Analytical solutions of the simplified Mathieu's equation

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DOI: 10.13111/2066-8201.2016.8.1.11

Received: 13 January 2016/ Accepted: 05 February 2016

Abstract: *Consider a second order differential linear periodic equation. The periodic coefficient is an approximation of the Mathieu's coefficient. This equation is recast as a first-*order *homogeneous system. For this system we obtain analytical solutions in an explicit form. The first solution is a periodic function. The second solution is a sum of two functions, the first is a continuous periodic function, but the second is an oscillating function with monotone linear increasing amplitude. We give a formula to directly compute the slope of this increase, without knowing the second numeric solution. The periodic term of the second solution may be computed directly. The coefficients of fundamental matrix of the system are analytical functions.*

Key Words: *linear differential equation with Mathieu coefficient, parametric resonance, periodic term of the solution.*

1. PROBLEM FORMULATION

Consider the following second order non-linear differential equation with respect to real dimensionless time *t*,

$$\frac{d^2 z}{dt^2} + Q(t)\, z = 0 \tag{1}$$

In what follows we assume the coefficient Q to be a real positive continuous periodic function with t real argument. The following function is a reasonable approximation of the Mathieu's coefficient [1], [2], [3].

$$Q = 1 + \frac{8q\,(1 - 2\cos 2t)}{1 + q - 2q\cos 2t}, \qquad -1/9 < q < 1/9. \tag{2}$$

The set of solutions is two-dimensional real space [4], [5], [6], [7]. The function x is a periodic solution.

$$x = \frac{\cos t - q\cos 3t}{1 - q}, \qquad \frac{d^2 x}{dt^2} + Q\, x = 0, \qquad x(0) = 1, \qquad \frac{dx}{dt}(0) = 0 \tag{3}$$

Let y be a second solution which satisfies,

$$\frac{dy^2}{dt^2}+Q\,y=0,\qquad y(0)=0,\qquad \frac{dy}{dt}(0)=1. \tag{4}$$

For the characteristic coefficient $\sigma(q)$, [8], the y_p function is a periodic term of the solution y.

$$y_p=y-\sigma\,t\,x. \tag{5}$$

The problem is to give the analytical formula for this coefficient.

2. EXPLICIT CHARACTERISTIC COEFFICIENT

We recast the equation (1) as a first-order system. The following system is obtained.

$$\frac{dz}{dt}=w,\qquad \frac{dw}{dt}=-\,Q\,w \tag{6}$$

Let u be the derivative of the periodic solution x. Except the case that q is zero, the derivative v of the oscillating solution y is not a periodic function.

$$\frac{dx}{dt}=u,\quad \frac{du}{dt}=-Q\,x,\qquad \frac{dy}{dt}=v,\quad \frac{dv}{dt}=-Q\,y\,. \tag{7}$$

The functions x, u, y, v have the following property

$$x\,v-u\,y\;=1 \tag{8}$$

Consequently, it results the expression of the fundamental matrix [3] for system (6).

$$\Phi=\begin{bmatrix}x & y\\ u & v\end{bmatrix},\qquad \Phi^{-1}=\begin{bmatrix}v & -\,y\\ -\,u & x\end{bmatrix}. \tag{9}$$

The y function is also a solution of fist-order linear equation

$$x\frac{d\,y}{dt}-u\,y=1,\qquad y(0)=0,\quad \frac{d\,y}{dt}(0)=1. \tag{10}$$

The periodic term y_p is the solution of the following first-order linear differential equation. Indeed, substituting the function y in the equation above, we find this linear equation (11).

$$y=y_p+\sigma\,t\,x,\;\; v_p\equiv\frac{dy_p}{dt},\;\;\Rightarrow\;\; v=v_p+\sigma\,t\,u+\sigma\,x,\;\; x\,(v_p+\sigma\,t\,u+\sigma\,x)-u\,(y_p+\sigma\,t\,u)=1$$

$$x\left(\frac{d\,y_p}{dt}+\sigma x\right)-u\,y_p\;=1,\qquad y_p(0)=0,\quad \frac{d\,y_p}{dt}(0)=1-\sigma. \tag{11}$$

where

$$x=\frac{\cos t}{1-q}\left(1+3q-4q\cos^2 t\right),\qquad u=\frac{dx}{dt}=-\frac{\sin t}{1-q}\left(1+3q-12q\cos^2 t\right) \tag{12}$$

