# Frequency Analysis of Uncertain System

Gabriela STROE*[,1], Irina-Carmen ANDREI[2]

*Corresponding author
*[,1]"POLITEHNICA" University of Bucharest, Faculty of Aerospace Engineering,
Gh. Polizu Street 1-7, Sector 1, Bucharest, 011061, Romania
ing.stroe@yahoo.com
[2]INCAS – National Institute for Aerospace Research "Elie Carafoli",
B-dul Iuliu Maniu 220, Sector 6, Bucharest, 061126, Romania
andrei.irina@incas.ro, icandrei28178@gmail.com

DOI: 10.13111/2066-8201.2016.8.2.9

*Received: 04 April 2016 / Accepted: 24 May 2016*


***4th International Workshop on Numerical Modelling in Aerospace Sciences, NMAS 2016,***
*11-12 May 2016, Bucharest, Romania, (held at INCAS, B-dul Iuliu Maniu 220, sector 6)*
*Section 3 – Modelling of structural problems in aerospace airframes*

***Abstract:*** *This paper is dedicated to the study and analysis of a successfully designed control system, whose capability should always maintain the stability and performance level in spite of the uncertainties in system dynamics and within the working environment to a certain degree. Design requirements such as gain margin and phase margin in using classical frequency-domain techniques are solely for the purpose of robustness. A constrained optimization is then performed to maximize the robust stability of the closed-loop system to the type of uncertainty chosen, the constraint being the internal stability of the feedback system. In most cases, it would be sufficient to search a feasible controller such that the closed-loop system should achieve certain robust stability. Performance objectives can also be included in the optimization cost function. Elegant solution formulae have been developed, which are based on the solutions of certain Algebraic Riccati Equations.*

***Key Words:*** *Design control system, robust stability, optimization cost function, uncertain system.*

## 1. INTRODUCTION

The system dynamics is usually governed by a set of differential equations in either open-loop or closed-loop systems. In the case of linear, a time-invariant system, these differential equations are linear ordinary differential equations (modeling aspects are detailed in Etkin, [4], McLean [5]). By introducing appropriate state variables and simple manipulations, a linear, time-invariant, continuous-time control system can be described by the following model:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in \boldsymbol{R}^n$ is the state vector, $\boldsymbol{u}(t) \in \boldsymbol{R}^m$ the input (control) vector, and $\boldsymbol{y}(t) \in \boldsymbol{R}^p$ the output (measurement) vector, [4], [5].

With the assumption of zero initial condition of the state variables and using Laplace transform, a transfer function matrix corresponding to the system in (1) can be derived as:

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I_n} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \tag{2}$$

and can be further denoted in a short form by:

$$\boldsymbol{G}(s) = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \tag{3}$$

## 2. BALANCED TRUNCATION METHOD

### 2.1 Theoretical Development

Due to the increasing demands on quality and productivity of industrial systems and with deeper understanding of these systems, the mathematical models derived to represent the system dynamics are more complete, usually of multi-input-multi-output form, and are of high orders. Consequently, the controllers designed are complex. The order of such controllers designed using, for instance, the $H_\infty$ optimization approach or the $\mu$-method, is higher or at least similar to, that of the plant. On the other hand, in the implementation of controllers, high-order controllers will lead to high cost, difficult commissioning, poor reliability and potential problems in maintenance. Lower-order controllers are always welcomed by practicing control engineers. [4], [5]. In general, there are three directions in getting a lower-order controller for a relatively high-order plan:

(**1**) Plant model reduction followed by controller design;

(**2**) Controller design followed by controller-order reduction;

(**3**) Direct design of low-order controllers.

Approaches (1) and (2) are widely used and can be used together. When a controller is designed using a robust design method, Approach (1) would usually produce a stable closed loop, though the reduction of the plant order is likely to be limited. [1], Auger and Lemoine [2]. In Approach (2), there is freedom in choosing the final order of the controller, but the stability of the closed-loop system should always be verified. The third approach usually would heavily depend on some properties of the plant, and require numerous computations.

The general idea of truncation methods is to neglect those parts of the original system that are less observable and less controllable, Rauw [3]. Hopefully, this would lead to a system of lower order and retaining the important dynamic behavior of the original system. However, in some systems, a mode would be weakly observable but highly controllable, or vice versa. To delete such a mode may be inappropriate with regard to the whole characteristics of the system.

Hence, in the balanced truncation method, a state similarity transformation is applied first to "balance" the controllability and observability features of the system, Moysis and collab. [6], Balas [7] and Chiang [7-8]. A stable system $\boldsymbol{G}(s)$ is called balanced if the solutions $\boldsymbol{P}$ and $\boldsymbol{Q}$ to the following Lyapunov equations:

$$\boldsymbol{AP} + \boldsymbol{PA^T} + \boldsymbol{BB^T} = 0 \tag{4}$$

$$\boldsymbol{A^TQ} + \boldsymbol{QA} + \boldsymbol{C^TC} = 0 \tag{5}$$

are such that:

$$\boldsymbol{P} = \boldsymbol{Q} = diag(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{\sigma_3}, \dots, \boldsymbol{\sigma_n}) = \boldsymbol{\Sigma} \tag{6}$$

with:

$$\boldsymbol{\sigma_1} \geq \boldsymbol{\sigma_2} \geq \boldsymbol{\sigma_3} \geq \cdots \geq \boldsymbol{\sigma_n} > 0 \tag{7}$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are called the controllability grammian and observability grammian, respectively, and $\boldsymbol{\sigma}_i$, $i = 1, \dots, n$ represents the $i^{th}$ Hankel singular value of the system, Rawley [9], Franklin [10]. When the system is balanced, both grammians are diagonal and equal. Next steps refer to calculating the grammians $\boldsymbol{P}$ and $\boldsymbol{Q}$, a Cholesky factor $\boldsymbol{R}$ of $\boldsymbol{Q}$ (8), to forming a positive-definite matrix $\boldsymbol{RPR^T}$ and afterwards to turn this matrix to its diagonal form:

$$\boldsymbol{Q} = \boldsymbol{R^T R} \tag{8}$$

$$\boldsymbol{RPR^T} = \boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{U^T} \tag{9}$$

$$\boldsymbol{UU^T} = \boldsymbol{I} \tag{10}$$

where $\boldsymbol{U}$ is an ortho-normal matrix (10) and $\boldsymbol{T}$ is obtained from relation (13):

$$\boldsymbol{\Sigma} = diag(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \dots, \boldsymbol{\sigma_n}) \tag{11}$$

$$\boldsymbol{\sigma_1} \geq \boldsymbol{\sigma_2} \geq \cdots \geq \boldsymbol{\sigma_n} > 0 \tag{12}$$

$$\boldsymbol{T} = \boldsymbol{\Sigma}^{\left(\frac{1}{2}\right)}\boldsymbol{U^T R} \tag{13}$$

As one can notice, $[\boldsymbol{T}, \boldsymbol{T}^{-1}]$ is the required state similarity transformation (balanced transformation). That is, $[\boldsymbol{TAT}^{-1}, \boldsymbol{TB}, \boldsymbol{CT}^{-1}]$ is a balanced realization, Datta [11], Doyle, Francis and Tannenbaum [12].

