A Numerical Solution for the Lifting Surface Integral Equation

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Abstract: *In this paper, we present a new numerical treatment of the lifting surface integral equation (LSIE) in the case of rectangular wing planform. The kernel of this equation possesses a strong singularity in Hadamard sense. First the equation is transformed into one containing weaker singularities, of Cauchy-type, and then the 2D singular integral is discretized by the aid of the Gauss-type quadrature formulae.*
Thus the problem is reduced to a finite system of linear algebraic equations. The numerical simulation reveals a very good agreement, in terms of jump pressure over the wing and aerodynamic coefficients, with the exact solution in the case of the low aspect ratio wing and also with other numerical solutions.

Key Words: *Gauss type quadrature, singular integral equation, rectangular wing planform*

1. INTRODUCTION

In the Multhopp's paper [7] we are led to the lifting surface integral equation (LSIE) in subsonic flow for the unknown dimensionless jump pressure over a wing which is supposed to be infinitely thin:

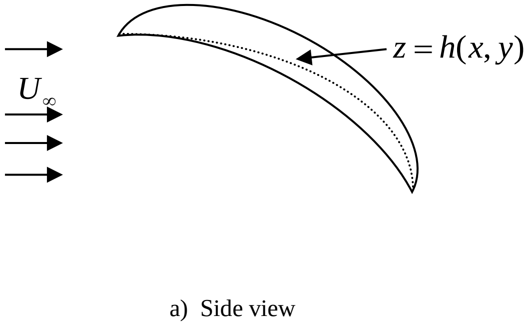
$$-\frac{1}{4\pi}\overset{*}{\iint_D}\frac{f(\xi,\eta)}{(y-\eta)^2}\left(1+\frac{x-\xi}{\sqrt{(x-\xi)^2+\beta^2(y-\eta)^2}}\right)d\xi d\eta = h_x{}'(x,y),\ (x,y)\in \overset{\circ}{D} \tag{1}$$

where D is the wing planform namely projection of the wing on XOY plane, $f(x,y)=p(x,y,+0)-p(x,y,-0)$ is unknown, $\beta=\sqrt{1-M_\infty^2}$ is the compressibility coefficient, M_∞^2 is the Mach number of the undisturbed flow and $z=h(x,y)$ is the equation of the mean surface of the wing such that $\left|h_x^{'}\right| << 1$.

In the case of the flat plate at uniform angle of attack $\varepsilon << 1$, $h(x,y)=-\varepsilon x$.

The leading and trailing edges of the wing planform are described by the equations $x=x_-(y)$ and $x=x_+(y)$ respectively.

The wing's geometry is shown in the Figure 1 below.



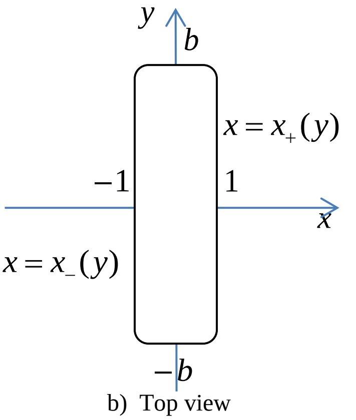


Fig. 1 The wing's geometry

The integral contains a strong singularity at $\eta = y$, the asterisk standing for "Finite Part" integral in Hadamard sense defined by

$$\int_{-b}^{*b} \frac{g(x,y,\eta)}{(y-\eta)^2} d\eta = \lim_{\varepsilon \to 0_+} \left(\int_{-b}^{y-\varepsilon} + \int_{y+\varepsilon}^{b} \frac{g(x,y,\eta)}{(y-\eta)^2} d\eta - 2\frac{g(x,y,y)}{\varepsilon} \right) \tag{2}$$

This integral must be solved taking into account the boundary behavior of the jump pressure:

$$\begin{aligned} &f(x,y) \propto \sqrt{x_+(y) - x} \text{ at the trailing edge} \\ &f(x,y) \propto \frac{1}{\sqrt{x - x_-(y)}} \text{ at the leading edge} \\ &f(x,y) \propto \sqrt{y \mp b} \text{ at the lateral edges y = } \pm b \end{aligned} \tag{3}$$

There are numerous papers devoted to solving numerical LSIE. We mention here Multhopp's method [7],[9], Vortex Lattice Method (VLM) [4] or Gauss-type quadrature formulae method [2]. In the present paper we deal with a method of the latter type for integrals with Cauchy-type singularities that are weaker than those of Hadamard-type and hence the quadrature formulae for spanwise integral are more accurate.