In order to compute the periodic term we can also consider the fourth-order system (13).

$$\frac{dv}{dt} = \frac{dv_{\mathrm{p}}}{dt} + \sigma\, t\, \frac{du}{dt} + 2\sigma\, u, \qquad -Q\, y = \frac{dv_{\mathrm{p}}}{dt} + Q(-\sigma t x) + 2\sigma\, u$$

$$\begin{array}{llll} \dfrac{dx}{dt} = u, & \dfrac{du}{dt} = -Q\,x, & \dfrac{dy_{\mathrm{p}}}{dt} = v_{\mathrm{p}}, & \dfrac{dv_{\mathrm{p}}}{dt} = -Q\, y_{\mathrm{p}} - 2\sigma u \\ x(0) = 1, & u(0) = 0, & y_{\mathrm{p}}(0) = 0, & v_{\mathrm{p}}(0) = 1 - \sigma \end{array} \tag{13}$$

Let C be the variable constant of integration for the equation (11), when

$$y_{\mathrm{p}} = C\, x \quad \Rightarrow \quad \frac{dC}{dt} = \frac{1}{x^2} - \sigma \qquad C(0) = 0. \tag{14}$$

We shall consider α constant

$$\alpha = \sqrt{\frac{1-q}{1+3q}}$$

We introduce the equivalent expressions for the exact x solution,

$$x(t) = \frac{\cos t}{1-q}(1+3q)\left[1 + (\alpha^2 - 1)\cos^2 t\,\right] = \frac{\left(\alpha^2 + \tan^2 t\right)\cos t}{\alpha^2\left(1 + \tan^2 t\right)} \tag{15}$$

If we make the change of the variable of integration, then the function $h(s)$ is the solution of the following equation,

$$C(t) = h(s), \quad s = \tan t, \qquad \frac{dh}{ds} = \alpha^4 r(s) - \frac{\sigma}{1+s^2}, \quad r(s) = \left(\frac{1+s^2}{\alpha^2 + s^2}\right)^2 \tag{16}$$

The rational function r has the equivalent expression

$$r(s) = \left(1 + \frac{1-\alpha^2}{\alpha^2 + s^2}\right)^2 = 1 + 2\,\frac{1-\alpha^2}{\alpha^2+s^2} + \frac{(1-\alpha^2)^2}{2\alpha^2}\left[\frac{\alpha^2 - s^2}{(\alpha^2+s^2)^2} + \frac{1}{\alpha^2+s^2}\right] \tag{17}$$

Hence we have an appropriate formula for r function.

$$r(s) = 1 + \frac{\beta^*}{\alpha^2+s^2} + \gamma\,\frac{\alpha^2 - s^2}{(\alpha^2+s^2)}, \quad \beta^* = \frac{(1-\alpha^2)(1+3\alpha^2)}{2\alpha^2}, \quad \gamma = \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right)^2. \tag{18}$$

Consequently, it results the equation of the function h.

$$\frac{dh}{ds} = \alpha^4\,\frac{d}{ds}\left[s + \gamma\,\frac{s}{\alpha^2+s^2}\right] + \alpha^3\beta^*\,\frac{\alpha}{\alpha^2+s^2} - \frac{\sigma}{1+s^2}, \quad h(0) = 0 \tag{19}$$

In order to obtain the value of the coefficient σ it is necessary to impose the following integral condition.

$$\frac{2}{\pi}\int_0^\infty \left(\alpha^3\beta^*\,\frac{\alpha}{\alpha^2+s^2} - \frac{\sigma}{1+s^2}\right) ds = \alpha^3\,\beta^* - \sigma = 0 \tag{20}$$

The characteristic coefficient may be found using the formula

$$\sigma = \beta(\alpha) \equiv \frac{1}{2}\alpha\left(1-\alpha^2\right)\left(1+3\alpha^2\right) = \frac{8\alpha q}{(1+3q)^2} \tag{21}$$

when $$\alpha = \sqrt{\frac{1-q}{1+3q}}, \qquad \gamma = \frac{8q^2}{(1-q)(1+3q)}$$

3. SECOND EXPLICIT SOLUTION

Let Γ_α be the integral periodic function

$$\Gamma_\alpha(t) = \int_0^t \left(\frac{\alpha}{1+(\alpha^2-1)\cos^2 t} - \text{sign } \alpha \right) dt \tag{22}$$

