An alternative flight control system for an unmanned aircraft whose flight control system fails during a longitudinal flight with constant forward velocity

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Abstract: *In this paper we build up a flight control system for an unmanned aircraft whose flight control system fails during a longitudinal flight with constant forward velocity. This task is accomplished using only the system of differential equations, which governs the movement of the aircraft around its center of mass. Numerical simulation is given.*

Key Words: Oscillations; Unmanned aircraft; Flight control; Longitudinal flight with constant forward velocity.

1. INTRODUCTION

The simplified system of differential equations, which governs the motion around the center of mass in a longitudinal flight with constant velocity of an unmanned aircraft, whose flight control system fails during the flight, is given by:

$$\begin{cases} \overset{\circ}{\alpha} = z_\alpha\,\alpha + q + \dfrac{g}{V}\cos\,\theta + z_{\delta_e}\,\delta_e \\ \overset{\circ}{q} = m_\alpha\,\alpha + m_q\,q + \dfrac{g}{V}\left(\overline{m}_{\dot{\alpha}}\cos\,\theta - \dfrac{c_2}{a}\cdot a_2\sin\,\theta\right) + m_{\delta_e}\,\delta_e \\ \overset{\circ}{\theta} = q \end{cases} \tag{1}$$

In this system the state parameters are: angle of attack α, pitch rate q and Euler pitch angle θ.

The control parameter is the elevator deflection δ_e. V is the forward velocity of the aircraft, considered constant and g is the gravitational acceleration.

The aerodynamical data are $z_\alpha, z_{\delta_e}, m_\alpha, m_q, \overline{m}_{\dot{\alpha}}, c_2, a, a_2, m_{\delta_e}$ and their numerical values in a specific case are given in Section 3 (Table 1). In the system (1) δ_e is a parameter.

Till the flight control system is in operation, the elevator deflection δ_e is a linear function of the state parameters α, q, θ, given by:

$$\delta_e = k_\alpha \alpha + k_q q + k_\theta \theta \tag{2}$$

In (2) k_α, k_q, k_θ are constants and for the considered numerical case their values are given in Section 3 (Table 1).

When the flight control system fails, δ_e becomes constant.

In [1] it was shown that $(\alpha, q, \theta)^T$ is an equilibrium state of the system (1), corresponding to δ_e, if and only if α is a solution of the equation:

$$A\alpha^2 + B\delta_e \alpha + C\delta_e^{\,2} + D = 0 \tag{3}$$

q is equal to zero and θ is a solution of the equation:

$$\cos\theta = -\frac{V}{g}\left[z_\alpha \alpha + z_{\delta_e} \delta_e\right] \tag{4}$$

where A, B, C, D are given by:

$$\begin{aligned}
A &= \left(m_\alpha - \overline{m}_{\dot\alpha} z_\alpha\right)^2 + \frac{c_2^{\,2}}{a^2} a_2^{\,2} z_\alpha^{\,2} \\
B &= 2\left(m_\alpha - \overline{m}_{\dot\alpha} z_\alpha\right)\left(m_{\delta_e} - \overline{m}_{\dot\alpha} z_{\delta_e}\right) + 2\frac{c_2^{\,2}}{a^2} a_2^{\,2} z_\alpha z_{\delta_e} \\
C &= \left(m_{\delta_e} - \overline{m}_{\dot\alpha} z_{\delta_e}\right)^2 + \frac{c_2^{\,2}}{a^2} a_2^{\,2} z_{\delta_e}^{\,2} \\
D &= -\frac{g^2}{V^2}\frac{c_2^{\,2}}{a^2} a_2^{\,2}
\end{aligned} \tag{5}$$

Equation (3) has real solutions if and only if δ_e satisfies:

$$\left|\delta_e\right| \le \sqrt{\frac{4AD}{B^2 - 4AC}} \tag{6}$$

For $z_\alpha < 0$ and a real solution α of the Eq. (3), it was shown that the Eq. (4) has a solution if and only if for α the following inequality holds:

$$\frac{1}{z_\alpha}\left[\frac{g}{V} - z_{\delta_e} \delta_e\right] \le \alpha \le -\frac{1}{z_\alpha}\left[\frac{g}{V} + z_{\delta_e} \delta_e\right] \tag{7}$$

