An Expedite Approximate Algorithm for Calculating the Controls in the Longitudinal Maneuver

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Abstract: *This paper discusses an expedite approximate algorithm for obtaining the controls in the longitudinal maneuver of the aerospace vehicles. The equations of motion are written in terms of the flight path (the trajectory that is desired is also given in terms of flight path, that is the local radius of curvature is given as a function of flight path) and the controls required for following a desired trajectory obtained accordingly. A finite terms integration procedure is subsequently presented.*

Key Words: *Flight Dynamics Simulation, Controls, Longitudinal*

1. INTRODUCTION

The values of the controls in a maneuver are a key parameter in the design of an airplane. Various methods exist to estimate the response of the airplane to controls.

This paper discusses an expedite approximate algorithm for obtaining the controls in the longitudinal maneuver of the aerospace vehicles.

The equations of motion are written in terms of the flight path (the trajectory that is desired is also given in terms of flight path, that is the local radius of curvature is given as a function of flight path) and the controls required for following a desired trajectory are obtained accordingly. A finite terms integration procedure is subsequently presented.

The equations of motion of an airplane or rocket are presented in many references, for example References [1-3].

Nevertheless, the modeling of the aircraft's motion is still an area of open research, and a large number of papers have been dedicated to this topic.

A comprehensive review is far beyond the purpose of this paper; older references, like, for example, Ref. [4], as well as more recent papers, like, for example, Ref. [5] briefly illustrate some basic research in the field. In the symmetric flight of an airplane, the governing equations for the variables describing the longitudinal motion can be separated from the equations of the lateral motion. The equations of motion of the longitudinal channel, written using time as an independent variable are, Refs. [1-4]

$$\begin{cases} m\dfrac{dV}{dt} = T\cos\alpha - R - mg\sin\gamma \\ mV\dfrac{d\gamma}{dt} = T\sin\alpha + P - mg\sin\gamma \\ J\dfrac{d\omega}{dt} = M + M_{am} \\ \dot{z} = V\sin\gamma \\ \dot{x} = V\cos\gamma \end{cases} \tag{1}$$

Next,

$$\frac{1}{r} = \left|\frac{d\gamma}{ds}\right| = \left|\frac{d\gamma}{dt}\right|\frac{dt}{ds} = \frac{d\gamma}{dt}\cdot\frac{1}{V} \tag{2}$$

and, with the slope as an independent variable, the equations of motion for the longitudinal channel in the symmetric flight become

$$\begin{cases} \dfrac{dV}{d\gamma} = \dfrac{r}{mV}\left[T\cos\alpha - q^{*}SM^{2}C_{x} - G\sin\gamma\right] \\ \dfrac{d\omega}{d\gamma} = \dfrac{q^{*}SM^{2}c}{J\,V}r(C_{m0} + C_{m}^{\alpha}\alpha + C_{m}^{\beta_0}\beta_0 + C_{m}^{\omega}\,\omega + C_{m}^{\dot{\alpha}}\dot{\alpha}) \\ \dfrac{d\alpha}{d\gamma} = \omega\,\dfrac{r}{V} - 1 \\ \dfrac{dz}{d\gamma} = r\sin\gamma \\ \dfrac{dx}{d\gamma} = r\cos\gamma \\ \dfrac{dt}{d\gamma} = \dfrac{r}{V} \\ \alpha = \theta - \gamma \end{cases} \tag{3}$$

The elevator deflection can be readily obtained form the equation that describes the motion in transverse to the velocity vector (normal to the trajectory),

$$\beta_0 = \left[\frac{m\dfrac{V^2}{r} - T\sin\alpha + G\cos\gamma}{q^{*}SM^{2}} - (C_{z0} + C_{z}^{\alpha}\alpha + C_{z}^{\omega}\omega + C_{z}^{\dot{\alpha}}\dot{\alpha})\right]\frac{1}{C_{z}^{\beta_0}} \tag{4}$$

where

$$\begin{aligned} \frac{d\alpha}{dt} &= \dot{\alpha} = \omega - \frac{V}{r} \\ q^{*} &= \frac{\rho}{2}a^{2} \\ q &= \frac{\rho}{2}V^{2} = q^{*}M^{2} \end{aligned} \tag{5}$$

Equations (3)-(5) can be can be integrated numerically (using any numerical integration method, e.g. Runge-Kutta, Adams, etc) to study the motion along a given trajectory, described by the equation

$$r = r(\gamma) \tag{6}$$

where, once again, r is the local radius of curvature of the trajectory and γ is the local slope of the trajectory.

Numerical integrations are easily done and a wealth of literature has been dedicated to the topic, see, for example Ref. [6].

