon + \mathrm{i})\, u]\} = \mathrm{Im}\{(\varepsilon - \mathrm{i})\Sigma_{n>0}\, (\varepsilon + \mathrm{i})^n u^n / n!\}$$

$$V(u,\varepsilon) = \frac{1}{2}\sum_{n\ge 2} c_n(\varepsilon)u^n, \qquad c_n(\varepsilon) = \frac{2}{n!}\mathrm{Im}\left\{(\varepsilon + \mathrm{i})^{n-1}\right\}, \quad c_n(-\varepsilon) = (-1)^n c_n(\varepsilon). \tag{15}$$

Let P be sixth order approximating Taylor polynomial

$$2P(u,\varepsilon) = u^2 \{ 1 + c_3 u + c_4 u^2 + c_5 u^3 + c_6 u^4 \}, \qquad V(u,\varepsilon) = P(u,\varepsilon) + \ldots, \tag{16}$$

where

$$c_2 = 1, \quad c_3 = \frac{2}{3}\varepsilon, \quad c_4 = \frac{1}{12}\left(3\varepsilon^2 - 1\right), \quad c_5 = \frac{1}{15}\varepsilon\left(\varepsilon^2 - 1\right), \quad c_6 = \frac{1}{360}\left(1 - 10\varepsilon^2 + 5\varepsilon^4\right).$$

The Q function given from (9) is an approximating Taylor polynomial of y useful solution of polynomial equation

$$P(y, \varepsilon) = P(-x, \varepsilon) \tag{17}$$

Using series expansions we deduce successively

$$\begin{aligned} 2V(x,\varepsilon) - 2V(u,\varepsilon) &= x^2 - u^2 + \sum_{n\ge 3} c_n(\varepsilon)\left(x^n - u^n\right) \\ 2V(x,\varepsilon) - 2V(vx,\varepsilon) &= x^2\left(1 - v^2\right)\left[1 + A(x,\varepsilon,v)\right] \\ A(x,\varepsilon,v) &= \frac{1}{1+v}\sum_{n\ge 3}\left\{c_n(\varepsilon)x^{n-2}\left(1 + \sum_{k=1}^{n-1} v^k\right)\right\}. \end{aligned} \tag{18}$$

If we make change of variables $u = x\, v = x \sin\sigma$, then the value of improper integral I can be calculated as integral of continuous function on the interval $(0,\pi/2)$.

$$I(x,\varepsilon) = \int_0^1 \sqrt{\frac{\exp(\varepsilon x v)}{\left(1-v^2\right)\left[1 + A(x,\varepsilon,v)\right]}}\, dv = \int_0^{\frac{\pi}{2}} \sqrt{\frac{\exp(\varepsilon x \sin\sigma)}{\left[1 + A(x,\varepsilon,\sin\sigma)\right]}}\, d\sigma \tag{19}$$

The last integral has the following approximating Taylor polynomial

$$I(x, \varepsilon) = (\pi/2)\, p(x, \varepsilon) - (\, \varepsilon\, x / 6\,)\, r(x, \varepsilon) + \ldots \tag{20}$$

$$p(x, \varepsilon) \equiv 1 + x^2\, (\, 1 + k_1\, \varepsilon^2) / 16 + x^4\, (\, 11/12 - k_2\, \varepsilon^2 - k_3\, \varepsilon^4\,) / 256 \tag{21}$$

$$r(x, \varepsilon) \equiv 1 + \; x^2\, [\, 1.7 - \pi / 2 - k_0\, \varepsilon^2\,] / 4 \tag{22}$$

$$\begin{aligned} k_0 &= \; \pi/3 - 1 - 11/270 \cong 0.006457 \\ k_1 &= 2[1\text{-}8/(3\pi)\,]/3 \cong 0.100782 \\ k_2 &= 2(19+11/135) / \pi - 12 - 1/9 \cong 0.036537 \\ k_3 &= (17+59/405) / \pi - 5 - 4/9 \cong 0.013194 \end{aligned} \tag{23}$$

The momentary "half-period" is given from (13).

$$t_{2n+2} - t_{2n} \; = \; [\, I(a_n, \varepsilon\,) + I(-a_n, \varepsilon\,)] / \omega \;\; \cong (\pi/\omega)\, p(a_n, \varepsilon\,) \tag{24}$$

The instantaneous "period" has the following expression

$$t_{2n+4} - t_{2n} \;\; \cong (\pi/\omega)\, [\, p(a_n, \varepsilon\,) + p(a_{n+1}\,, \varepsilon)\,]. \tag{25}$$

4. COMMENT

Suppose now that f is a real continuous odd function on space $\boldsymbol{R}$. Consider the second order differential equation

$$\frac{d^2u}{dt^2} + \frac{1}{2}\varepsilon\left|\frac{du}{dt}\right|\frac{du}{dt} + \omega^2 f(u) = 0 \tag{26}$$

The integral of this equation has the expression (4), but

$$V(u, \varepsilon) = \int_{(0,\, u)} \; f(u) \exp(\varepsilon u)\, du = V(-u, -\varepsilon).$$

For exemple

$$f(u) = u \quad \Rightarrow \quad V(u, \varepsilon) = [\, 1 + (\, \varepsilon\, u - 1\,) \exp\, (\varepsilon\, u)\,]\, \varepsilon^{-2}.$$

Between the local successive amplitudes x and y it results an algebraic equation.

$$y = M(x, y) = \{\, 1 - (\, \varepsilon\, x + 1\,) \exp[-\varepsilon(\, x+y\,)]\,\}/\varepsilon.$$

If we assume that $0 < \varepsilon < 1/4$ and $0 < x < 1/2$ than

$$y \cong Q_0(x) = x\, \{\, 1 - 2\varepsilon x(\, 1 + 22\varepsilon^2 x^2/45) / 3 + 4\varepsilon^2 x^2(\, 1 + 26\, \varepsilon^2 x^2/45) / 9\, \}.$$

The second order differential equation (26) can be recast and study as a first-order system [1], [2], [4].

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