Since for $\delta_e = 0$ the solutions of Eq.(3) are:

$$\alpha = \pm\sqrt{\frac{\dfrac{g^2}{V^2}\dfrac{c_2^{\,2}}{a^2} a_2^{\,2}}{\dfrac{c_2^{\,2}}{a^2} a_2^{\,2} z_\alpha^{\,2} + \left(m_\alpha - \overline{m}_{\dot\alpha} z_\alpha\right)^2}} \tag{8}$$

and both verify (7) for $z_\alpha < 0$, the values $\underline{\delta_e}$, $\overline{\delta_e}$, defined by:

$$\begin{aligned} \underline{\delta_e} &= \inf\left\{\delta_e \,\middle|\, \delta_e < 0,\ \exists\ a\ real\ solution\ of\ Eq.(1.3)\ for\ which\ (1.7)\ holds\right\} \\ \overline{\delta_e} &= \sup\left\{\delta_e \,\middle|\, \delta_e > 0,\ \exists\ a\ real\ solution\ of\ Eq.(1.3)\ for\ which\ (1.7)\ holds\right\} \end{aligned} \tag{9}$$

and the closed interval $I = [\underline{\delta_e}, \overline{\delta_e}]$ were considered.

It was shown [1] that the following statements hold:

a.) If $\delta_e \in I$, then for the system (1) there exists a countable infinity of equilibriums corresponding to δ_e, namely for any $n \in Z$

$$\left(\begin{aligned} &\alpha_{1,2} = -\frac{B}{2A}\delta_e \pm \frac{1}{2A}\cdot\sqrt{B^2\delta_e^{\,2} - 4A\left(C\delta_e^{\,2} + D\right)}; \quad q = 0; \\ &\theta_{1,2} = 2\pi n \pm \arccos\left[\frac{V}{g}\left(-z_\alpha\,\alpha_n^{\,\pm} - z_{\delta_e}\,\delta_e\right)\right] \end{aligned}\right) \tag{10}$$

b.) If $\delta_e \in \partial I = \left\{\underline{\delta_e}, \overline{\delta_e}\right\}$, then the equilibriums corresponding to δ_e be saddle-node bifurcation points.

c.) If $\delta_e \notin I$, then for the system (1) there are no equilibriums corresponding to δ_e.

Also in [1], it was shown that:

- if the following inequalities hold:

$$(z_\alpha + m_q)^2 > 4(z_\alpha m_q - m_\alpha) + \frac{4g}{V}\sqrt{\overline{m}_{\dot\alpha}^{\,2} + \frac{c_2^{\,2}}{a^2}a_2^{\,2}} \tag{11}$$

$$\left(m_\alpha z_{\delta_e} - z_\alpha m_{\delta_e}\right)\delta_e + \frac{g}{V}z_\alpha\left|\frac{c_2}{a}a_2\right| > \frac{g}{V}\left|m_\alpha - z_\alpha\overline{m}_{\dot\alpha}\right| \tag{12}$$

then a longitudinal flight with constant forward velocity and constant state parameters becomes increasing oscillatory, i.e. $\dot\theta(t)$ is a positive $2n\pi$-periodic function;

- if beside inequality (11) the following inequality holds:

$$\left(m_\alpha z_{\delta_e} - z_\alpha m_{\delta_e}\right)\delta_e - \frac{g}{V}z_\alpha\left|\frac{c_2}{a}a_2\right| < -\frac{g}{V}\left|m_\alpha - z_\alpha\overline{m}_{\dot\alpha}\right| \tag{13}$$

then a longitudinal flight with constant forward velocity and constant state parameters becomes decreasing oscillatory, i.e. $\dot\theta(t)$ is a negative $2n\pi$-periodic function.