Many researchers have been directing their efforts to improve existing numerical algorithms, see, for example, Refs. [7] and [8], however, numerical integrations are significantly slower that the analytical calculations, hence, in spite of the tremendous progress of computers and numerical algorithms, a wide interest in analytical (or, at least semi-analytical) solutions still exists in all engineering fields, as illustrated, for example, in Refs. [9] and [10].

Numerical integration is widely used in flight dynamics, however, an analytical tool able to provide a fast estimate of the variables that describe the motion would be very beneficial, both for allowing a fast evaluation during early design stages, and for generating fast, yet sufficiently accurate data for more advanced analyses. Consequently, the finite terms integration of the aircraft's equations of motion has been continuously present in literature during the last decade, Refs. [4], [11] and [12].

The subsequent part of this paper presents a finite terms approximate integration of the longitudinal equations of motion for the airplane in symmetric flight. The algorithm is only valid for integration along a short arc of the trajectory, however, the time-step is much larger than the Δt required for regular numerical integration.

The algorithm was briefly presented in an older paper, Ref.[12], however, the main equations are repeated here for a better readability of the paper. Typos that unfortunately were not detected prior to publishing the older version of this paper, Ref.[12] are also corrected. A new numerical example is presented and discussed in greater details, to better illustrate the advantages of the method.

2. THE MATHEMATICAL MODEL

The left hand side of the equation of motion along the velocity

$$\frac{dV}{dt} = \frac{1}{m}\left[T\cos\alpha - qSC_x\right] - g\sin\gamma$$

can be written as

$$\frac{dV}{dt} = \frac{dV}{d\gamma}\cdot\frac{d\gamma}{dt} = \frac{dV}{d\gamma}\cdot V\cdot\frac{1}{r} = \frac{1}{2r}\cdot\frac{d(V^2)}{d\gamma} \tag{7}$$

so the equation of motion along the tangent to trajectory becomes

$$\frac{1}{2r}\frac{dV^2}{d\gamma} = \frac{1}{m}\left[T\cos\alpha - \frac{\rho SC_x}{2}V^2\right] - g\sin\gamma \tag{8}$$

Next, letting

$$\frac{T}{m} = a_t$$
$$\frac{\rho S C_x}{2m} = K_x \tag{9}$$

yields

$$\frac{d(V^2)}{d\gamma} + 2rK_x V^2 = 2ra_t - 2rg\sin\gamma \tag{10}$$

The coefficients of equation (10) varies slow with time, so, for a short trajectory arc, equation (10) can be integrated as a constant coefficients linear equation, hence the solution can be written as a sum of a homogenous solution and of a non-homogeneous solution,

$$V^2 = \left(V^2\right)_{\text{homogeneous}} + \left(V^2\right)_{particular}$$

After some algebra

$$V^2 = \left(V_I^2 - A\cos\gamma_I - B\sin\gamma_I - C\right)e^{-2rK_x(\gamma-\gamma_I)} + A\sin\gamma + B\cos\gamma + C \tag{11}$$

where

$$C = \frac{a_t}{K_x};\quad B = \frac{2rg}{1+(2rK_x)^2};\quad A = -\frac{4r^2 gK_x}{1+(2rK_x)^2}$$

while γ_I and V_I are the "initial conditions", that is the slope and the velocity at the beginning of the short trajectory arc over which the equation is integrated.

Next, the variation of the Cartesian coordinates over the same short trajectory segment, can be obtained from

$$\begin{cases} z = z_I + \int_{\gamma_I}^{\gamma} r\sin\gamma d\gamma \cong z_I + \dfrac{r\sin\gamma + r_I\sin\gamma_I}{2}(\gamma-\gamma_I) \\ x = x_I + \int_{\gamma_I}^{\gamma} r\cos\gamma d\gamma \cong x_I + \dfrac{r\cos\gamma + r_I\cos\gamma_I}{2}(\gamma-\gamma_I) \end{cases}$$

and the current time at the (r, γ) position is

$$t = t_I + \frac{1}{2}\left(\frac{r}{V} + \frac{r_I}{V_I}\right)(\gamma-\gamma_I)$$

The angular pitch speed is often small, and so the angular-related-pitch terms (at least over a short trajectory arc), so the trajectory can be seen as a series of "quasi-steady-states".

3. RESULTS AND DISCUSSIONS

The numerical example discussed below utilizes data for a hypothetical fighter whose aerodynamic coefficients are given in Table 1. The reference surface of the wing is 13.18 m^2 and the aerodynamic chord is 2.45 m. The mass of the airplane is 9874kg and it is assumed that during the maneuver it remains constant. It is further assumed that during the maneuver the thrust also remains constant at 65000 N.