2. NEW FORM OF THE LSIE

In reference [5] is given an equivalent form of the LSIE, containing only Cauchy-type singularities.

$$\frac{1}{4\pi}\frac{\partial}{\partial y}\iint_D^{\#\,\#} \frac{f(\xi,\eta)}{y-\eta}\left(1+\frac{\sqrt{(x-\xi)^2+\beta^2(y-\eta)^2}}{x-\xi}\right) d\xi d\eta = h_x'(x,y),\ (x,y)\in \mathring{D} \tag{4}$$

where the mark # stands for the integral in the sense of *Cauchy Principal Value*:

$$\int_a^{\#b} \frac{f(\xi)}{x-\xi} d\xi = \lim_{\varepsilon \to 0_+} \left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^{b} \right) \frac{f(\xi)}{x-\xi} d\xi, \quad x \in (a,b) \tag{5}$$

and

$$\overset{\#}{\int}\overset{\#}{\int}_{D}(\ldots)d\xi d\eta=\lim_{\varepsilon_1,\varepsilon_2\to 0_+}\iint_{D_{\varepsilon_1,\varepsilon_2}}(\ldots)d\xi d\eta \tag{6}$$

where $D_{\varepsilon_1,\varepsilon_2}$ is the area D from which two bands centered at (x,y) parallel to X-axis of width $2\varepsilon_1$ and Y-axis of width $2\varepsilon_2$ respectively, where removed (Figure 2).

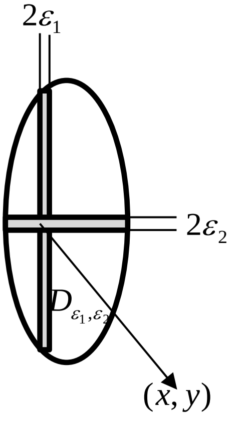


Fig. 2

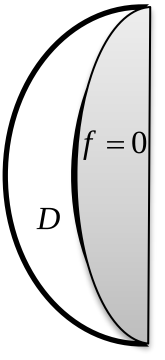


Fig. 3

In what follows we make the following assumptions:

i) the wing planform D is a convex set

ii) the X-axis is symmetry line of the set D

If the first assumption is not fulfilled for some wing configurations we set $f=0$ beyond the trailing edge (Figure 3).

Integrating the equation (4) with respect to y we get

$$\frac{1}{4\pi}\overset{\#}{\int}\overset{\#}{\int}_{D}\frac{f(\xi,\eta)}{y-\eta}\left(1+\frac{\sqrt{(x-\xi)^2+\beta^2(y-\eta)^2}}{x-\xi}\right)d\xi d\eta=\varphi(x)+\int_0^y h_x{}'(x,t)dt,\ (x,y)\in\overset{\circ}{D} \tag{7}$$

where $\varphi(x)$ is an arbitrary function of integration. We shall prove that $\varphi(x)=0$ for any arbitrary $x\in(-1,1)$.

Indeed, from the first assumption it follows that the functions $y\mapsto x_{\pm}(y)$ and $y\mapsto f(x,y)$ are even whence the function $\eta\mapsto\overset{\#x_+(\eta)}{\int_{x_-(\eta)}}f(\xi,\eta)\left(1+\frac{\sqrt{(x-\xi)^2+\beta^2\eta^2}}{x-\xi}\right)d\xi$ is even and therefore $\overset{\#b}{\int_{-b}}\left[\overset{\#x_+(\eta)}{\int_{x_-(\eta)}}\frac{f(\xi,\eta)}{\eta}\left(1+\frac{\sqrt{(x-\xi)^2+\beta^2\eta^2}}{x-\xi}\right)d\xi\right]=0$.