Assuming that the state-space model of the original system $\boldsymbol{G}(s)$, $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}]$ is already in the balanced realization form, and also assuming the relation (14), based on equations (15) and (16), the model can be further developed, Balas [7], Franklin [10].

$$\boldsymbol{\Sigma} = diag(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2) \tag{14}$$

$$\boldsymbol{\Sigma_1} = diag(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \dots, \boldsymbol{\sigma_r}) \tag{15}$$

$$\boldsymbol{\Sigma_2} = diag(\boldsymbol{\sigma_{r+1}}, \boldsymbol{\sigma_{r+2}}, \dots, \boldsymbol{\sigma_n}), \ \boldsymbol{\sigma_r} > \boldsymbol{\sigma_{r+1}} \tag{16}$$

The matrices **A, B** and **C** can be compatibly partitioned as:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A_{11}} & \boldsymbol{A_{12}} \\ \boldsymbol{A_{21}} & \boldsymbol{A_{22}} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B_1} \\ \boldsymbol{B_2} \end{bmatrix}, \boldsymbol{C} = [\boldsymbol{C_1} \quad \boldsymbol{C_2}] \tag{17}$$

Then, a reduced-order system $\boldsymbol{G_r}(s)$ can be defined by relation (18):

$$\boldsymbol{G_r}(s) = \boldsymbol{C_1}(s\boldsymbol{I_n} - \boldsymbol{A_{11}})^{-1}\boldsymbol{B_1} + \boldsymbol{D} = \begin{bmatrix} \boldsymbol{A_{11}} & \boldsymbol{B_1} \\ \boldsymbol{C_1} & \boldsymbol{D} \end{bmatrix} \tag{18}$$

Such a $\boldsymbol{G_r}(s)$ is of *r*-th order and is called a balanced truncation of the full order *n*-th system $\boldsymbol{G}(s)$. It can be shown that a $\boldsymbol{G_r}(s)$ is stable in the balanced realization form, and the condition (19) is fulfilled:

$$\|\boldsymbol{G}(s) - \boldsymbol{G_r}(s)\|_\infty \leq 2Tr(\boldsymbol{\Sigma_2}) \tag{19}$$

$$Tr(\boldsymbol{\Sigma_2}) = \boldsymbol{\sigma_{r+1}} + \boldsymbol{\sigma_{r+2}} + \cdots + \boldsymbol{\sigma_n} \tag{20}$$

where $Tr(\boldsymbol{\Sigma_2})$ denotes the trace (20) of the matrix $\boldsymbol{\Sigma_2}$, the sum of the last $(n-r)$ Hankel singular values, Balas [7], Chiang [7, 8].

Assume that $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}]$ is a minimal and balanced realization of a stable system $\boldsymbol{G}(s)$. It can be shown that $\boldsymbol{A_{22}}$ is stable and thus invertible. [11]

For such purpose, the definitions expressed by the relations (21) ÷ (24) are necessary:

$$\boldsymbol{A_r} = \boldsymbol{A_{11}} - \boldsymbol{A_{12}}\boldsymbol{A_{22}}^{-1}\boldsymbol{A_{21}} \tag{21}$$

$$\boldsymbol{B_r} = \boldsymbol{B_1} - \boldsymbol{A_{12}}\boldsymbol{A_{22}}^{-1}\boldsymbol{B_2} \tag{22}$$

$$\boldsymbol{C_r} = \boldsymbol{C_1} - \boldsymbol{C_2}\boldsymbol{A_{22}}^{-1}\boldsymbol{A_{21}} \tag{23}$$

$$\boldsymbol{D_r} = \boldsymbol{D} - \boldsymbol{C_2}\boldsymbol{A_{22}}^{-1}\boldsymbol{B_2} \tag{24}$$

A reduced-order system $\boldsymbol{G_r}(s)$ defined by (25):

$$\boldsymbol{G_r}(s) = \boldsymbol{C_r}(s\boldsymbol{I} - \boldsymbol{A_r})^{-1}\boldsymbol{B_r} + \boldsymbol{D_r} \tag{25}$$

is called a singular perturbation approximation (i.e. a balanced residualization) of $\boldsymbol{G}(s)$:

$$-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} + \boldsymbol{D} = -\boldsymbol{C_r}\boldsymbol{A_r}^{-1}\boldsymbol{B_r} + \boldsymbol{D_r} \tag{26}$$

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{A_{22}}^{-1}\boldsymbol{A_{21}} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{A_{22}}^{-1}\boldsymbol{A_{21}} & \boldsymbol{I} \end{bmatrix} = \boldsymbol{I} \tag{27}$$

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{A_{12}}\boldsymbol{A_{22}}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{A_{12}}\boldsymbol{A_{22}}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} = \boldsymbol{I} \tag{28}$$

$$\left( \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A_{12}}\boldsymbol{A_{22}}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{A_{11}} & \boldsymbol{A_{12}} \\ \boldsymbol{A_{21}} & \boldsymbol{A_{22}} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{A_{22}}^{-1}\boldsymbol{A_{21}} & \boldsymbol{I} \end{bmatrix} \right)^{-1} = \begin{bmatrix} \boldsymbol{A_r}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A_{22}}^{-1} \end{bmatrix} \tag{29}$$

It can also be shown that such a reduction $\boldsymbol{G_r}(s)$ is a stable and balanced realization, [7, 8]

$$\|\boldsymbol{G}(s) - \boldsymbol{G_r}(s)\|_\infty \le 2\,(\boldsymbol{\sigma_{r+1}} + \boldsymbol{\sigma_{r+2}} + \cdots + \boldsymbol{\sigma_n}) \tag{30}$$

It can be seen that instead of discarding the "less important" part totally as in the balanced truncation method, the derivative of $x_2$ in the following equation is set to zero, in the singular perturbation approximation (balanced residualization) method, [7, 8].

$$\dot{\boldsymbol{x}}_{\boldsymbol{2}} = \boldsymbol{A_{21}}\boldsymbol{X_1} + \boldsymbol{A_{22}}\boldsymbol{X_2} + \boldsymbol{B_2}\boldsymbol{U} \tag{31}$$

$x_2$ is then solved in terms of $x_1$ and *u*, and is substituted as residual into the state equation of $x_1$ and output equation to obtain the reduced-order system $\boldsymbol{G_r}(s)$ as given above, [7, 8].