In other words, if the elevator deflection δ_e is not in the interval I at the moment when the flight control system fails, then the flight necessarily becomes oscillatory.

In the next section we will build up an alternative flight control, which is able to bring back the aircraft in a longitudinal flight with constant forward velocity and constant state parameters defined by:

$$\alpha^* = \frac{g}{V}\cdot\frac{1}{z_\alpha m_{\delta_e} - z_{\delta_e} m_\alpha}\left[z_{\delta_e}\overline{m}_{\dot\alpha} - m_{\delta_e}\right],\ q^* = 0,\ \theta^* = 0$$

2. DESIGN OF THE ALTERNATIVE FLIGHT CONTROL SYSTEM

Remark that for every $\theta^* \in R'$ there is a unique steady state of the system (1) having the pitch angle θ equal to θ^*; namely $X^* = [\alpha^*, 0, \theta^*]^T$ where α^* is given by:

$$\alpha^* = \frac{1}{z_\alpha m_{\delta_e} - z_{\delta_e} m_\alpha} \cdot \frac{g}{V} \cdot \left[(z_{\delta_e} \overline{m}_{\dot\alpha} - m_{\delta_e}) \cos\theta^* - z_{\delta_e} \frac{c_2}{a} a_2 \sin\theta^* \right] \tag{14}$$

The steady state X^* corresponds to the elevator deflection δ_e^* given by:

$$\delta_e^* = \frac{1}{z_\alpha m_{\delta_e} - z_{\delta_e} m_\alpha} \cdot \frac{g}{V} \cdot \left[(m_\alpha - z_\alpha \overline{m}_{\dot\alpha}) \cos\theta^* + z_\alpha \frac{c_2}{a} \cdot a_2 \sin\theta^* \right] \tag{15}$$

In the following, an alternative flight control system which stabilizes an arbitrary oscillatory longitudinal flight bringing back the aircraft at the longitudinal flight equilibrium $X^* = [\alpha^*, 0, 0]^T$ with α^* given by:

$$\alpha^* = \frac{g}{V} \cdot \frac{1}{z_\alpha m_{\delta_e} - z_{\delta_e} m_\alpha} \left[z_{\delta_e} \overline{m}_{\dot\alpha} - m_{\delta_e} \right] \tag{16}$$

will be attempted.

For that consider the elevator deflection δ_e as a function of the pitch angle θ, suggested by (15), and given by:

$$\delta_e(\theta) = \frac{1}{z_\alpha m_{\delta_e} - z_{\delta_e} m_\alpha} \cdot \frac{g}{V} \cdot \left[(m_\alpha - z_\alpha \overline{m}_{\dot\alpha}) \cos\theta + z_\alpha \frac{c_2}{a} \cdot a_2 \sin\theta \right] \tag{17}$$

approximating $\delta_e(\theta)$ by its first order Taylor polynomial at $\theta^* = 0$:

$$\delta_e(\theta) \approx \frac{g}{V} \cdot \frac{m_\alpha - z_\alpha \overline{m}_{\dot\alpha}}{z_\alpha m_{\delta_e} - z_{\delta_e} m_\alpha} - \frac{g}{V} \cdot \frac{\frac{c_2}{a} a_2 z_\alpha}{z_{\delta_e} m_\alpha - z_\alpha m_{\delta_e}} \theta \tag{18}$$

The Taylor polynomial provides the idea of considering the alternative flight control defined as:

$$\delta_e = \frac{g}{V} \cdot \frac{m_\alpha - z_\alpha \overline{m}_{\dot\alpha}}{z_\alpha m_{\delta_e} - z_{\delta_e} m_\alpha} - \mu\theta \tag{19}$$

where $\mu \in R'$ is a parameter called amplifier.

For μ equal to μ_0, given by:

$$\mu_0 = \frac{g}{V} \cdot \frac{\frac{c_2}{a} a_2 z_\alpha}{z_{\delta_e} m_\alpha - z_\alpha m_{\delta_e}} \tag{20}$$

the alternative flight control, defined by (19), coincides with the first order Taylor polynomial given by (18).

