A review of some basic aspects related to integration of airplane's equations of motion

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Abstract: *Numerical integration of the airplane's equations of motion has long been considered among the most fundamental calculations in airplane's analysis. Numerical algorithms have been implemented and experimentally validated. However, the need for superior speed and accuracy is still very topical, as, nowadays, various optimization algorithms rely heavily on data generated from the integration of the equations of motion and having access to larger amounts of data can increase the quality of the optimization. Now, for a number of decades, engineers have relied heavily on commercial codes based on automatically selected integration steps. However, optimally chosen constant integration steps can save time and allows for larger numbers of integrations to be performed. Yet, the basic papers that presented the fundamentals of numerical integration, as applied to airplane's equations of motion are nowadays not easy to locate. Consequently, this paper presents a review of basic aspects related to the integration of airplane's equation of motion. The discussion covers fundamentals of longitudinal and lateral-directional motion as well as the implementation of some numerical integration methods. The relation between numerical integration steps, accuracy, computational resource usage, numerical stability and their relation with the parameters describing the dynamic response of the airplane is considered and suggestions are presented for a faster yet accurate numerical integration.*

Key Words: *Flight Dynamics Simulation, Controls*

1. INTRODUCTION

The equations of motion of an airplane (or rocket) that will be subsequently utilized within this paper are written in the classical manner, presented, for example, in Ref. [1-3], and also utilize d in Refs. [4] and [5].

Numerical integration of the airplane's equations of motion has long been considered among the most fundamental calculations in airplane's analysis. Since the 1950s and, on a larger scale, in the 1960s, many numerical algorithms have been implemented and experimentally validated.

The numerical integration of the airplane's equations of motion has been considered fundamental, with only basic issues being covered by most engineering classes at a medium level, and students typically obtain numerical solutions using different (user-friendly) commercial packages, such as Matlab.

Solutions based on commercial software packages, however, utilize built-in integration-step calculation algorithms, so, in most cases, engineers no longer deal with manually calculating best integration steps for the problems they work on.

Such an approach allows engineers to focus on other aspects, (as the software package works automatically, after the equations to be solved are defined), and in many cases, the additional time required for solving the problems using variable step algorithms are not of a great concern.

On the other hand, the need for superior speed and accuracy is more actual than ever as, nowadays, various optimization algorithms Ref. [6-8] rely heavily on data generated from the integration of the equations of motion.

In such a case, adopting accurately calculated constant integration steps can be very beneficial, yielding reductions of the integration time.

Choosing the best constant step for a numerical integration, although considered trivial in the not that distant past, seems to create difficulties to engineers that, for decades, have been used to relay, for most problems, on user-friendly code development environments and easy to use variable step commercial routines.

On the other hand, papers from 1960s or early 1970s that laid the foundations of numerical integration of airplane's equations of motion seem to be hard to find. Consequently, this paper covers some basic aspects related to the numerical integration methods as they apply to the equation of motion of an airplane.

2. THE MATHEMATICAL MODELS

In the symmetric flight of an airplane, the governing equations for the variables describing the longitudinal motion can be separated from the equations of the lateral motion. The equations of motion of the longitudinal and lateral channel, respectively, can be written, for example, as, Refs. [1-4]

$$
\begin{aligned}
&\frac{G}{g}\frac{dV}{dt} = T\cos(\alpha+\tau) - qSC_D - G\sin\gamma \\
&\frac{G}{g}V\dot{\gamma} = T\sin(\alpha+\tau) - qS(C_{L0} + C_{L\alpha}\alpha + C_{L\delta}\delta + C_{Lp}p + C_{L\dot{\alpha}}\dot{\alpha}) \\
&\qquad - G\cos \\
&J_y\dot{q} = qSc(C_{m0} + C_{m\alpha}\alpha + C_{m\delta}\delta + C_{mq}q + C_{m\dot{\alpha}}\dot{\alpha}) \\
&\dot{\alpha} = q - \dot{\gamma} \\
&\dot{z} = V\sin\gamma \\
&\dot{x} = V\cos\gamma
\end{aligned} \tag{1}
$$

and

$$
\begin{aligned}
\dot{v} &= \left(\frac{Y_v}{m}\right)v + \left(\frac{Y_p}{m}\right)p + \left(\frac{Y_r}{m} - u_0\right)r + \left(g\cos\theta_0\right)\phi \\
\dot{p} &= \left(\frac{L_v}{I_x^{'}} + I_{zx}^{'}N_v\right)v + \left(\frac{L_p}{I_x^{'}} + I_{zx}^{'}N_p\right)p + \left(\frac{L_r}{I_x^{'}} + I_{zx}^{'}N_r\right)r \\
\dot{r} &= \left(\frac{N_v}{I_x^{'}} + I_{zx}^{'}L_v\right)v + \left(\frac{N_p}{I_x^{'}} + I_{zx}^{'}L_p\right)p + \left(\frac{N_r}{I_x^{'}} + I_{zx}^{'}L_r\right)r \\
\dot{\phi} &= p + r\tan(\theta_0) \\
\dot{\psi} &= r\sec(\theta_0) \\
\Delta\dot{y}_E &= u_0\psi\cos\theta_0 + v
\end{aligned}
\tag{2}
$$