This idea is similar to what happens in analysis of singular perturbation systems with:

$$\varepsilon \cdot (\dot{x}_2) = \boldsymbol{A_{21}}\boldsymbol{X_1} + \boldsymbol{A_{22}}\boldsymbol{X_2} + \boldsymbol{B_2}\boldsymbol{U} \tag{31'}$$

where $0 < \varepsilon \ll 1$, and hence the term of singular perturbation approximation.

Let $\boldsymbol{G}(s)$ represent a stable and square system with a state space model $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}]$ of minimal and balanced realization. Then, let the grammians $\boldsymbol{P} = \boldsymbol{Q}$ (32), where $\boldsymbol{\sigma}$ is the smallest Hankel singular value with multiplicity *l* and every diagonal element of $\boldsymbol{\Sigma_1}$ is larger than $\boldsymbol{\sigma}$, [7, 8].

$$\boldsymbol{P} = \boldsymbol{Q} = diag(\boldsymbol{\Sigma_1}, \boldsymbol{\sigma}\boldsymbol{I_1}) \tag{32}$$

### 2.2 The Obtaining of a General Hankel Approximation

Let $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}]$ be compatibly partitioned. Then, a $(n-1)^{\text{th}}$-order system $\boldsymbol{G_h}(s)$ can be constructed with the consideration of the definitions (33) ÷(36) and equations (37)÷(38), as (39) and (42); the way of obtaining a Hankel approximation $\widehat{\boldsymbol{G}}_{\boldsymbol{k}}(s)$ (43) is shown below, while getting an optimal Hankel approximation, as expressed by relations (47), (48) and (51), is further detailed, Datta [11], Verhulst [14].

$$\widehat{\boldsymbol{A}} = \boldsymbol{\Gamma}^{-1}\left(\boldsymbol{\sigma}^{2}\mathbf{A}_{11}^{\mathrm{T}} + \boldsymbol{\Sigma}_{1}\boldsymbol{A}_{11}\boldsymbol{\Sigma}_{1} - \boldsymbol{\sigma}\boldsymbol{C}_{1}^{T}\boldsymbol{U}\boldsymbol{B}_{1}^{T}\right) \tag{33}$$

$$\widehat{\boldsymbol{B}} = \boldsymbol{\Gamma}^{-1}\left(\boldsymbol{\Sigma}_{1}\boldsymbol{B}_{1} + \boldsymbol{\sigma}\boldsymbol{C}_{1}^{T}\boldsymbol{U}\right) \tag{34}$$

$$\widehat{\boldsymbol{C}} = \boldsymbol{C}_{1}\boldsymbol{\Sigma}_{1} + \boldsymbol{\sigma}\boldsymbol{U}\boldsymbol{B}_{1}^{T} \tag{35}$$

$$\widehat{\boldsymbol{D}} = \boldsymbol{D} - \boldsymbol{\sigma}\boldsymbol{U} \tag{36}$$

where $\boldsymbol{U}$ is an ortho normal matrix satisfying:

$$\boldsymbol{B}_{2} = -\boldsymbol{C}_{2}^{T}\boldsymbol{U} \tag{37}$$

$$\boldsymbol{\Gamma} = \boldsymbol{\Sigma}_{1}^{2} - \boldsymbol{\sigma}^{2}\boldsymbol{I} \tag{38}$$

The reduced-order system $\boldsymbol{G}_{\boldsymbol{h}}(s)$ is defined as [13, 14]:

$$\boldsymbol{G}_{\boldsymbol{h}}(s) = \widehat{\boldsymbol{C}}\left(s\boldsymbol{I} - \widehat{\boldsymbol{A}}\right)^{-1}\widehat{\boldsymbol{B}} + \widehat{\boldsymbol{D}} \tag{39}$$

The $(n-l)^{\text{th}}$-order $\boldsymbol{G}_{\boldsymbol{h}}(s)$ is stable and it is an optimal approximation of $\boldsymbol{G}_{\boldsymbol{h}}(s)$ satisfying:

$$\|\boldsymbol{G}(s) - \boldsymbol{G}_{\boldsymbol{h}}(s)\|_{H} = \boldsymbol{\sigma} \tag{40}$$

It is also true that $\boldsymbol{G}(s) - \boldsymbol{G}_{\boldsymbol{h}}(s)$ is all-pass with the infinity-norm, (41):

$$\|\boldsymbol{G}(s) - \boldsymbol{G}_{\boldsymbol{h}}(s)\|_{\infty} = \boldsymbol{\sigma} \tag{41}$$

It can be shown that the Hankel singular values of $\boldsymbol{G}_{\boldsymbol{h}}(s)$ are correspondingly equal to those first $(n-1)$ Hankel singular values of $\boldsymbol{G}_{\boldsymbol{h}}(s)$. Hence, the above reduction formula can be repeatedly applied to get further reduced-order systems with known error bounds. [15]

Let the Hankel singular values of $\boldsymbol{G}_{\boldsymbol{h}}(s)$ be $\boldsymbol{\sigma}_{1} > \boldsymbol{\sigma}_{2} > \cdots > \boldsymbol{\sigma}_{r}$ with multiplicities $m_{\boldsymbol{i}}$, $i = 1, \ldots, r$, i.e. $m_{1} + m_{2} + \cdots + m_{n} = n$. After successive iterations, therefore relation (42) comes out:

$$\boldsymbol{G}(s) = \boldsymbol{D}_{\mathbf{0}} + \boldsymbol{\sigma}_{1}\boldsymbol{E}_{1}(\boldsymbol{s}) + \boldsymbol{\sigma}_{2}\boldsymbol{E}_{2}(\boldsymbol{s}) + \cdots + \boldsymbol{\sigma}_{r}\boldsymbol{r}(\boldsymbol{s}) \tag{42}$$

where $\boldsymbol{D}_{\mathbf{0}}$ is a constant matrix and $\boldsymbol{E}_{\boldsymbol{i}}(s)$, $i = 1, \ldots, r$ are stable, norm-1, all-pass transfer function matrices; $\boldsymbol{E}_{\boldsymbol{i}}(s)$ are the differences at each approximation. We may define reduced-order models, for $k = 1, \ldots, r-1$

$$\widehat{\boldsymbol{G}}_{\boldsymbol{k}}(s) = \boldsymbol{D}_{\mathbf{0}} + \sum_{i=1}^{k}\boldsymbol{\sigma}_{\boldsymbol{i}}\boldsymbol{E}_{\boldsymbol{i}}(\boldsymbol{s}) \tag{43}$$

