On the synthesis of the pilot optimal control model

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Abstract: *The study continues some work of the authors, this time performing a synthesis of optimal control model of the human pilot in systems with input delay, by removing the Padé or Hess approximations characterizing the pilot structural central nervous block and their introduction as a pure delay block. On the one hand, the method ensures a better accuracy of synthesis and on the other hand is advantageous with respect to general results to date for time delay systems since: a) the optimal control law is given explicitly and b) the Riccati equations for the gain matrices do not contain any time advanced or delayed arguments. The approach is stimulated by recent works of M. Basin and his collaborators.*

Key Words: *LQG control, optimal model of human pilot, PIO, input delay, Kalman filtering, separation principle, duality principle.*

1. INTRODUCTION

Recent papers of the authors [1]-[3] have considered some aspects of defining and evaluating the so called optimal control model (OCM) of the human pilot [4], [5], as a general chapter of the "**P**ilot-**I**nduced-**O**scillation" PIO) phenomenon study [6]. The present paper continues the work [1]-[3], this time proposing a synthesis of OCM in systems with input delay [7]-[16]. The basic novelty consists of removing the Padé or Hess [4] approximations for characterizing the "central nervous block" component and of their introduction as a pure delay block, $u(t) \to u(t-\tau)$.

The paper is organized as follows. In Section 2, the synthesis of pilot OCM is defined as a problem for linear systems with time delay in control input. In Section 3, the solution of the problem is proved, in two steps, by making reference to two principles in the control theory: a) the separation principle and b) the duality principle. A conclusion Section 4 ends the paper.

2. SYNTHESIS OF THE PILOT OCM IN THE FRAMEWORK OF THE SYSTEMS WITH TIME DELAY IN CONTROL INPUT

The block diagram for the pilot-vehicle system is shown in Fig. 1. The aircraft dynamics is written in the form of the well known invariant linear system [1]-[3]

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{Ax}(t) + \boldsymbol{B}\delta(t) + \boldsymbol{Ew}(t),\ \boldsymbol{y}_o(t) = \boldsymbol{Cx}(t) + \boldsymbol{D}\delta(t) + \boldsymbol{v}_y(t) := \boldsymbol{y}(t) + \boldsymbol{v}_y(t) \tag{1}$$

The approach of the pilot modeling is based on the argument, experimentally proved, that the pilot behaves "optimally". Thus, the conceptual pilot dynamics supposes (Fig. 1): a) a "mental" component, analogous to a Linear Quadratic Gaussian (LQG) controller, b) a decisional, central nervous component and c) an actuating, neuromuscular component. In [1], the "central nervous block" of the pilot model was placed after the neuromuscular block and was described as a Padé approximation, in accordance with the approach in [4] concerning the so-called "Modified Optimal Control Model" (MOCM) (s is the Laplace operator)

$$\delta(s)/u_d(s) = \left(1 - \tau s/2 + (\tau s)^2/8\right) \Big/ \left(1 + \tau s/2 + (\tau s)^2/8\right) \tag{2}$$

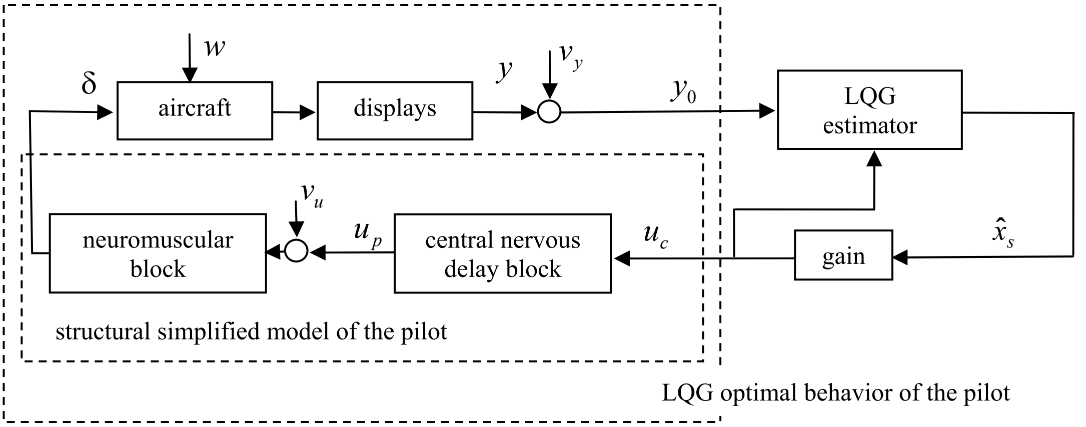

Fig. 1 – Conceptual block diagram for the pilot-vehicle system

In [2], in the so-called "Hess's LQG based pilot model", the same block was placed before the neuromuscular block, (Fig. 1), and the approximation (2) used is slightly different

$$u_p(s)/u_c(s) = \left(1 - \tau s/2 + (\tau s)^2/16\right) \Big/ \left(1 + \tau s/2 + (\tau s)^2/16\right) \tag{2'}$$