Hence, if we set $y=0$ in (7) it follows that $\varphi(x)\equiv 0$. Using this approach we obtained a new form of the LSIE under the aforementioned assumptions:

$$\frac{1}{4\pi}\overset{\#\,\#}{\iint_D}\frac{f(\xi,\eta)}{y-\eta}\left(1+\frac{\sqrt{(x-\xi)^2+\beta^2(y-\eta)^2}}{x-\xi}\right)d\xi d\eta=\int_0^y h_x{}'(x,t)dt,\ (x,y)\in\overset{\circ}{D} \tag{8}$$

3. DISCRETIZATION IN THE CASE OF THE RECTANGULAR PLANFORM

If D is a the rectangle $[-1,1]\times[-b,b]$ then the above assumptions are met. We perform the change of variables $x=x', y=by', \xi=\xi', \eta=b\eta'$ but for the sake of simplicity we shall denote the new variables by the same notations as the old. Taking into account the boundary behavior of the jump pressure we seek the solution in the form

$$f(x,y)=\sqrt{\frac{1-x}{1+x}}\sqrt{b^2-y^2}\,F(x,y),\ (x,y)\in\overset{\circ}{D} \tag{9}$$

In the new variables the equation (8) is written

$$\int_{-1}^{1\#}\frac{\sqrt{1-\eta^2}}{y-\eta}\left(\int_{-1}^{1\#}\sqrt{\frac{1-\xi}{1+\xi}}F(\xi,b\eta)\left[1+\frac{\sqrt{(x-\xi)^2+\beta^2b^2(y-\eta)^2}}{x-\xi}\right]d\xi\right)d\eta=4\pi\int_0^{by}h_x{}'(x,t)dt \tag{10}$$

In paper [6] is given a general Gauss-type quadrature formula for singular integrals which is particularized in [3] for various weights. Thus

$$\begin{gathered}\int_{-1}^{1}\sqrt{\frac{1-\xi}{1+\xi}}F(\xi)d\xi\approx\frac{2\pi}{2m+1}\sum_{k=1}^{m}(1-\xi_k)F(\xi_k)\\ \int_{-1}^{\#1}\sqrt{\frac{1-\xi}{1+\xi}}\frac{F(\xi)}{x_i-\xi}d\xi\approx\frac{2\pi}{2m+1}\sum_{k=1}^{m}\frac{(1-\xi_k)}{x_i-\xi_k}F(\xi_k),\ i=\overline{1,m}\\ \int_{-1}^{1}\sqrt{1-\eta^2}F(\eta)d\eta\approx\frac{\pi}{n+1}\sum_{l=1}^{n}(1-\eta_l^2)F(\eta_l)\\ \int_{-1}^{\#1}\sqrt{1-\eta^2}\frac{F(\eta)}{y_j-\eta}d\eta\approx\frac{\pi}{n+1}\sum_{l=1}^{n}\frac{1-\eta_l^2}{y_j-\eta_l}F(\eta_l),\ j=\overline{1,n+1}\end{gathered} \tag{11}$$

where

$$\begin{gathered}\xi_k=\cos\frac{2k}{2m+1}\pi,\ x_i=\cos\frac{2i-1}{2m+1}\pi,\ i,k=\overline{1,m}\\ \eta_l=\cos\frac{l\pi}{n+1},\ y_j=\cos\frac{2j-1}{n+1}\frac{\pi}{2},\quad j=\overline{1,n+1},l=\overline{1,n}\end{gathered} \tag{12}$$

Using the formulae (12) the following discretized form of the equation (11) is obtained:

$$\sum_{k=1}^{m}\sum_{l=1}^{n}A_{ijkl}F_{kl}=H_{ij},\ i=\overline{1,m},\ j=\overline{1,n+1} \tag{13}$$

where $F_{kl}=F(\xi_k,b\eta_l)$ are the unknowns,

$$A_{ijkl} = \frac{\pi}{2(2m+1)(n+1)} \frac{(1-\eta_l^2)(1-\xi_k)}{y_j - \eta_l} \left[1 + \frac{\sqrt{(x_i - \xi_k)^2 + \beta^2 b^2 (y_j - \eta_l)^2}}{x_i - \xi_k} \right] \tag{14}$$

and $H_{ij} = \int_0^{by_j} h_x{}'(x_i, t)dt$

4. NUMERICAL RESULTS AND COMPARISONS

The linear system (13) has $m(n+1)$ equations and mn unknowns. In order to fit the number of equations with the number of unknowns we introduce a set of n fictitious unknowns $(C_i)_{i=\overline{1,m}}$ such that $\sum_{k=1}^{m}\sum_{l=1}^{n} A_{ijkl}F_{kl} + C_i = H_{ij}$, $i = \overline{1,m}$, $j = \overline{1,n+1}$.