The $r = r(\gamma)$ function that describes the trajectory is given in Fig.1, where the ordinate is the natural logarithm of *r*, in meters (the corresponding figure in Ref.[12] shows the decimal logarithm or the radius of curvature). The radius of curvature of the trajectory is very high at the beginning and the end of the maneuver; this corresponds to a rectilinear flight (*r* goes, theoretically, to infinity). Next, for the most part of the maneuver, the radius of curvature remains constant, so the trajectory that is sought is a circle.

The finite terms integration algorithm can be applied as an explicit scheme, as a predictor-corrector method, as well as an implicit algorithm. The results shown below were obtained using a simple explicit method, meant to illustrate method's capacity of providing fast evaluations of the required parameters, using relatively rough grids without running in numerical instability.

The $r = r(\gamma)$ curve was discretized using a 40 points mesh (unevenly distributed), for $0.2^0 \le \gamma \le 360^0$. The grid size is 1 deg for $\gamma \le 5^0$, next $\Delta\gamma = 5$ deg until γ reaches 75deg. The grid size is held constant at 20deg until $\gamma = 280^0$ and it is next reduced at 10deg until $\gamma = 350^0$. After that, the grid size is held at 2 deg until the end of the maneuver, $\gamma = 360^0$.

Table 1. Aerodynamic Properties

M	C_{x0}	$C_x^{\alpha^2}$	$C_x^{\beta^2}$	C_z^{α}	C_z^{β}	C_m^{α}	C_m^{β}
0.4	0.0267	5.0317	0.9844	8.0317	1.9786	- 1.9815	- 3.4472
0.6	0.0268	5.3437	1.0262	8.3437	2.0627	- 2.1578	- 3.5943
0.8	0.0269	5.8662	1.1163	8.8662	2.2326	- 2.4084	- 3.8921
0.9	0.0302	6.2016	1.1581	9.2016	2.3162	- 2.4932	- 4.0407
1.0	0.0444	7.7422	1.3737	10.7422	2.7473	- 3.1000	- 4.8079
1.1	0.0736	8.5505	1.4573	11.5505	2.9146	- 3.3549	- 5.1137
1.2	0.0736	7.8253	1.3737	10.8253	2.7473	- 3.3222	- 4.9237
1.3	0.0714	7.4830	1.3610	10.4830	2.5991	- 3.8270	- 4.7863
1.4	0.0694	6.3469	1.1420	9.3469	2.2726	- 3.4826	- 4.2046
1.6	0.0693	4.5706	0.9008	7.5706	1.7475	- 2.8550	- 3.2631
2	0.0473	2.6207	0.6531	5.6207	1.2081	- 1.7648	- 2.2649

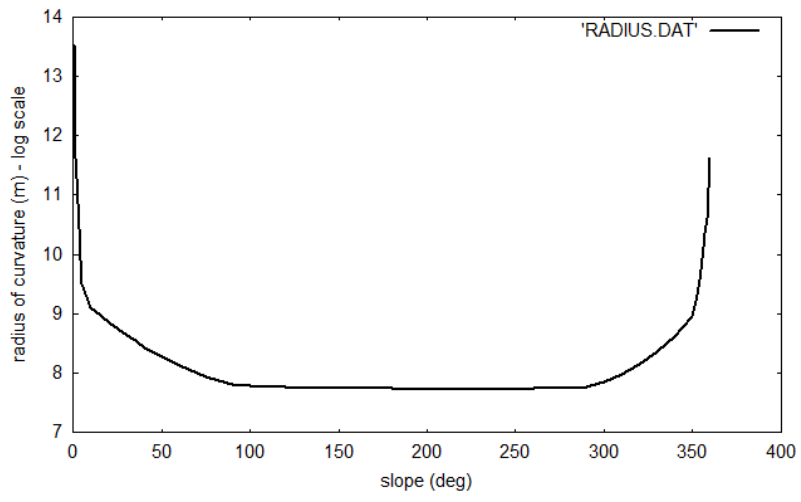

Fig.1 Radius of curvature versus slope

Figure 2 shows the trajectory of the airplane, as provided by the algorithm described above.

The coefficients of the equations were obtained using a simple explicit algorithm that utilizes the values of the aerodynamic date at the beginning of the “short arc”. The orbit is a closed loop, as desired.

Figure 3 shows the variation of the angle of attack during the vertical loop that is calculated.

The plot has a few sharp corners, unlikely to appear in real flight, that are due to quite coarse mesh utilized.

Nevertheless, the algorithm can provide a fast estimate of the parameter without running into numerical instabilities and the values that .are obtained are realistic.

Figure 4 presents the elevator deflection during the maneuver. Again, the graph shows a few sharp corners, unlikely to appear in real flight, that are due to quite coarse mesh utilized.