(τ is the effective time delay in the decisional, central nervous component, incorporating the "computing" time , u_p is the pilot's delayed control input, u_c is the pilot's commanded control).

Another difference between the two methods concerns the definition of the cost index. In the pilot OCM, the pilot's control task is modeled as the minimization of a standard quadratic performance index negotiating the pilot's observations and the pilot's commanded control u_c .

In the pilot MOCM, the pilot's control task contains a supplementary component – the pilot's commanded control-rate; see details in [1], [2].

The approach in this paper (Fig. 1) is to eliminate the Padé or Hess approximations of the central nervous block and to assume the natural representation

$$u_p(t) = u_c(t - \tau) \tag{2''}$$

Further on, the neuromuscular block will be modeled, as usually

$$\dot{\delta}(t) = -\delta(t)/\tau_\eta + u_c(t - \tau)/\tau_\eta \,,\, u_d(t) = u_p(t) + v_u(t) \tag{3}$$

(τ_η is the neuromuscular lag) and methodologically will be considered as part of the plant dynamics (1).

Thus, the two blocks (1) and (3), in state space form, are given by the system

$$\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\ \dot{\delta}(t)\end{bmatrix}=\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\boldsymbol{C}_d\\ 0 & -1/\tau_\eta\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \delta(t)\end{bmatrix}+\begin{bmatrix}0\\ 1/\tau_\eta\end{bmatrix}u_c(t-\tau)+\begin{bmatrix}\boldsymbol{E} & 0\\ 0 & 1/\tau_\eta\end{bmatrix}\begin{bmatrix}\boldsymbol{w}(t)\\ \boldsymbol{v}_u(t)\end{bmatrix}$$
$$\boldsymbol{y_o}(t)=\begin{bmatrix}\boldsymbol{C} & \boldsymbol{D}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \delta(t)\end{bmatrix}+\boldsymbol{v}_y(t):=\boldsymbol{y}(t)+\boldsymbol{v}_y(t) \tag{4}$$

or, in matrix form

$$\dot{\boldsymbol{x}}_s(t)=\boldsymbol{A}_s\boldsymbol{x}_s(t)+\boldsymbol{B}_s u(t-\tau)+\boldsymbol{E}_s\boldsymbol{w}_1(t),\ \boldsymbol{y_o}(t)=\boldsymbol{C}_s\boldsymbol{x}_s(t)+\boldsymbol{v}_y(t) \tag{4'}$$

For the sake of notation simplicity, $u(t-\tau):=u_c(t-\tau)$. $\boldsymbol{w}_1$ and $\boldsymbol{v}_y$ are zero mean Gaussian white noises with strictly positive intensities $\boldsymbol{W}_1,\ \boldsymbol{V}_y$.

Pilot OCM as a LQG control problem. *Given the system (4'), find the control $u(t)$ that minimizes the cost*

$$\min J(u)$$
$$J(u)=\lim \mathrm{E}\left(\int_{t_0}^{t_f}\left(\boldsymbol{x}_s^{\mathrm{T}}\boldsymbol{C}_s^{\mathrm{T}}\boldsymbol{Q}_J\boldsymbol{C}_s\boldsymbol{x}_s+u^2R_J\right)dt+\boldsymbol{x}_s^{\mathrm{T}}(t_f)\boldsymbol{P}_f\boldsymbol{x}_s(t_f)\right)\Big/t_f,\ \text{for}\ t_f\to\infty \tag{5}$$

The symbol $E\left[f(x)\right]$ means the expectation of the function f of a random variable x. $\boldsymbol{Q}:=\boldsymbol{C}_s^{\mathrm{T}}\boldsymbol{Q}_J\boldsymbol{C}_s\ge 0$ and $R_J>0$ are cost function weights. $\boldsymbol{a}^{\mathrm{T}}$ denotes the transpose to a vector (matrix) $\boldsymbol{a}$.