Solving this square linear system numerically we get $C_i = 0, i = \overline{1,m}$ showing that the system (13) is consistent. The consistency is proved also if we solve the linear system (13) using the NSolve function from MATHEMATICA Package. Due to the spanwise symmetry only half of the unknowns are used, hence the time consumption is halved. Once F_{kl} determined it is easy to evaluate the lift and moment coefficients by means of formulae (11).

$$c_L = -\frac{2}{A}\iint_D f(\xi,\eta)d\xi d\eta \approx -\frac{4\pi^2 b}{A(2m+1)(n+1)} \sum_{k=1}^{m}\sum_{l=1}^{n} (1-\xi_k)(1-\eta_l^2)F_{kl} \tag{15}$$

$$c_x = -\frac{2}{Aa_0}\iint_D \eta f(\xi,\eta)d\xi d\eta \approx -\frac{4\pi^2 b^2}{a_0 A(2m+1)(n+1)} \sum_{k=1}^{m}\sum_{l=1}^{n} \eta_l (1-\xi_k)(1-\eta_l^2)F_{kl} \tag{16}$$

$$c_y = \frac{2}{Aa_0}\iint_D \xi f(\xi,\eta)d\xi d\eta \approx \frac{4\pi^2 b}{a_0 A(2m+1)(n+1)} \sum_{k=1}^{m}\sum_{l=1}^{n} \xi_k (1-\xi_k)(1-\eta_l^2)F_{kl} \tag{17}$$

where A is the area of the wing planform and a_0 is the wing chord (dimensionless).

In order to study the efficiency of the present numerical method we consider the incompressible flow past a rectangular wing planform at uniform angle of attack ε and aspect ratio $AR = b = 1$. We choose the same number n of collocation points along the span and along the chord. The values of the lift slope coefficient C_L / ε in terms of n are shown in Table 1.

Fitting this set of data we obtain an asymptotic behavior of the form $C_L / \varepsilon = 1.4602265 + 0.1169319 / n^{3.5}$. Thus we predict that for the square wing $C_L / \varepsilon = 1.460227$ correct to 6 decimal places. This value is the same with that obtained by Tuck in [8] up to the 6th decimal digit.

A second example is the incompressible flow past a low aspect ratio $(b << 1)$ rectangular wing at uniform angle of attack ε. The distribution of the jump pressure over the wing in the considered cases are shown in Figure 4 and Figure 5 respectively.

In reference [1] is given the analytical expression (in the frame of the theory of the wings of low aspect ratio) of the lift slope coefficient as $C_L / \varepsilon = \pi b / 2$. For $b = 0.1$ Dragos' exact solution gives $C_L / \varepsilon = 0.15708$ whilst our method provides $C_L / \varepsilon = 0.15702$.

These two examples show a very good agreement between analytical or numerical solutions and our method fact which validates the latter one.

Table 1 Convergence rate with number n of spanwise collocation points

m=n	c_L / ε
10	1.4601313
20	1.4602118
30	1.4602224
40	1.4602244
50	1.4602254
60	1.4602259
80	1.4602263
100	1.4602264

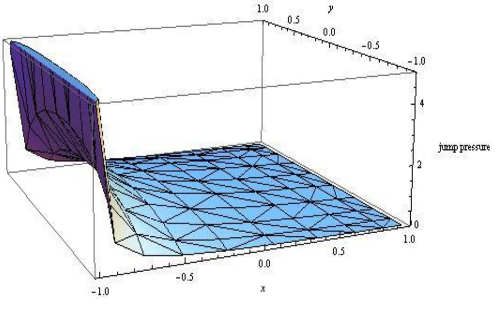


Fig. 4 (square wing)

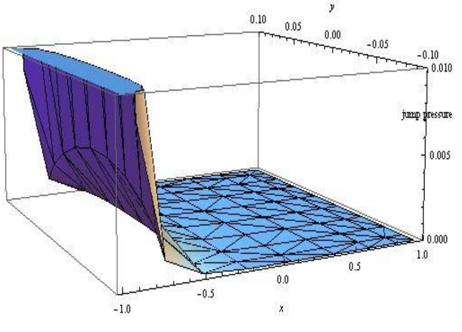


Fig. 5 (low aspect ratio wing $b = 0.1$)

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