Such a $\widehat{\boldsymbol{G}}_{\boldsymbol{k}}(s)$ (43) is stable, with the order $m_1 + \ldots + m_k$, and satisfies the condition (44):

$$\left\|\boldsymbol{G}(s) - \widehat{\boldsymbol{G}}_{\boldsymbol{k}}(s)\right\|_{\infty} = \left(\boldsymbol{\sigma}_{\boldsymbol{k}+1} + \boldsymbol{\sigma}_{\boldsymbol{k}+2} + \cdots + \boldsymbol{\sigma}_{\boldsymbol{r}}\right) \tag{44}$$

However, $\widehat{\boldsymbol{G}}_{\boldsymbol{k}}(s)$ (43) is not an optimal Hankel approximation, for $k < r - 1$, [7, 8].

### 2.3 The Obtaining of an Optimal Hankel Approximation

The method to obtain an **optimal Hankel approximation** with “general” order, as expressed by relations (47), (48) and (51), is given below.
Let the Hankel singular values of $\boldsymbol{G}(s)$ be expressed by relation (45):

$$\boldsymbol{\sigma}_{1} \geq \boldsymbol{\sigma}_{2} \geq \cdots \geq \boldsymbol{\sigma}_{\boldsymbol{k}} > \boldsymbol{\sigma}_{\boldsymbol{k}+1} = \cdots = \boldsymbol{\sigma}_{\boldsymbol{k}+\boldsymbol{l}} > \boldsymbol{\sigma}_{\boldsymbol{k}+\boldsymbol{l}+1} \geq \cdots \boldsymbol{\sigma}_{\boldsymbol{n}} \tag{45}$$

Apply appropriate state similarity transformations to make the grammians of $\boldsymbol{G}(s)$ be arranged as according to [7,8]: $\boldsymbol{\Sigma} = diag(\boldsymbol{\sigma}_{1}, \boldsymbol{\sigma}_{2}, \ldots, \boldsymbol{\sigma}_{\boldsymbol{k}}, \boldsymbol{\sigma}_{\boldsymbol{k}+\boldsymbol{l}+1}, \ldots, \boldsymbol{\sigma}_{\boldsymbol{n}}, \boldsymbol{\sigma}_{\boldsymbol{k}+1}, \ldots, \boldsymbol{\sigma}_{\boldsymbol{k}+\boldsymbol{l}})$

Define the last *l* Hankel singular values to be $\boldsymbol{\sigma}$. This $\widehat{\boldsymbol{G}}(s)$ is not stable but has exactly *k* stable poles. The *k*th-order stable part $\boldsymbol{G}_{h.k}(s)$ of $\widehat{\boldsymbol{G}}(s)$, obtained by using modal decompositions say, is a *k*th-order Hankel optimal approximation of $\boldsymbol{G}(s)$ and satisfies the condition (46):

$$\|\boldsymbol{G}(s)-\boldsymbol{G}_{h.k}(s)\|_H=\boldsymbol{\sigma} \tag{46}$$

This may be due to robust controller design methods used subsequently that leads to better closed-loop performance even with a reduced-order plant. The Balanced Truncation method and the Hankel-norm Approximation usually perform better at high frequency, while the singular perturbation approximation (balanced residualization) method performs better in the low- and medium-frequency ranges. [7, 8] If a system is unstable, modal decomposition can be applied first. That is, find a stable $\boldsymbol{G}_s(s)$ and an unstable $\boldsymbol{G}_{us}(s)$ (with all the poles in the closed right-half complex plane) such that: $\boldsymbol{G}(s)=\boldsymbol{G}_s(s)+\boldsymbol{G}_{us}(s)$

$$\boldsymbol{G}(s)=\boldsymbol{G}_s(s)+\boldsymbol{G}_{us}(s) \tag{47}$$

Then, $\boldsymbol{G}_s(s)$ can be reduced to $\boldsymbol{G}_{sr}(s)$, by using any methods presented, and a reduced-order system of the original $\boldsymbol{G}(s)$ can be formed as:

$$\boldsymbol{G}_r(s)=\boldsymbol{G}_{sr}(s)+\boldsymbol{G}_{us}(s) \tag{48}$$

The formulae introduced are for continuous-time systems. In the case of discrete-time systems, the grammians are calculated from the discrete Lyapunov equations instead, [7]:

$$\boldsymbol{APA^T}-\boldsymbol{P}+\boldsymbol{BB^T}=\boldsymbol{0} \tag{49}$$

$$\boldsymbol{A^TQA}-\boldsymbol{Q}+\boldsymbol{C^TC}=\boldsymbol{0} \tag{50}$$

The balanced truncation method can then be applied similar to the case of continuous time. However, it should be noted that the reduced-order system is no longer in a balanced realization form, though the same error bound still holds, [7, 8]

$$\boldsymbol{G}_r(s)=[\boldsymbol{A_r},\boldsymbol{B_r},\boldsymbol{C_r},\boldsymbol{D_r}] \tag{51}$$

For using the singular perturbation approximation (balanced residualization) on a system with zero $\boldsymbol{D}$-matrix, the reduced-order system $\boldsymbol{G}_r(s)$ can be instead defined (51) by the means of equations (52)÷(55):

$$\boldsymbol{A_r}=\boldsymbol{A_{11}}+\boldsymbol{A_{12}}(\boldsymbol{I}-\boldsymbol{A_{22}})^{-\boldsymbol{1}}\boldsymbol{A_{21}} \tag{52}$$

$$\boldsymbol{B_r}=\boldsymbol{B_1}+\boldsymbol{A_{12}}(\boldsymbol{I}-\boldsymbol{A_{22}})^{-\boldsymbol{1}}\boldsymbol{B_2} \tag{53}$$

$$\boldsymbol{C_r}=\boldsymbol{C_1}+\boldsymbol{C_2}(\boldsymbol{I}-\boldsymbol{A_{22}})^{-\boldsymbol{1}}\boldsymbol{A_{21}} \tag{54}$$

$$\boldsymbol{D_r}=\boldsymbol{C_2}(\boldsymbol{I}-\boldsymbol{A_{22}})^{-\boldsymbol{1}}\boldsymbol{B_2} \tag{55}$$

Such a reduced-order system is still in a balanced realization and enjoys the same error bound. For instance, in case of balanced transformation, in order to avoid numerical instability of forming products $\boldsymbol{BB^T}$ and $\boldsymbol{C^TC}$, algorithms for direction calculation of the Choleski factors and improved balanced truncation schemes are used, Datta [11], Doyle [12].