3. PROBLEM SOLUTION

Two principles are recalled when we attempt to solve a stochastic problem of control synthesis, such as LQG problem: separation principle and duality principle (see [17], [18] for details). These principles, valid for systems without delays, remain also valid for linear systems with delays [15]. So, due to the separation principle, the original control synthesis problem, stated as an LQG problem, can be split into an optimal **Linear Quadratic Regulator (LQR) problem** for linear system states and an **optimal estimation (filtering) problem** for linear system states over linear observations. Also, the theory establishes the duality between the solution of the optimal filtering problem for linear system states over observations and the solution of the optimal LQR control problem.

LQR problem. A first step in the synthesis of the pilot OCM similar to previous LQG based approaches [1]-[3] is the building of LQR for input delay. More exactly, in terms of the Linear Quadratic Regulator (LQR) paradigm applied to the following linear system with time delay in control input

$$\dot{\boldsymbol{x}}_s(t)=\boldsymbol{A}_s\boldsymbol{x}_s(t)+\boldsymbol{B}_s u(t-\tau),\ \boldsymbol{x}_s(r)=0,\ r\in\left[t_0-\tau,t_0\right] \tag{4''}$$

the pilot, based on the observation output considered here as an performance output (the observation is "measured" on the screen)

$$\boldsymbol{y}(t)=\boldsymbol{C}_s\boldsymbol{x}_s(t) \tag{6}$$

aims to minimize the index

$$\begin{gathered}\min J(u)\\ J(u):=J\left(\boldsymbol{x}_s\left(t_0\right),u(.),t_0\right)=\int_{t_0}^{t_f}\left(\boldsymbol{x}_s^{\mathrm{T}}\boldsymbol{C}_s^{\mathrm{T}}\boldsymbol{Q}_J\boldsymbol{C}_s\boldsymbol{x}_s+u^2R_J\right)dt+\boldsymbol{x}^{\mathrm{T}}\left(t_{\mathrm{f}}\right)\boldsymbol{P}_f\boldsymbol{x}\left(t_{\mathrm{f}}\right),\ \text{for}\ t_f\to\infty\end{gathered}\tag{5′}$$

The representation (4") is characterized by the fact that the state space is dimensionally finite but the input operator $u(t-\tau)$ involves infinite dimensional extensions, given the transcendental structure of the transfer matrices [9], [10]. Such control problems have been addressed since 1950 and the mostly known result is that based on the Smith predictor [11]. Also, the Artstein's state transformation

$$\boldsymbol{z}(t)=\boldsymbol{x}(t)+\int_{-\tau}^{0}e^{-\boldsymbol{A}_s(\theta+\tau)}\boldsymbol{B}_s u(t+\theta)d\theta \tag{7}$$

was introduced in [12] to map (4") into an input-delay free system, namely in

$$\dot{\boldsymbol{z}}(t)=\boldsymbol{A}_s\boldsymbol{z}(t)+\boldsymbol{B}_s e^{-\boldsymbol{A}_s\tau}u(t) \tag{7'}$$

Unfortunately, this simplifying transformation has as secondary effect an equivalent complication of the observation equation (6)

$$\boldsymbol{y}(t)=\boldsymbol{C}_s\left(\boldsymbol{z}(t)-\int_{-\tau}^{0}e^{-\boldsymbol{A}_s(\theta+\tau)}\boldsymbol{B}_s u(t+\theta)d\theta\right) \tag{7''}$$

Therefore, in this paper will be preferred another approach, having as starting point the works of M. Basin and his collaborators [13]-[16].

Proposition 3.1. *The solution to the LQR problem for linear time invariant system (4"), (5'), (6) with input delay and infinite horizon,* $t_f\to\infty$, *is given by*

$$\begin{gathered}u^*(t-\tau)=R_J^{-1}\boldsymbol{B}_s^{\mathrm{T}}\exp\left(-\boldsymbol{A}_s^{\mathrm{T}}\tau\right)\boldsymbol{P}\boldsymbol{x}_s(t-\tau)\\ \boldsymbol{A}_s^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}_s+\boldsymbol{P}\exp\left(-\boldsymbol{A}_s\tau\right)\boldsymbol{B}_s R_J^{-1}\boldsymbol{B}_s^{\mathrm{T}}\exp\left(-\boldsymbol{A}_s^{\mathrm{T}}\tau\right)\boldsymbol{P}=\boldsymbol{Q}\end{gathered}\tag{8}$$

The difference with respect to the case without delay [17], [18], consists in the presence of the factors expressing the delay effect in the two equations. The proof adjusts to the problem (4"), (5′), (6) the reasoning of [14], [15].

