Dichotomic Structure of DAEs Solutions for the Aircraft Control

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Abstract

The paper has its roots in earlier studies focused on DAEs solutions, for the aircraft flight control and intends to be a synthesis of them. The main goal is to structure the solution for the control laws so as to derive its components, which control any significant mechanical phenomenon for the controlled flight. The basic method used becomes from a unified manner of finding the solution of DAEs using a rigorous guideline stated as "necessary and sufficient condition" in an algebraic equation form that is used in an algorithmic procedure and for statement of the equations, which emphasises the dichotomic structure. The viewpoint considers an extended DAE system, including the differential equations of control variables, that allows to formulate this question as an inverse problem and to regard the algebraic equation, for constraints, as a singular implicit solution of the differential subsystem. Stating the necessary and sufficient condition for an implicit equation be a singular implicit solution of the extended differential system, we use it to approach the solution for flight control and for its dichotomic structure with additive components.

1 Introduction

The usual configuration of an aircraft uses a control system of aerodynamic type, with aerodynamic devices such as rudder, elevator, and aileron. This type of controls provides mainly (aerodynamic) couples around the mass centre. The differential model for the controlled flight includes the moment equations which have the following structure, [3]:

$$\vec{J} \cdot \dot{\vec{\omega}} = \vec{M}_P + \vec{M}_c^{am} + \vec{M}_C + \vec{M}_{I0} + \vec{M}_{Ic} \qquad (1)$$

with low indices meaning: P – main effect; c – complementary effect; C – control effect; I – inertia effect; 0 – basic effect; am – damping effect. To avoid specific or very specific skills of the pilot devoted to a given aircraft and to simplify the manner to control the aircraft, regarding only to the leading effects, it is necessary to derive from (1) a simplified model, with only the general leading terms of every aircraft. The only way to approach this goal is to use the control system with a devoted subsystem to provide additional terms by side of the pilot controls. A natural, and perhaps the most easy, way is to consider the control variable (the deflection of aerodynamic devices), denoted by δ, with additive structure; each additive component by side of the main one in this structure is designated to avoid terms that are outside the general leading effects in (1). But current practice of airplane piloting and mathematical reasons show that the flight is not stable around the desired flight evolution, considering analytical properties of the solutions for the differential model of the controlled flight using such structure of the control variable. That is why this paper states necessary and sufficient conditions to solve this unpleasant fact.

Tacking into account that $\vec{M}_C$ has a linear dependence on the control variable, as presented in [1], [2], [3], we will consider as a generic differential model for the controlled flight:

$$\begin{aligned} \dot{x} &= f_0(x) + f_1(x) + h(x,u) \\ 0 &= w(x,u) \end{aligned} \qquad (2)$$

with the starting conditions:

$$\begin{aligned} x(t_0) &= x_0 \\ 0 &= w(x_0, u(t_0)) \end{aligned} \qquad (3)$$

and w (.) the implicit function for the constraints of the desired flight evolution. The term $f_1(x)$ represents the effect to be avoided by an additional component, u_1, of the control variable, u, that is:

$$\begin{aligned} u &= u_0 + u_1 \\ 0 &= f_1(x) + h(x) \cdot u_1 \end{aligned} \qquad (4)$$

To solve this mathematical problem, preserving the desired flight evolution, we use previous results derived in [3], [7], [8], [11], [12], and [13].

2 Basic method for solving DAEs

The method to solve the problem (1)-(4) is based on the idea of [4], developed in [3] and used in [13] as a general tool for systems control. A first application to the aerospace domain is proposed in [11] and in [12] as a slight extension.

As we remark, the problem just formulated may be regarded as equivalent to the following one:

$$\begin{aligned} \dot{\sigma} &= F(\sigma,u) \\ 0 &= W(\sigma,u) \end{aligned} \tag{5}$$

collecting the constraints and the differential modelling function in a more abstract form, without regarding at their structure. This is the semiexplicit differential algebraic equation, denoted in the sequel DAE, with:

$\sigma\in\Sigma$ differential variables, $u\in U$ algebraic variables and $\Sigma \subseteq \mathbf{R}^n$, $\mathbf{U} \subseteq \mathbf{R}^m$ connected open sets; $(m, n) \in \mathbf{N}^2$; $m \le n$