## 3. FREQUENCY ANALYSIS OF UNCERTAIN SYSTEM

### 3.1 A Summary of the Frequency Analysis

The frequency responses of the perturbed open-loop system may be computed by using a few different values of the perturbation parameters. Values of each perturbation are chosen, the corresponding open-loop transfer function matrices are generated and frequency responses calculated and plotted. Such as gain and phase margins (and their generalizations) help to quantify the sensitivity of stability and performance versus the model uncertainty, which is the imprecise knowledge of the way and the extent that the control input directly affects the feedback variables. Reducing the effects of some forms of uncertainty (low-frequency disturbances) without increasing the problematic effects of other dominant forms (model uncertainty, sensor noise) is the primary goal of the feedback control system. Closed-loop stability is the way to deal with the uncertainty in initial conditions or arbitrarily small disturbances. Frequency-domain uncertainty is a form of model uncertainty, which often quantifies model uncertainty by describing absolute or relative uncertainty in the process's frequency response. The real problem in robust multivariable feedback control system design is to synthesize a control law which maintains system response and error signals to within pre-specified tolerances, despite the effects of uncertainty on the system. Uncertainty may take many forms, but the most significant include noise / disturbance signals and transfer function modeling errors. Another source of uncertainty is the simplicity and lack of accuracy in modeling the nonlinear distortion. Uncertainty in any form is no doubt the major issue in most control system designs. For a continuous time system, the state-feedback law $\boldsymbol{u}$ (56) minimizes the quadratic cost function (57), [1]:

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x} \tag{56}$$

$$\boldsymbol{J}(\boldsymbol{u}) = \int_{\boldsymbol{0}}^{\infty} \left(\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^{\boldsymbol{T}}\boldsymbol{R}\boldsymbol{u} + \boldsymbol{2}\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{N}\boldsymbol{u}\right)dt \tag{57}$$

subject to the system dynamics, (58) and (59):

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{58}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \tag{59}$$

The solution $\boldsymbol{S}$ of the associated Riccati equation is given by equation (60):

$$\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{S} + \boldsymbol{S}\boldsymbol{A} - (\boldsymbol{S}\boldsymbol{B} + \boldsymbol{N})\boldsymbol{R}^{-\boldsymbol{1}}\left(\boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{S} + \boldsymbol{N}^{\boldsymbol{T}}\right) + \boldsymbol{Q} = \boldsymbol{0} \tag{60}$$

and the closed-loop eigenvalues $e$ (61); $\boldsymbol{K}$ is derived from $\boldsymbol{S}$ using the relation (62):

$$e = eig(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}) \tag{61}$$

$$\boldsymbol{K} = \boldsymbol{R}^{-\boldsymbol{1}}\left(\boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{S} + \boldsymbol{N}^{\boldsymbol{T}}\right) \tag{62}$$

### 3.2 Bode Plot in Case of a Dynamic System Model

In case of a dynamic system model, Bode plot will reveal the frequency response, magnitude and phase of frequency response. The plot displays the magnitude (in dB) and phase (in degrees) of the system response as a function of frequency. When system is a multi-input, multi-output (MIMO) model, bode produces an array of Bode plots, each plot showing the frequency response of one I/O pair. Bode determines the plot frequency range based on system dynamics. For continuous-time systems, *bode* evaluates the frequency response on the imaginary axis $s = j\omega$ and considers only positive frequencies. Bode computes the frequency response by evaluating the gain and phase of the frequency response based on the zero, pole, and gain data for each input/output channel of the system. The phase margin measures the system tolerance to time delay. If there is a time delay greater than $180/W_{pc}$ in the loop (where $W_{pc}$ is the frequency where the phase shift is 180 deg), the system will become unstable in closed-loop. The time delay $\tau_d$ can be thought of as an extra block in the forward path of the block diagram that adds phase to the system but has no effect on the gain. That is, a time delay can be represented as a block with magnitude of 1 and phase $\omega_{\tau d}$ [rad/s]. The phase margin is the difference in phase between the phase curve and -180 degrees at the point corresponding to the frequency that gives us a gain of 0 dB (the gain crossover frequency, $W_{gc}$). The gain margin is the difference between the magnitude curve and 0 dB at the point corresponding to the frequency that gives us a phase of -180 degrees (the phase crossover frequency, $W_{pc}$). The major point of interest is steady-state error. The steady-state error can be observed directly off the Bode plot as well.

The constant $(K_p, K_v, K_\alpha)$ is found from the intersection of the low frequency asymptote with the $w=1$ line. The graphical interpretation is based on extending the low frequency line to the $w=1$ line, where the magnitude at this point is a constant. Since the Bode plot of this system is a horizontal line at low frequencies (slope = 0), this indicates us that this system is of type zero. The phase margin is defined as the change in open-loop phase shift required at unity gain to make a closed-loop system unstable. Gain margin is a factor by which the gain of a stable system is allowed to increase before the system reaches instability. Gain margin is defined as the magnitude of reciprocal of the open-loop transfer function evaluated at the frequency $\omega_2$ at which the phase angle is – 180º. Phase margin of a stable system is the amount of additional phase lag required to bring the system to point of instability. For a stable system both gain margin and phase margin should be positive.

Using frequency methods, it is possible to determine a great deal of information from the open-loop transfer function. One of the most important facts about a given system which may be determined via frequency methods is the relative stability of the system. The gain and phase margins are given in terms of how much either may be increased before instability results. When these metrics are negative, the system is already unstable, and the values given represent the amount by which either must be reduced to achieve stability.

The gain margin is determined by locating the phase crossing of the -180 degrees -line, and drawing a vertical line up to the corresponding magnitude plot. The distance between the 0 -dB line and the magnitude plot is the gain margin. The frequency at which this crossing occurs is called the gain margin frequency. The root-locus can be used to determine the value of the loop gain, which results in a satisfactory closed-loop behavior.

## 4. NUMERICAL SIMULATIONS AND CONCLUSIONS

The state-space model as an LTI object has two inputs (rudder, aileron) and two outputs (yaw, bank angle). Compute the open-loop eigenvalues and plot them in the ***s***-plane.

Open-loop analysis is very important and study time response. The aircraft is oscillating around a nonzero bank angle. Thus, the aircraft is turning in response to an aileron impulse.