The restriction of Γ on the interval $[0, \pi/2)$ is a known function. Integrating the given equation (14) above, we obtain the expression of C constant variable.

$$C(t) = C^*(t) + \beta\,\Gamma_\alpha(t), \qquad C^*(t) = \alpha^4\left[1+\frac{\gamma\cos^2 t}{1+(\alpha^2-1)\cos^2 t}\right]\tan t. \tag{23}$$

From (15) we can use the equivalent expression of the x exact solution

$$C^* x = \alpha^4\left[1+\frac{\gamma\cos^2 t}{1+(\alpha^2-1)\cos^2 t}\right]\tan t\,\frac{\cos t}{\alpha^2}\left[1+\left(\alpha^2-1\right)\cos^2 t\right] \tag{24}$$

Hence

$$y^* \equiv C^* x = \alpha^2\,[1+(\alpha^2-1)\cos^2 t+\gamma\cos^2 t\,]\sin t = \alpha^2\,[\,1+(\,\alpha^2-1+\gamma\,)\cos^2 t\,]\sin t$$

Consider the following identity

$$\alpha^2\,(\,\alpha^2-1+\gamma\,) \equiv 1-\alpha^2-\beta/\alpha$$

Therefore y^* can be written as

$$y^*(t) :\equiv [\,\alpha^2+(\,1-\alpha^2-\beta/\alpha\,)\cos^2 t\,]\sin t \tag{25}$$

From this it results the periodic term of the y solution.

$$y_p(t) = y^*(t) + \beta\,x(t)\,\Gamma_\alpha(t) \tag{26}$$

Therefore

$$y(t) = y^*(t) + \beta\,x(t)\,[\Gamma_\alpha(t) + t\,] \tag{27}$$

It is easy to find the derivative v^*

$$v^* = \frac{dy^*}{dt} = \left[\,1-\frac{\beta}{\alpha}+3\left(\alpha^2-1+\frac{\beta}{\alpha}\right)\sin^2 t\right]\cos t \tag{28}$$

Consequently, we deduce successively

$$v_p = \frac{d\,y_p}{dt} = v^* + \beta x \left[\frac{\alpha}{1 + (\alpha^2 - 1)\cos^2 t} - sign\alpha \right] + \beta\, u\, \Gamma_\alpha(t)$$

$$v_\mathrm{p} = v^* + (\beta/\alpha)\cos t - \beta\, x + \beta\, u\, \Gamma_\alpha(\mathrm{t}), \qquad v = v_\mathrm{p} + \beta\, t\, u \tag{29}$$

The characteristic matrix is the sum of a periodic matrix and an oscillating matrix

$$\Phi = \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} x & y_\mathrm{p} \\ u & v_\mathrm{p} + \beta\, x \end{bmatrix} + \beta\, t \begin{bmatrix} 0 & x \\ 0 & u \end{bmatrix} = \Phi^* + \beta\Gamma_\alpha(t) \begin{bmatrix} 0 & x \\ 0 & u \end{bmatrix} + \beta\, t \begin{bmatrix} 0 & x \\ 0 & u \end{bmatrix} \tag{30}$$

$$\Phi^* = \begin{bmatrix} x & y^* \\ u & v^* + (\beta/\alpha)\cos t \end{bmatrix} \tag{31}$$

The matrix Φ and Φ* are unimodular matrix. Finally, the solution of the system (6) has the following expression

$$\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = \Phi(t) \begin{bmatrix} z(0) \\ w(0) \end{bmatrix}$$

4. RESULTS

The coefficients of the two-order differential linear system are real continuous functions. The unique variable coefficient is a function $Q\,(q,\,t)$ in which q is real small parameter. In this particular case it is known an analytical explicit solution $(x,\, u)^\mathrm{T}$. In order to find the expression of the characteristic coefficient, [8], [9], it was imposed an explicit necessary improper integral condition. So it was found the explicit analytical formula for the characteristic coefficient of one particular two-order differential system. Consequently, it was found the second explicit solution $(y,v)^\mathrm{T}$.

The vector $(y{-}\beta tx, v{-}\beta tu)$ is a periodic vector. The components of the fundamental matrix have explicit expressions in which y^* and v^* are trigonometric polynomial functions and $\Gamma_\alpha(t)$ is a definite integral on the real interval $(0,\, t)$. For directly calculating the periodic term y_p, the fourth-order system (13) is useful, but if and only if the parameter σ is equal with the characteristic coefficient β, [10], [11].

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