The flight of the aircraft under the action of the alternative flight control, defined by (19), is governed by the system of differential equations:

$$\begin{cases} \dot{\alpha} = z_\alpha\,\alpha + \dfrac{g}{V}\cos\theta + z_{\delta_e}\left[\dfrac{g}{V}\cdot\dfrac{m_\alpha - z_\alpha \overline{m}_{\dot{\alpha}}}{z_\alpha\, m_{\delta_e} - z_{\delta_e}\, m_\alpha} - \mu\,\theta\right] \\ \dot{q} = m_\alpha\,\alpha + m_q\, q + \dfrac{g}{V}\left[\overline{m}_{\dot{\alpha}}\cos\theta - \dfrac{c_2}{a}a_2\sin\theta\right] + m_{\delta_e}\left[\dfrac{g}{V}\cdot\dfrac{m_\alpha - z_\alpha \overline{m}_{\dot{\alpha}}}{z_\alpha\, m_{\delta_e} - z_{\delta_e}\, m_\alpha} - \mu\,\theta\right] \\ \dot{\theta} = q \end{cases} \tag{21}$$

Obviously, the state $X^* = [\alpha^*, 0,\ 0]^T$ is a steady state of the system (21). The Jacobean matrix of the system (21) at $X^* = [\alpha^*, 0,\ 0]^T$ is

$$\begin{pmatrix} z_\alpha & 1 & -\mu\, z_{\delta_e} \\ m_\alpha & m_q & -\dfrac{g}{V}\dfrac{c_2}{a}a_2 - m_{\delta_e}\,\mu \\ 0 & 1 & 0 \end{pmatrix} \tag{22}$$

and its characteristic equation is:

$$\begin{gathered} \lambda^3 - (z_\alpha + m_q)\,\lambda^2 + \left(z_\alpha\, m_q + \dfrac{g}{V}\dfrac{c_2}{a}a_2\, m_{\delta_e}\,\mu - m_\alpha\right)\lambda - \\ \left[\left(z_\alpha\, m_{\delta_e} - m_\alpha\, z_{\delta_e}\right)\mu + z_\alpha\,\dfrac{g}{V}\dfrac{c_2}{a}a_2\right] = 0 \end{gathered} \tag{23}$$

Let be:

$$\mu_1 = z_\alpha + m_q\,;\ \mu_2 = z_\alpha\, m_q + \frac{g}{V}\frac{c_2}{a}a_2 + m_{\delta_e}\,\mu - m_\alpha\,;$$

$$\mu_3 = z_\alpha\,\frac{g}{V}\frac{c_2}{a}a_2 + \left(z_\alpha\, m_{\delta_e} - m_\alpha\, z_{\delta_e}\right)\mu$$

and the characteristic equation (23) written in the form:

$$\lambda^3 - \mu_1\,\lambda^2 + \mu_2\,\lambda - \mu_3 = 0 \tag{24}$$

Assuming as in [1] that the following inequalities hold

$$\begin{cases} z_\alpha + m_q < 0 \\ \dfrac{g}{V}\dfrac{c_2}{a}a_2 > 0 \\ m_{\delta_e} < 0 \end{cases} \tag{25}$$

it follows that $\mu_1 < 0$ and $\mu_2 > 0$ for μ satisfying:

$$\mu < \frac{\left(m_\alpha - z_\alpha\, m_q\right) a}{c_2\, a_2\, m_{\delta_e}} \cdot \frac{V}{g} \tag{26}$$