There are practical limits as to how large the gain can be made. In fact, very high gains lead to instabilities. If the root-locus plot is such that the desired performance cannot be achieved by the adjustment of the gain, then it is necessary to reshape the root-loci by adding the additional controller to the open-loop transfer function. Additional controller must be chosen so that the root-locus will pass through the proper region of the *s*-plane. In many cases, the speed of response and the damping of the uncompensated system must be increased in order to satisfy the specifications. This requires moving the dominant branches of the root locus to the left.

Plot the impulse response and time-domain specifications. The compensator is a static gain; try to determine appropriate gain values using the root locus technique. Plot the root locus for the rudder to yaw channel. Plot the closed-loop impulse response for duration of few seconds, and compare it to the open-loop impulse response.

Now close the loop on the full MIMO model and see how the response from the aileron looks. The feedback loop involves input 1 and output 1 of the plant. Plot the MIMO impulse response. Connect the washout in series with the design model - relation between input 1 and output 1- to obtain the open-loop model and draw another root locus for this open-loop model. Plot the closed-loop response from rudder to yaw rate. Finally, plot the closed-loop impulse response.

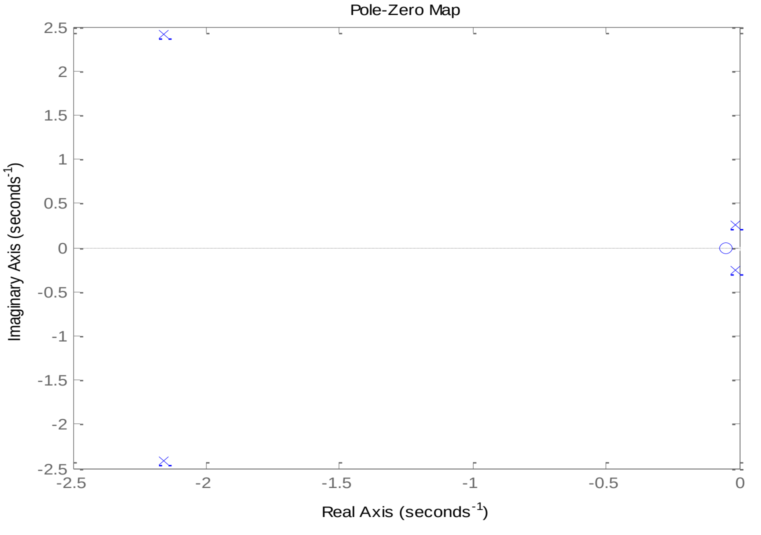


Fig. 1 - Low frequencies line indicating a zero type system

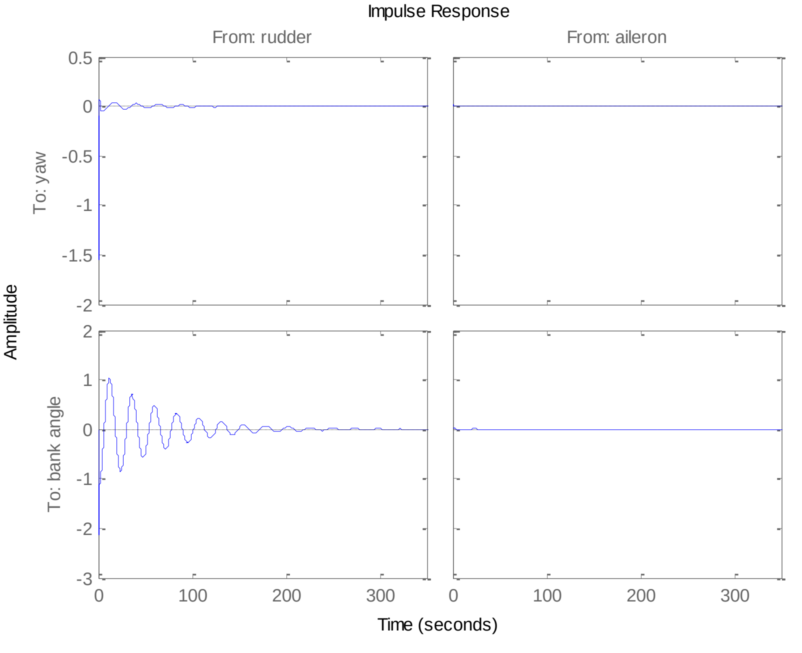


Fig. 2 - Impulse response, from rudder (*left*) and aileron (*right*), case # 1

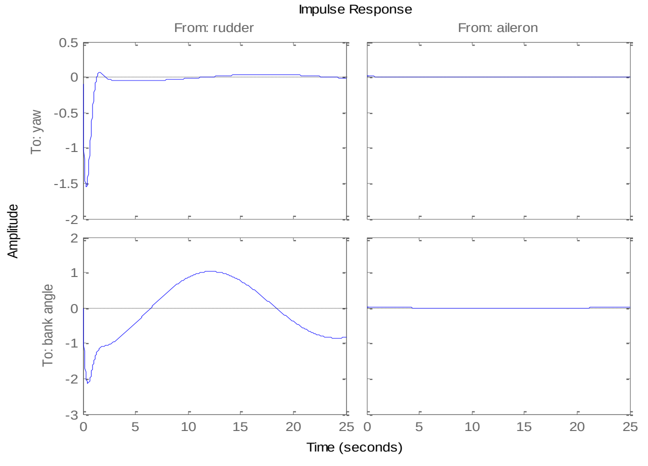


Fig. 3 - Impulse response, from rudder (*left*) and aileron (*right*), case # 2

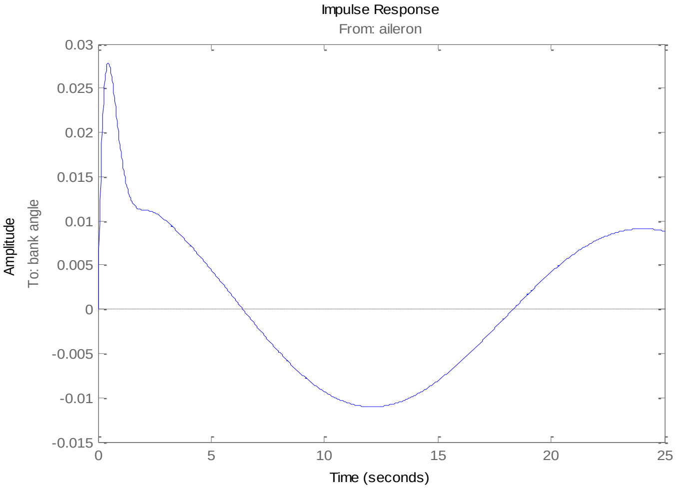


Fig. 4 - Impulse response, from aileron to bank angle

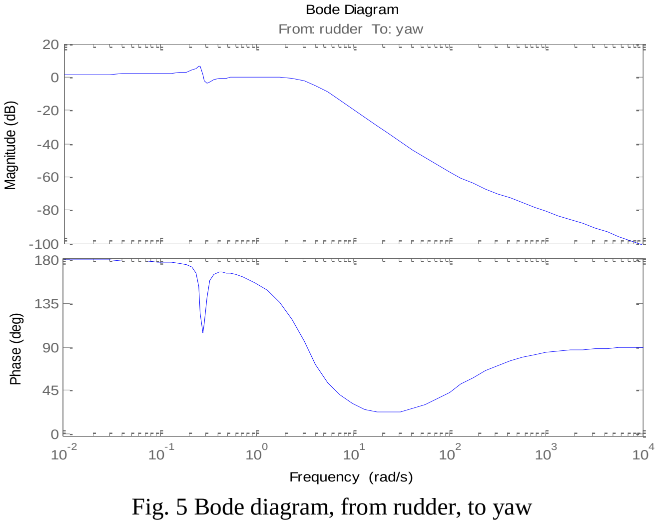


Fig. 5 Bode diagram, from rudder, to yaw

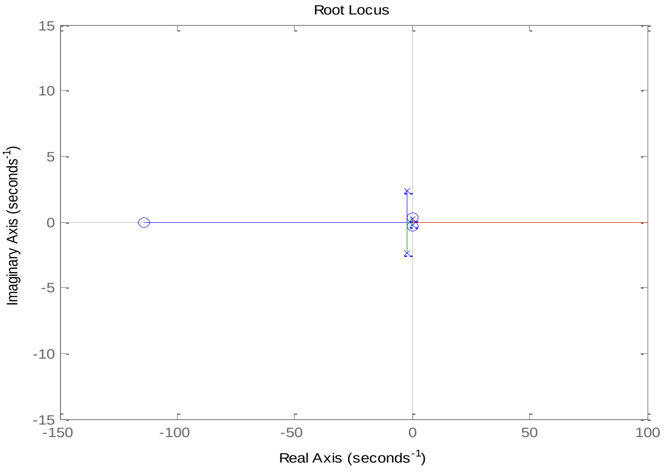


Fig. 6 - Root locus

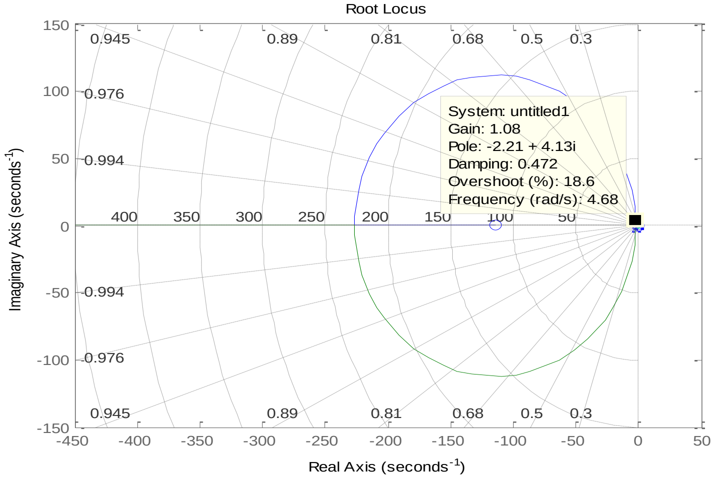


Fig. 7 - Root locus, detailed: gain 1.08, damping 0.472

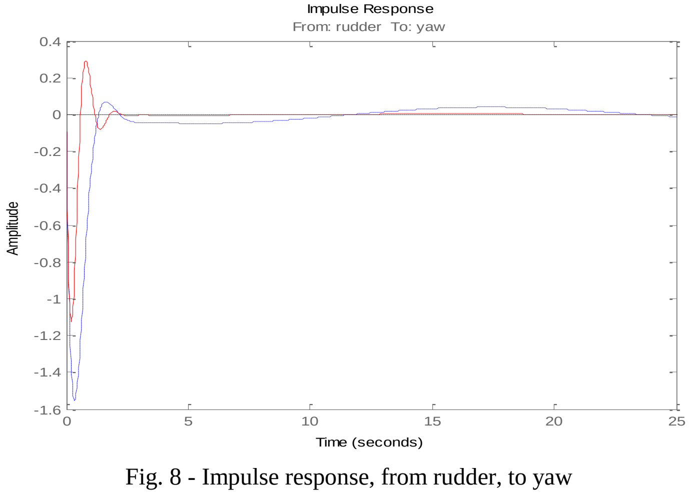


Fig. 8 - Impulse response, from rudder, to yaw

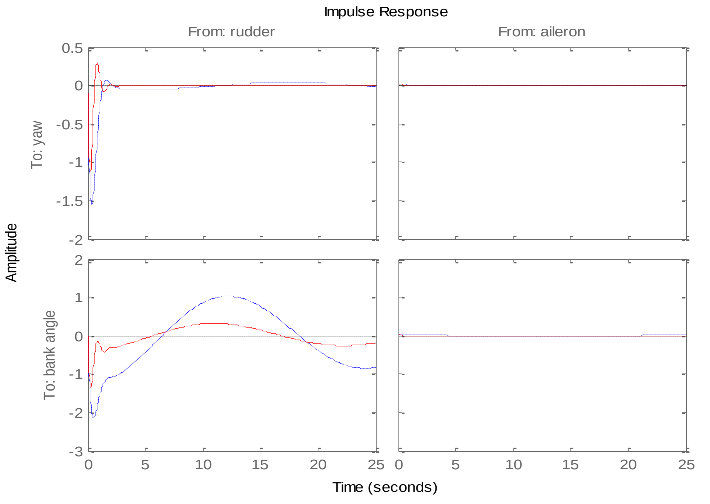


Fig. 9 - Impulse response, from rudder (*left*) and aileron (*right*), to yaw (*up*) and bank angle (*down*)