$rank\,(F_\sigma) = n$, $F \subseteq C^\nu(\Sigma\times\mathbf{U},\mathbf{R}^n)$ having finite values

$rank\,([W_\sigma \mid W_u]) = q \le m$, $W \subseteq C^{\nu+1}(\Sigma\times\mathbf{U},\mathbf{R}^q)$ and has finite values

$\nu \ge 2 \in \mathbf{N}$ is adequate for the necessary smoothness of F and W

having: $x=[\sigma\ \ u]^T \subseteq \mathbf{X} \subseteq \mathbf{R}^{n+m}$ the state variable, $\dot{x} \subseteq \mathbf{Y} \subseteq \mathbf{R}^{n+m}$, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{P}$ open sets; $(m, n, p) \in \mathbf{N}^3$, $G \subseteq C^\nu(\mathbf{Y}\times\mathbf{X}\times\mathbf{P},\mathbf{R}^{n+m})$, $d \in \mathbf{N}$ adequate for the necessary smoothness of G, $rank(G_x)=n+m$, $rank(G_{\dot{x}}) = n$.

Following the control theory view point, the algebraic subsystem represents for the differential subsystem the requirements regarding the physical system evolution, and by consequence W will be named *"restriction function"*. Regarding the algebraic variables as parameters (having values that varies along the solutions of (5)), the manifold represented by algebraic equations of (5) is in fact an integral manifold as a singular implicit solution of the differential equations of (5), [4]. It is not a particular implicit solution derived from a general implicit solution, $C = W(\sigma,u)$, by adequate values of constants C, because this situation becomes a degenerate one regarding the significance of algebraic variables, u.

Tacking into account that from any point of a singular solution comes at least one regular solution, derived from a general implicit solution, it becomes almost evident that at each point (σ, u) the σ values are directed to those corresponding for a regular solution, when it is performed a numerical integration or it is implemented the direct algebraic solution of u to the physical controlled object modeled by (5). Such attempts to solve DAEs were tried in aeronautics for the so called "constrained flight". The results were not highly satisfactory, being very clear the "drift-off" phenomena stated for many particular DAEs. Hence it becomes that the u values derived from the algebraic subsystem (5) must be in accordance with the dynamic behaviour of the restriction function to assure its zero value stability and with a necessary and sufficient requirement for W be a singular implicit solution of (5).

The recent 20 years many work has done to stabilize the DAE solutions obtained using numerical algorithms. All of them, in a way or another, transform the DAE (5) to another mathematical system that has the manifold identified by restriction function as an attractor for its solutions. If these stabilisation procedures would be in a consistent (reciprocal) relation with a necessary and sufficient requirement for W be a singular implicit solution of (5), then all will must become equivalent each other. Considering the observations made upon singular solution and the detailed explanations presented in [4], we will state such a requirement.

From (5) we state the following inverse problem, [4]:

"Find *the closing function U so that the differential system*

$$\begin{cases} \dot{\sigma} = F(\sigma,u) \\ \dot{u} = U(\sigma,u) \end{cases} \tag{DS}$$

with the initial state variable values $\sigma(t_0) = \sigma_0$, $u(t_0) = u_0$

has as stable singular implicit solution

$$0 = W(\sigma,u) \text{ ''} \tag{SS}$$

It must be pointed out that the starting consistency condition $0 = W(\sigma(t_0),u(t_0))$ is implicitly supposed accomplished and (SS) equation has solutions over an open set of $\mathbf{R}^{n+m}$, at least.

Among other methods, the specific way of this approach of finding the solution to this inverse problem is based on the following:

Fundamental lemma. The necessary and sufficient condition for W to be a singular implicit solution of (DS) is:

$$W(\sigma,u) = 0 \ \Leftrightarrow\ W_\sigma\cdot F(\sigma,u) + W_u\cdot U(\sigma,u) \tag{NSC}$$

according to [13]. It now becomes clear that the values $W(\sigma,u)$ and $W_\sigma\cdot F(\sigma,u) + W_u\cdot U(\sigma,u)$ appear to be related if the function (W, $\Sigma\times U$, R^q) really defines an invariant integral manifold for (DS). Considering Lemma NSC, the relationship is of a function type in the point ($W(\sigma,u) = 0$, $W_\sigma\cdot F(\sigma,u) + W_u\cdot U(\sigma,u) = 0$), more over being of bijective type. Taking into account the meaning of control variables for reaching the singular solution, in an asymptotic way at least, we extend the function type of this relationship all over the values $W(\sigma,u)$, in order to have a deterministic relation from these values toward the time rate values of function (W, Σ×U, R^q). So, we are able to define the following function:

Φ:*Im(W)* $\times$ *Dom(W)* $\rightarrow$ R^q

having the values for (w, σ , u) ∈ Im(W)×Dom(W) defined by the analytical formula:

$$\Phi(W,\sigma,u) = W_\sigma \cdot F(\sigma,u) + W_u \cdot U(\sigma,u) \quad \text{(RE)}$$

with the following features:

$\forall(\sigma,u) \in \mathsf{R}^{n+m} \rightarrow 0 \equiv \Phi(0,\sigma,u)$ and

$\forall(\sigma,u) \in \mathsf{R}^{n+m} \rightarrow 0 \neq \Phi(W \neq 0,\sigma,u)$ (f_1)

$\Phi \in C^{\nu}(\mathsf{R}^{m+n+m}, \mathsf{R}^q)$ (f_2)

This way, the necessary and sufficient condition (NSC) is expressed by the relation (RE), with (f_1), (f_2) defining features.

Some complementary results regarding the properties of the function Φ are given in [7] and [13].

The previous results enable us to state a procedure to find the solution of the inverse problem formulated as equivalent to (5). If *rank*(*Wu*)=*m* , (RE) determines the differential modeling function *U*. When *rank*(*Wu*)=q<*m* it is necessary to find out a new/ supplementary adequate restriction function following the same requirements and rules used to obtain (RE). Because (RE) is, in fact the essential necessary and sufficient condition upon *W* to be singular implicit solution for (DS), it becomes in a very natural way the new restriction function to continue the algorithm for finding all the components of *U* function. So:

$$0 = W_\sigma \cdot F(\sigma,u) + W_u \cdot U(\sigma,u) - \Phi(W,\sigma,u) \equiv \widetilde{W}(W,\sigma,u) \quad \text{(WS)}$$

and not $W_\sigma \cdot F(\sigma,u) + W_u \cdot U(\sigma,u) \equiv \widetilde{W}(\sigma,u)$ as used by other authors. This way, the intrinsic characteristics of (RE) and the stability of *W* function are preserved. Also (WS) provide a sequential algorithm at each step using the fundamental lemma for a new inverse problem. The main truth that states this algorithmic procedure to find all the components of differential modelling function for control variable *u* is, as in [[3], [7], [8], and 13]:

Recurrence lemma. If *W* is the current restriction function, and $\widetilde{W}$ is the new one defined by (WS) with Φ the current perturbation function, then (RE) constructed for $\widetilde{W}$ maintain the (RE) for *W* with stable and asymptotic behaviour around zero value.

The steps of a sequential algorithm, to determine the differential modelling function for every component of *u* , are:

s_1) Construct/ build the restriction equation (RE) for (DS)-(SS) system, denote $\tilde{u} = u$ and $\widetilde{W}(.) = W(.)$

s_2) Determine the $rank(\widetilde{W}_{\tilde{u}})$

s_3) Decide the differential determination of *u* components:

d_1) If $rank(\widetilde{W}_{\tilde{u}}) = m$ and all the components of *u* (at any order time derivatives) are present in a rank matrix then $\widetilde{U}$ function is well determined and the algorithm is stopped.

d_2) If $rank(\widetilde{W}_{\tilde{u}}) = q < m$ then we need to continue the building of new restriction function, as (WS), and the derived restriction equation (RE_1). The procedure return to (s_2).