Assuming now that also the following inequalities are fulfilled:

$$\begin{cases} z_\alpha\, m_{\delta_e} - m_\alpha\, z_{\delta_e} > 0 \quad \textit{and} \quad \mu < \dfrac{-\, g\, z_\alpha\, c_2\, a_2}{a\, V\left(z_\alpha\, m_{\delta_e} - m_\alpha\, z_{\delta_e}\right)} \\ \left(z_\alpha + m_q\right) \dfrac{g}{V} \dfrac{c_2}{a} a_2\, m_{\delta_e} > z_\alpha\, m_{\delta_e} - m_\alpha\, z_{\delta_e} \ \textit{and} \\ \qquad \mu < \dfrac{z_\alpha \dfrac{g}{V} \dfrac{c_2}{a} a_2 - \left(z_\alpha + m_q\right)\left(z_\alpha\, m_q - m_\alpha + \dfrac{g}{V} \dfrac{c_2}{a} a_2\right)}{\left(z_\alpha + m_q\right) m_{\delta_e} - \left(z_\alpha\, m_{\delta_e} - m_\alpha\, z_{\delta_e}\right)} \end{cases} \tag{27}$$

it follows that we have:

$$\mu_1\, \mu_2 < \mu_3 < 0 \tag{28}$$

for every μ which satisfies:

$$\mu < \min \left\{ \begin{array}{l} \dfrac{\left(m_\alpha - z_\alpha\, m_q\right) a\, V}{c_2\, a_2\, m_{\delta_e} g}, \dfrac{-\, g\, z_\alpha\, c_2\, a_2}{a\, V\left(z_\alpha\, m_{\delta_e} - m_\alpha\, z_{\delta_e}\right)}, \\ \dfrac{z_\alpha \dfrac{g}{V} \dfrac{c_2}{a} a_2 - \left(z_\alpha + m_q\right)\left(z_\alpha\, m_q - m_\alpha + \dfrac{g}{V} \dfrac{c_2}{a} a_2\right)}{\left(z_\alpha + m_q\right) m_{\delta_e} - \left(z_\alpha\, m_{\delta_e} - m_\alpha\, z_{\delta_e}\right)} \end{array} \right\} \tag{29}$$

Applying now Lemma 2 from [2], we obtain that for every μ satisfying (29) the roots $\lambda_1, \lambda_2, \lambda_3$ satisfy the inequalities:

$$\lambda_1 < 0 \ ; \ \mathrm{Re}\left(\lambda_2\right) < 0 \ \text{and} \ \mathrm{Re}\left(\lambda_3\right) < 0 \tag{30}$$

We remark that conditions (27) are compatible with conditions (25) and are satisfied for the numerical data from Table 1.

We have obtained in this way that if μ satisfies (29), and then the steady state $X^* = \left[\alpha^*, 0,\ 0\right]^T$ is exponentially stable.

On the other hand, in order to be able to bring back the aircraft from an oscillatory movement in the steady state X^*, the amplification factor μ must insure that the steady state X^* is the unique steady state of the system (21), which governs the flight under the action of the alternative flight control system.

In this case for the asymptotically stable steady state $X^* = \left[\alpha^*, 0,\ 0\right]^T$ global attraction can be expected.

It is easy to see that if the amplification factor μ satisfies the inequality

$$\mu < \min\left\{ \frac{\frac{g}{V}\left(\left| m_\alpha - z_\alpha \, \overline{m}_{\dot{\alpha}} \right| + \left| z_\alpha \cdot \frac{c_2}{a} \cdot a_2 \right| \right)}{m_\alpha \, z_{\delta_e} - z_\alpha \, m_{\delta_e}}, \ \tilde{\mu} \right\} \tag{31}$$

then the last requirement is fulfilled.

Here $\tilde{\mu}$ is equal to the right hand member of the inequality (29).

In this way it is proved that for μ satisfying (31), the steady state $X^* = [\alpha^*, 0, 0]^T$ of the system (21) is exponentially stable and global attraction can be expected, i.e. the alternative flight control defined by (19) brings back the aircraft from an arbitrary oscillatory longitudinal flight with constant forward velocity to the steady state X^*.

3. NUMERICAL EXAMPLES

The numerical illustration of the above presented facts will be made in the case of the particular model plane described in [3]. Computations were made for the same data given in Table 1.