where, as known, the stability analysis usually shows both complex solutions (corresponding to oscillatory mode) and real solutions that indicate modes with stable or unstable exponential variations. As mentioned above, equations (1) and (2) are, nowadays, routinely solved using various numerical algorithms that are implemented in a multitude of subroutines included in many software packages. Fundamentals of the numerical integration are presented, for example, in Ref. [9], and improved algorithms are described, for example, in Refs. [10] and [11]. Analytical (or semi-analytical) solutions for flow and motion problems, illustrated, for example, in Refs. [12] and [13], are best choices, however, they are rarely available. So in most cases, engineers rely heavily on numerical integrations. References [9] and [14] contain, once again, wide descriptions of the most important aspects related to numerical methods. To illustrate the main steps and the complexity of the process some basic aspects related to the automatic selection of the integration steps are briefly listed below, as indicated in Refs. [9] and [14]. Ordinary differential equations in canonical form

$$y' = f(x, y(x)) \tag{3}$$

are, most usually, integrated using Runge-Kutta algorithms,

$$
\begin{aligned}
y_{n+1} &= y_n + \sum_{j=1}^{m} P_{mj} h k_j \\
k_j &= f\left(x_0 + h\alpha_j, y_0 + \sum_{i=1}^{j-1} \beta_{ji} h k_i\right)
\end{aligned}
\tag{4}
$$

or using Adams predictor-corrector algorithms,

$$
\begin{aligned}
& y_{n+1} = y_{n-k} + \int_{x_{n-k}}^{x_{n+1}} y'\,dx = y_{n-k} + \int_{x_{n-k}}^{x_{n+1}} f(x, y(x))dx \\
& x = x_n + \alpha h, \alpha \le 1 \\
& y_{n+1} = y_{n-k} + h\int_{-k}^{1} f_n + \alpha\nabla f_n + \ldots + (-1)^r C_{-\alpha}^{r} \nabla^r f_m d\alpha \\
& x = x_{n+1} + \alpha h, \alpha \le 0 \\
& y_{n+1} = y_{n-k} + h\int_{-k-1}^{1} f_{n+1} + \alpha\nabla f_{n+1} + \ldots + (-1)^r C_{-\alpha}^{r} \nabla^r f_{m+1} d\alpha
\end{aligned}
\tag{5}
$$

The integration step, *h*, must be small enough to allow for an accurate representation of fast variables.

The error of an *s* order Runge-Kutta method, Ref. [14]

$$e = h^{s+1} y^{(s+1)}(\xi)/(s+1)! \tag{6}$$

can be estimated as Ref. [14]

$$e = \frac{y_h - y_{2h}}{2^s - 1} \tag{7}$$

If the maximum acceptable error is e_0, the next step for reducing the current error toward e_0 is Ref. [14]

$$h_0 = Sh\left(\frac{e_0}{e}\right)^{\frac{1}{s+1}} \tag{8}$$

where according to Ref. [14], the recommended value for *S*,

$$S = \left|\frac{y^{(s+1)}(\xi_0)}{y^{(s+1)}(\xi)}\right|^{\frac{1}{s+1}} \tag{9}$$

is $S = 0.9$; alternately, for a fourth order Runge-Kutta algorithm, good results have been obtained if the size of the new step h is selected as, Ref. [14]

$$\left|\frac{K_2 - K_3}{K_1 - K_2}\right| \approx 0.01 \tag{10}$$

In the Adams algorithm, with finite differences up to order 3, Ref. [9], the predictor step

$$y_{n+1} = y_n + \frac{h}{24}\left(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right) \tag{11}$$

is followed by the corrector step

$$y_{n+1} = y_n + \frac{h}{24}\left(9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}\right) \tag{12}$$

The errors in the predictor and corrector step are, respectively, Ref. [9]

$$e = \frac{251}{720} h^5 y^{(5)}(\xi) \text{ and } e = \frac{19}{720} h^5 y^{(5)}(\xi) \tag{13}$$

Continuously selecting the integration steps implies calculating the parameters listed in Eqs. (6-10) and (11-13) at every time step and changing the integration steps until the desired accuracy is reached.

The process involves calculating data that, mostly, are not further used in the integration process itself, and although some “smart” coding rules can be used to allow some of the data generated during the step verification algorithm to be utilized in integration, the variable step integration still takes longer that a constant time integration.

Choosing the time steps in constant step integration is not straightforward; it is trivial to say that choosing a time step too large generates errors, while choosing too small time steps results in big data and long integration times without providing increases in accuracy (at the very best; if the number of steps increase way too much, the accuracy can even decrease due to accumulation of errors).

The best time step must be chosen according to the nature of the mathematical model that is studied.

Simply playing with the steps until the variations of the solution are small does not mean that the solution is accurate. A judgment must always be made based upon the nature of the equations that are studied.