3 Dichotomic structure of control laws

Before solving the problem (2), (3), (4) we will suppose that every function in (2), has all the adequate properties for the existence and oneness of solutions and for the algorithm just presented to solve the inverse problem equivalent to the DAE problem. We will try to use the previous results in order to simplify the DAE (2) considering the new constraint (4) and so:

$$f_0(x) + f_1(x) + h(x,u_0 + u_1) \equiv f_0(x) + h(x,u_0) \quad (6)$$

The problem (2) becomes:

$$\begin{aligned} \dot{x} &= f_0(x) + h_0(x) \cdot u_0 \\ 0 &= w(x,u_0 + u_1) \\ 0 &\equiv f_1(x) + h_0(x) \cdot u_1 \end{aligned} \quad (2_0)$$

with starting conditions:

$$\begin{aligned} x(t_0) &= x_0 \\ 0 &= w(x_0,u_0(t_0) + u_1(t_0)) \\ 0 &= f_1(x_0) + h_0(x_0) \cdot u_1(t_0) \end{aligned} \quad (3_0)$$

The identity (4) is, in fact, a new restriction function denoted:

$$w_1(x,u_1) \equiv f_1(x) + h_0(x) \cdot u_1$$

Thus, (2_0) is even a problem of (7) type but with a simplified differential modeling function and an extended restriction function. The solution of this problem is found using the method just presented above, but with the following goals:

g_1) to derive a reduced model for only the u_0

g_2) to overcome the existence of f_1 (.), as a parasite term of the primary model, without any lose of control laws accuracy

g_3) the control variable u_1 must be determined completely independent of the reduced model of (g_1); for this model of u_1, the state and control variable of the reduced model are external variables.

These goals, once attained, sustain the name "dichotomic structure of control laws". The (g_1) is reached assuming that:

$$u_1 \equiv (h_0^{-1} \circ f_1)(x, y) \tag{7}$$

and so deriving the reduced model:

$$\dot{x} = f_0(x) + h_0(x) \cdot u_0 \qquad (PR_0)$$
$$0 = w(x, u_0 + (h_0^{-1} \circ f_1)(x)) \equiv w_0(x, u_0)$$

The closing function for this problem is denoted $U_0(.)$ and (PR_0) becomes an ODE:

$$\dot{x} = f_0(x) + h_0(x) \cdot u_0$$
$$\dot{u}_0 = U_0(x, u_0) \qquad (RI_0)$$
$$x(t_0) = x_0$$

The solution of this system, $(\tilde{x}, \tilde{u}_0)$, is not in any case a stable solution for

$$\dot{x} = f_0(x) + h_0(x)u_0 + f_1(x) + h_0(x)(h \circ f_1)(x) \tag{8}$$
$$x(t_0) = x_0$$

because the restriction

$$w_1(x, u_1) = 0 \tag{9}$$

was not in any case a stable condition for (PR_0) and (RI_0). Tacking into account this restriction and the fundamental lemma, we consider the DAE model:

$$\dot{x} = f_0(x) + h_0(x) \cdot u_0$$
$$\dot{u}_0 = U_0(x, u_0) \qquad (PR_1)$$
$$w_1(x, u_1) = 0$$

Now, we have reached the next two goals, (g_2), (g_3), because $f_1(.)$ is not explicitly present in (PR_1) and u_1 is determined independently of (RI_0), where it is not involved. So, the closing function for u_1, denoted by $U_1(.)$, will depend on (x, u_0) only as external variables:

$$\dot{u}_1 = U_1(u_1; x, u_0)$$
$$w_1(x_0, u_1(t_0)) = 0 \qquad (RI_1)$$

The (8) model becomes:

$$\dot{x} = f_0(x) + h_0(x) \cdot u_0 + f_1(x) + h_0(x) \cdot u_1$$
$$\dot{u}_0 = U_0(x, u_0) \tag{10}$$
$$\dot{u}_1 = U_1(x, u_0, u_1)$$

for which the solution $(\tilde{x}, \tilde{u}_0)$ becomes stable.

The previous results prove the following

Theorem of additive control. The problem (2) with started condition (3) has a unique additive decomposition of control variable $u = u_0 + u_1$, whose components are the solutions of problems (PR_0) with starting conditions:

$$x(t_0) = x_0$$
$$0 = w_0(x_0, u(t_0))$$

and (PR1) with starting conditions:

$$x(t_0) = x_0$$
$$0 = w_0(x_0 u_0(t_0))$$
$$0 = f_1(x_0) + h_0(x_0) \cdot u_1(t_0)$$

Proof. The decomposition is valid by its own building. The oneness is derived considering the oneness of solutions of each differential system (RI_0) and (RI_1).