Table 1

Parameter	Value	Units	Parameter	Value	Units
z_α	-1.598075	s^{-1}	a	-0.485	s^{-1}
z_{δ_e}	-0.52089	s^{-1}	a_2	11.964	s^{-2}
m_α	1.72514652738	s^{-2}	V	84.5	m/s
m_{δ_e}	-9.97292276532	s^{-2}	g	9.81	m/s^2
m_q	-22.61196	s^{-1}	c_2	-0.029	
$\overline{m}_{\dot{\alpha}}$	-5.26416	s^{-1}	k_α	-0.401	
k_q	-1.284		k_p	1 - 8	

As it is shown in [2] the interval I in this case is $I = [-0.0467823, 0.0467823]$ rad.

Under the action of the flight control system, defined by (2) for k_α, k_q, k_p given in Table 1, the behavior of the aircraft was simulated numerically in [2].

In [1] the occurrence of two oscillatory solutions of the same aircraft is simulated in the case when the flight control system fails at $t = 0$.

The first is an increasing oscillatory movement and corresponds to the situation in which the elevator deflection δ_e at the moment of failing is blocked at $\delta_e = -0.05$ rad (Fig.1).

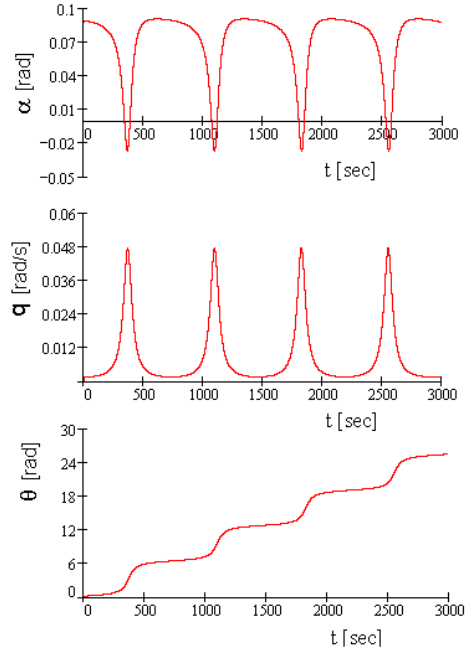

Figure 1. Increasing oscillatory solution when δ_e = - 0.05 [rad] at the moment of the failure of the automatic flight control system and the starting point is:
α_1 = 0.086974288419088 [rad]; q_1 = 0 [rad/sec]; θ_1= 0.159329728679884 [rad].

The second is a decreasing oscillatory movement and corresponds to the situation in which the elevator deflection δ_e at the moment of failing is blocked at $\delta_e = 0.048\ rad$ (Fig.2).

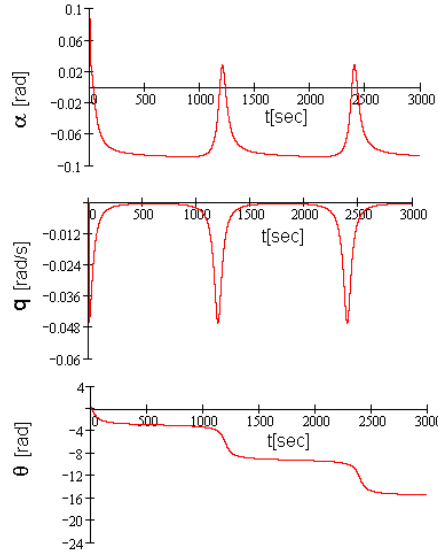

Figure 2. Decreasing oscillatory solution when $\delta_e = 0.048\ [rad]$ at the moment of the failure of the automatic flight control system and the starting point is:
α_1 = 0.086974288419088 [rad]; q_1 = 0 [rad/sec]; θ_1= 0.159329728679884 [rad].