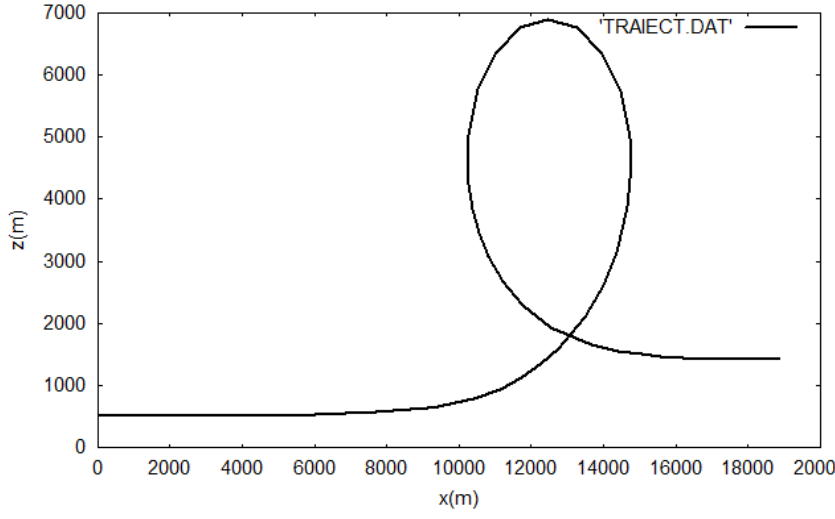

Fig. 2 Trajectory

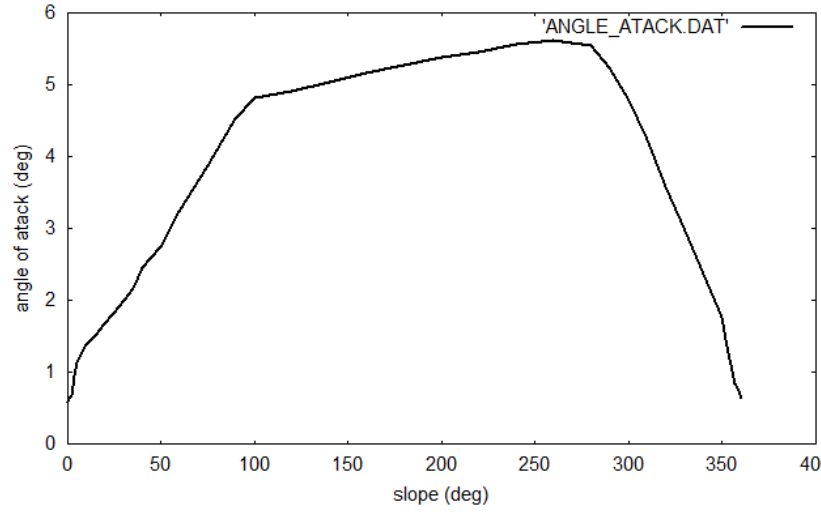

Fig. 3 The variation of the angle of attack versus slope

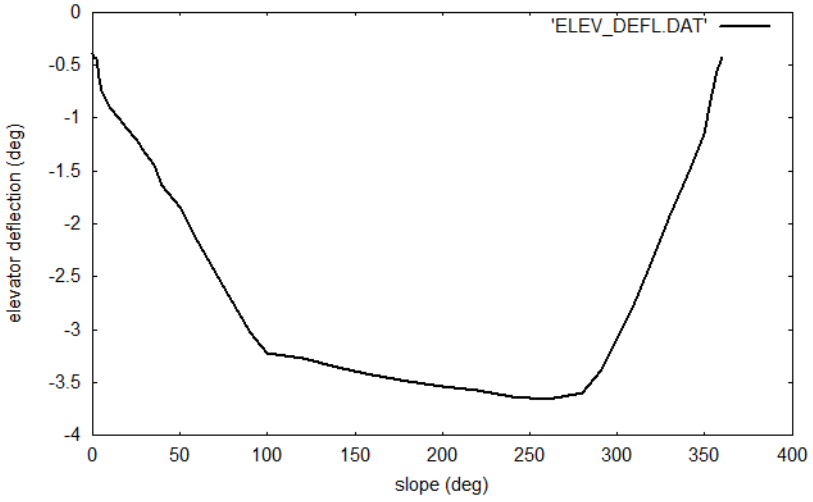

Fig. 4 The elevator deflection versus slope

4. CONCLUSIONS

This paper discusses an expedite approximate algorithm for obtaining the controls in the symmetrical longitudinal maneuver of airplanes and rockets. The equations of motion are written in terms of the flight path and the controls required for following a desired trajectory are also expressed as functions of γ. A finite terms integration procedure is subsequently discussed and a numerical example is presented, that shows that, even on a course grid, and using a simple explicit scheme, the method is capable to provide fast estimates of the flight parameters without running into numerical instability.

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