It is of a significant meaning to remark that (PR_1) maintains the same structure with a new additive component of u for a supplementary term f_2 in (2) and (RI_1) is not affected, too. Thus, we state:

Theorem of dichotomy. If the modeling function f presents the structure:

$$f(x, u) \equiv f_0(x) + f_1(x) + f_2(x) + h_0(x) \cdot u$$

then, the inverse problem (2) with starting condition (3) has an oneness additive structure $u = u_0 + u_1 + u_2$ whose components are the solutions of the three independent inverse problems:

- (PR_0) with starting conditions
  $x(t_0) = x_0 \ , 0 = w_0(x_0, u_0)$
- (PR_1) with starting conditions
  $x(t_0) = x_0 \ , 0 = w_1(x_0, u_0)$
- (PR_2) with starting conditions
  $x(t_0) = x_0 \ , 0 = w_2(x_0, u_0)$

Proof. Because the two restriction functions w_1 and w_2 does not interfere when we derive the differential modeling function for u_1 and u_2 the two problems (PR_1) and (PR_2) may be solved separately. On the other side, the problem (PR_0) is solved separately from the other two ones. The oneness of the control variables comes from the oneness of a solution for a differential equation.

These two theorems allow finding side by side the additive components of the control variables in order to counterbalance the undesired terms in (2). That fact is very important for a control system from its building and its analysis, considering each term set off in a separate manner.

4 Conclusions

The main concluding lines of this paper are:

1. The primary DAE may be equivalent to an ODE using the restriction equation (RE) and the recurrence lemma.
2. Using such a method it is possible to simplify the DAEs for control systems considering an additive structure of control variables. These components have dichotomic features.
3. The sequential algorithm seems to be valid for analysing the over determined DAE systems.

4. The algorithms provide stable solutions for state and control variables

APPENDIX

Some remarks for DAEs solutions

The systems of differential equations and algebraic equations have brought into evidence that their solutions imply/ require the consistency of differential solution provided by the extended differential system (derived from the original one and the algebraic equations) with the algebraic subsystem all over the time evolution. Analytical or numerical method to find such solutions shows some of their main problems: ill conditioning, stiffness, stability, consistency.

The most general form of DAE is as an implicit equation and, without loss of generality, we may consider the autonomous problem $G(\dot{x}, x, \lambda) = 0$, which may be put into the semiexplicit differential algebraic equation, denoted in the also DAE.

For DAEs the view point of control theory is to consider the algebraic variable as the control variable of the differential subsystem (5). The control variable must be determined so to steer the solution of the differential subsystem inside or, at least, in a very narrow neighborhood of the integral manifold identified by the implicit equation of restriction function. This way, become almost natural to consider for control variable a differential modelling function *U* (*σ, u*) to represent its necessary dynamic behaviour in order to achieve this desired goal. This function is named "*closing function*" according to [4].

The Φ function is a fitting indicator for *W* to be a singular implicit solution of (DS) and represent the time rate of *W* along the solution of (DS).

$$\left.\frac{DW(.)}{Dt}\right|_{(DS)} = \Phi(W(.), \sigma, u)$$

Hence one can prescribe the structure of this function and thus defining the dynamics of nonzero values of *W*. Tacking into account these meanings, the Φ function may be named "*perturbation function*". The basic requirement for the perturbation function is to provide a stable and asymptotic behaviour of *W* around the zero value:

$$\lim_{t\to\infty} W(\sigma, u)\Big|_{(DS)} = 0$$

Now the (RE) definition formula, having the perturbation function just defined, becomes an equation, named "*restriction equation*" that is equivalent to (NSC) under the features (f_1), (f_2). So, we get the:

(RE) corollary. The (RE) relation with perturbation function having the properties (f_1), (f_2) is the necessary and sufficient condition that restriction function define a singular implicit solution of (DS)

The lemma concerning the structure of the perturbation function for very small deviations of *W* provides the basis to use a linear structure like this:

$\Phi(W, \sigma, u) \equiv \Phi(W) = S \cdot W$

with S a stability matrix for the differential system:

$$\dot{W} = S \cdot W \qquad \text{(LC)}$$

used for the first establishing and proving of results regarding the method to solve DAEs and local properties of singular solutions or regular solutions that approach the singular solutions.

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