In order to illustrate that the alternative flight control system is able to bring back the aircraft from these oscillatory solutions to the steady state $X_0* = (0.08767715, 0, 0)^T$, we have integrated numerically the system (21) for $\mu = -4$ starting from the points of the above

oscillatory solutions after 25 s of oscillations. The results of these integrations are presented in Figs. 3 and 4, respectively.

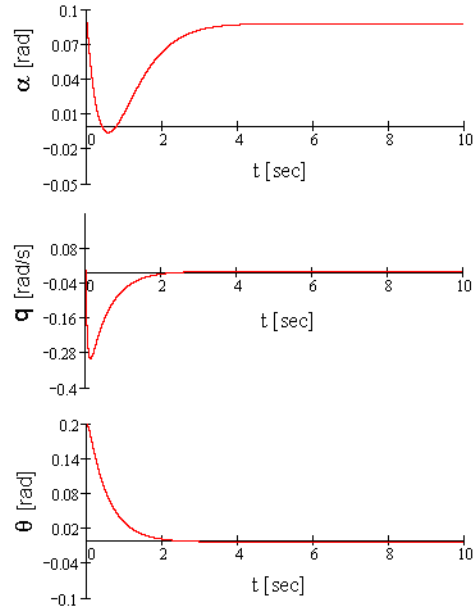

Figure 3. The alternative flight control system ($\mu = -4$) which begins to act after 25 s of oscillations, from the state α_1 = 0.088517237216094 [rad]; q_1 = 0.001584881972789 [rad/sec]; θ1= 0.19862398 [rad] (see Fig.1) brings back the aircraft to the steady state α* = 0.087677171 [rad]; q* = 0 [rad/sec]; θ*= 0 [rad] in 10 seconds.

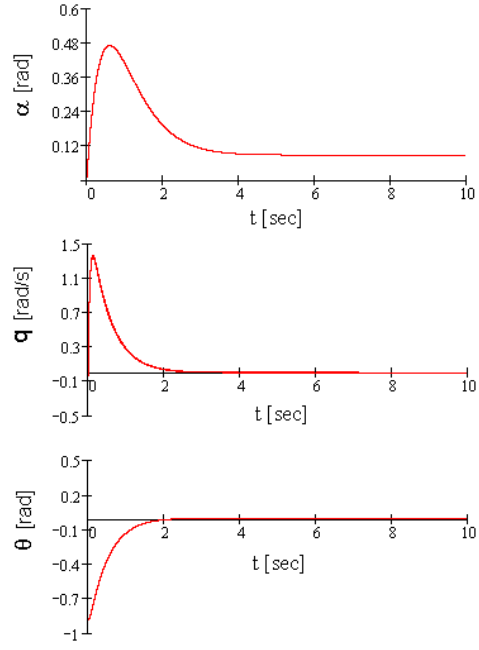

Figure 4. The alternative flight control system ($\mu = -4$) which begins to act after 25 s of oscillations, from the state α_1 = 0.00943807124362 [rad]; q_1 = -0.034712731272892 [rad/sec]; θ1= -0.886609702207 [rad] (see Fig.2) brings back the aircraft to the steady state α* = 0.087677066 [rad]; q* = 0 [rad/sec]; θ* = 0 [rad] in 10 seconds.

Figures 3 and 4 illustrate in which kind the aircraft is brought back to the steady state X^*.

4. CONCLUSION

For an unmanned aircraft whose automated flight control system fails during a longitudinal flight with constant forward velocity the following statements hold:

i) There exists a range $\left(\underline{\delta_e}, \overline{\delta_e}\right)$ having the property that if in the moment of failure the elevator deflection δ_e stops in this range, then after a period of transition the flight becomes longitudinal with constant angle of attack, constant pitch angle and zero yaw rate.

ii) If at the moment of failure the elevator deflection δ_e stops outside of this range, then the longitudinal flight becomes oscillatory,

iii) The alternative flight control system defined by (19) is able to bring back to the steady state movement $X^* = (\alpha^*, 0, 0)^T$ the aircraft which is in oscillatory longitudinal movement.

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