3. RESULTS AND DISCUSSIONS

As it is well known, both the modes of the longitudinal motion and the modes of the lateral-directional motion include complex solutions (corresponding to oscillatory modes) as well as real solutions (modes with stable or unstable exponential variations).

The steps of the numerical integration algorithms must therefore be chosen accordingly. One of the most important variables in the numerical studies of oscillations is the "Nyquist frequency", based on the "Nyquist sampling theorem", which states, trivially said, that it is impossible to detect an oscillation without being able to specify at least 2 points per period.

The short discussion that follows illustrates the relations between the Nyquist frequency, integration steps and accuracy.

Figures 1 to 3 illustrate the exact solution for an ODE similar to the equations that describe the motion of an airplane and the numerical solutions obtained using fourth order Runge-Kutta algorithms and various time-steps.

It can be easily seen that, limiting the time step to just twice the value corresponding to "Nyquist frequency" yields virtually no useful results (figure 1).

The time step that generates 6 points per semi-period provides results that are close to the exact solutions, however, phase errors are present (figure 2).

Choosing 10 points per cvasiperiod improves significantly the accuracy, without costly increases in integration time (figure 3).

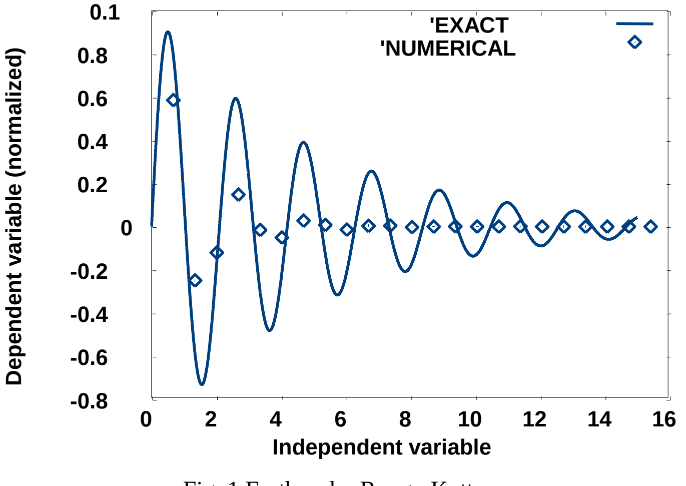

Fig. 1 Forth order Runge-Kutta,
time step = ("mode's period") /3, i.e. "4 points per period" (or "twice Nyquist")

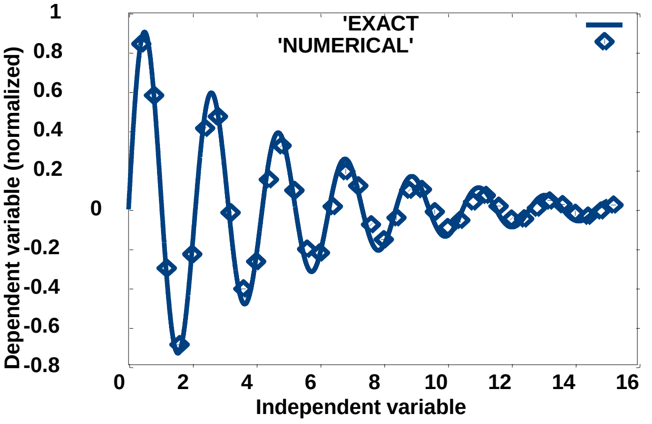

Fig. 2 Forth order Runge-Kutta,
time step = ("mode's period") /5 i.e. "6 points per period" (or "three times Nyquist")

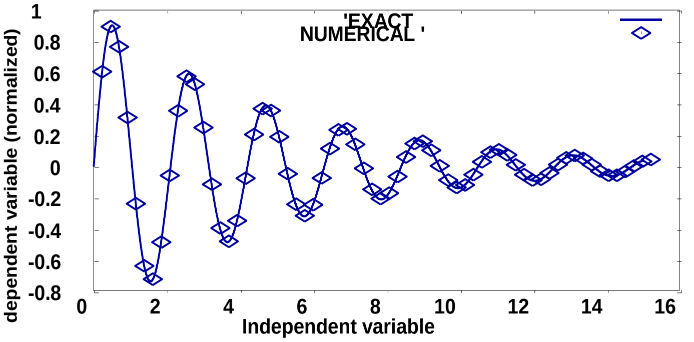

Fig. 3 Forth order Runge-Kutta,
time step = ("mode's period") /9 i.e. "10 points per period" (or "five times Nyquist")

3. CONCLUSIONS

Equations describing the airplane's motion are, nowadays, routinely solved using various variable step numerical integration algorithms that are implemented in a multitude of subroutines included in many software packages.

The need for superior speed and accuracy is very important in optimization the algorithms that utilize data generated from the integration of the equations of motion. In such a case, adopting accurately calculated constant integration steps can be very beneficial, yielding reductions of the integration time.

Choosing the time step to just twice the value corresponding to "Nyquist frequency" yields to no results; lowering the time step to 6 points per semi-period provides results that are close to the exact solutions, however, phase errors are present; using 10 points per cvasiperiod improves significantly the accuracy.

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