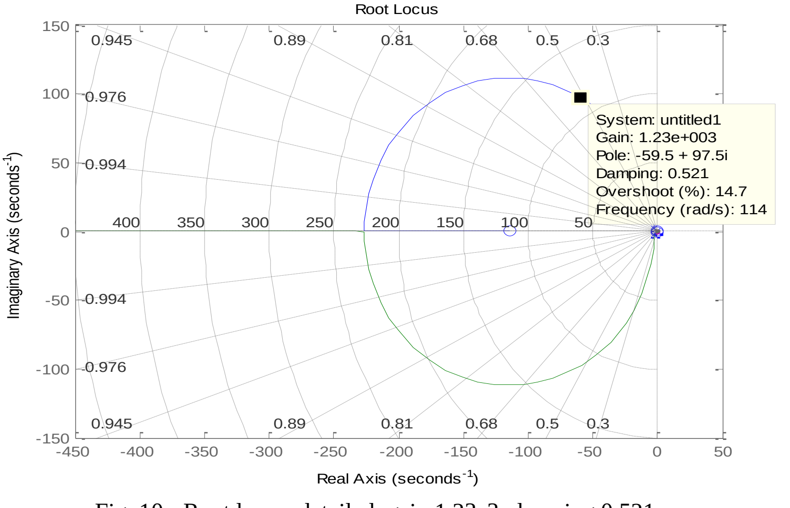


Fig. 10 - Root locus, detailed: gain 1.23e3, damping 0.521

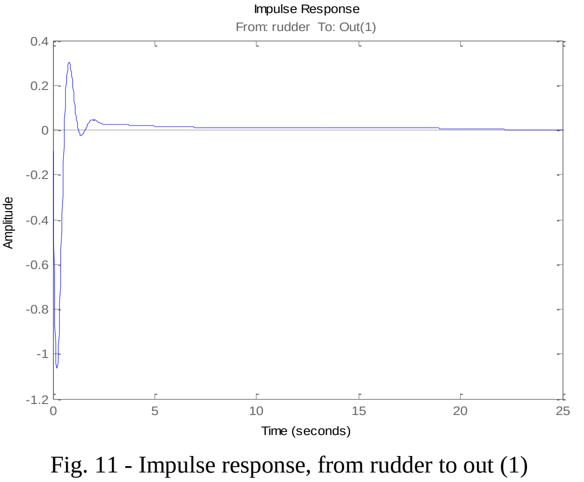


Fig. 11 - Impulse response, from rudder to out (1)

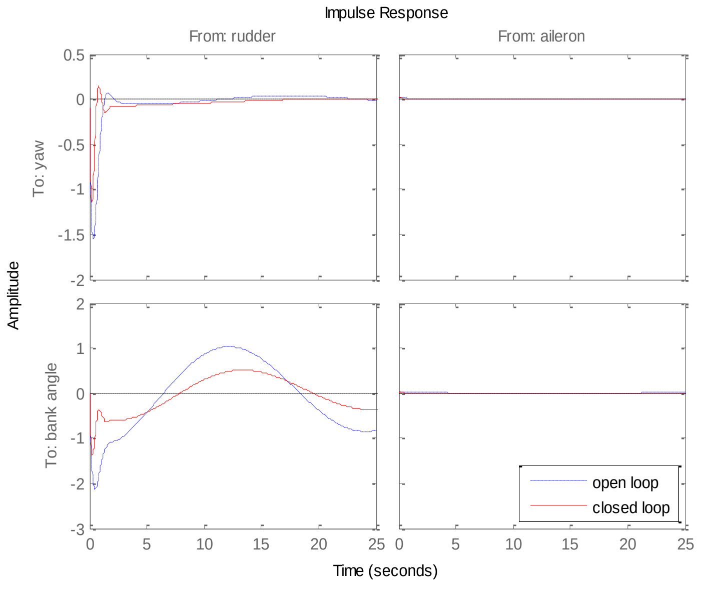


Fig. 12 - Impulse response, from rudder (*left*) and aileron (*right*), to yaw (*up*) and bank angle (*down*), open loop versus closed loop

Using frequency methods, it is possible to determine a large amount of information from the open-loop transfer function.

One of the most important facts related to a given system which may be determined via frequency methods is the relative stability of the system.

The frequency responses of the perturbed open-loop system may be computed by using a few different values of the perturbation parameters. The values of each perturbation are chosen, the corresponding open-loop transfer function matrices are generated and then frequency responses are calculated and plotted.

Gain and phase margins (and their generalizations) help to quantify the sensitivity of stability and performance versus the model uncertainty, which is due to the imprecise knowledge of how the control input directly affects the feedback variables. Reducing the effects of some forms of uncertainty (low-frequency disturbances) without the troublesome issue of increasing the effects of other dominant forms (model uncertainty, sensor noise) represents the primary goal of the feedback control system. Closed-loop stability is the way to deal with the uncertainty in initial conditions or arbitrarily small disturbances.

## REFERENCES

[1] * * * http://www.mathworks.com/control/ref/bode.html.
[2] F. Auger, P. Flandrin, P. Goncalves and O. Lemoine, *Time-Frequency Toolbox for use with MATLAB*, Rice University (USA).
[3] M. Rauw, *FDC 1.2. - A SYMULINK Toolbox for Flight Dynamics and Control Analisys*, 2001.
[4] B. Etkin, *Dynamics of Flight-Stability and Control*, Wiley, New York, USA, 2-nd edition, 1982.
[5] D. McLean, *Automatic Flight Control System*, Prentice Hall, Hertfordshire, UK, 1990.
[6] L. Moysis, M. Tsiaousis, N. Charalampidis, M. Eliadou, I. Kafetzis, *An Introduction to Control Theory Applications with MATLAB*, http://users.auth.gr/lazarosm/, 31 August 2015.
[7] G. Balas, R. Chiang, A. Packard, M. Safonov, *Robust Control Toolbox For Use with MATLAB*, by The MathWorks, Inc., 2006.
[8] R. Y. Chiang, M. Safonov, *MATLAB- Robust Control Toolbox*, by The MathWorks, Inc.

[9] C. W. Rowley, B. A. Batten, *Dynamic and Closed-Loop Control*, Fundamentals and Applications of Modern Flow Control, Progress in Astronautics and Aeronautics, AIAA, 2009.
[10] G. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, Prentice-Hall, 5th edition, 2005.
[11] B. N. Datta, *Numerical Methods for Linear Control Systems*, Elsevier, San Diego, CA, 2004.
[12] J. C. Doyle, B. A. Francis and A. R. Tannenbaum, *Feedback Control Theory*, Macmillan, 1992.
[13] K. J. Åström and R. M. Murray, *Feedback Systems: An Introduction for Scientists and Engineers*, 2005.
[14] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Universitext, Springer-Verlag, 2nd edition, 1996.
[15] G. E. Dullerud and F. Paganini, *A Course in Robust Control Theory- A Convex Approach*, Volume **36** of Texts in Applied Mathematics, Springer-Verlag, 1999.