Proof. Define de Hamiltonian function of the problem, with p the co-state

$$H(x_s,u,p)=\frac{1}{2}\left(u^2R_J+\boldsymbol{x}_s^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}_s\right)+\boldsymbol{p}^{\mathrm{T}}\left(\boldsymbol{A}_s\boldsymbol{x}_s+\boldsymbol{u}_1(u)\right),\boldsymbol{u}_1(u):=\boldsymbol{B}_s u(t-\tau) \tag{9}$$

The maximum principle condition $\partial H/\partial u=0$ gives

$$R_J u(t)+\left(\partial\boldsymbol{u}_1(t)/\partial u\right)^{\mathrm{T}}\boldsymbol{p}(t)=0$$

By denoting

$$\partial u(t-\tau)/\partial u(t)=\boldsymbol{M}(t) \tag{10}$$

one obtains

$$u^*(t)=-R_J^{-1}\boldsymbol{M}^{\mathrm{T}}(t)\boldsymbol{B}_s^{\mathrm{T}}\boldsymbol{p}(t) \tag{11}$$

Based on the linearity of the problem, $p(t)$ is searched as a linear function

$$\boldsymbol{p}(t)=-\boldsymbol{P}(t)\boldsymbol{x}_s(t) \tag{12}$$

with $P(t)$ a symmetric matrix. The transversality condition [20] implies that $\boldsymbol{p}(t_f)=\partial J/\partial \boldsymbol{x}_s(t_f)=\boldsymbol{P}_{t_f}\boldsymbol{x}_s(t_f)$, thus

$$\boldsymbol{P}(t_f)=-\boldsymbol{P}_{t_f} \tag{13}$$

The co-state equation gives

$$\begin{gathered}-d\boldsymbol{p}(t)/dt=-\partial H/\partial \boldsymbol{x}\rightarrow -d\boldsymbol{p}(t)/dt=\boldsymbol{Q}\boldsymbol{x}_s(t)+\boldsymbol{A}_s\boldsymbol{p}(t)\rightarrow\\ \dot{\boldsymbol{P}}(t)\boldsymbol{x}_s(t)+\boldsymbol{P}(t)d(\boldsymbol{x}_s(t))/dt=\boldsymbol{Q}\boldsymbol{x}(t)-\boldsymbol{A}_s\boldsymbol{P}(t)\boldsymbol{x}_s(t)\rightarrow\\ \dot{\boldsymbol{P}}(t)\boldsymbol{x}(t)+\boldsymbol{P}(t)\boldsymbol{A}_s\boldsymbol{x}_s(t)+\boldsymbol{P}(t)\boldsymbol{B}_s u(t-h)=\boldsymbol{Q}\boldsymbol{x}_s(t)-\boldsymbol{A}_s\boldsymbol{P}(t)\boldsymbol{x}_s(t)\end{gathered} \tag{14}$$

Again in view of the linearity, differentiating the last expression in x does not imply loss of generality

$$\dot{\boldsymbol{P}}(t)=\boldsymbol{Q}-\boldsymbol{P}(t)\boldsymbol{A}_s-\boldsymbol{A}_s^{\mathrm{T}}\boldsymbol{P}(t)-\boldsymbol{P}(t)\boldsymbol{B}_s\boldsymbol{M}R_J^{-1}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{B}_s^{\mathrm{T}}\boldsymbol{P}(t),\boldsymbol{P}(t_f)=-\boldsymbol{P}_{t_f} \tag{15}$$

The equation is similar to the well known differential Riccati equation providing the control [17], but herein is in the form specific to the input delay problem. For the infinite horizon case, $t_f\rightarrow\infty$, and for time invariant systems, the algebraic equation associated to (15), comporting a constant solution noted also $\boldsymbol{P}$ for convenience, will be taken into account

$$\boldsymbol{Q}-\boldsymbol{P}\boldsymbol{A}_s-\boldsymbol{A}_s^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}_s\boldsymbol{M}R_J^{-1}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{B}_s^{\mathrm{T}}\boldsymbol{P}=0 \tag{15'}$$

Further on, the analysis will show that the matrix $\boldsymbol{M}$ played only a fleeting role in the calculations above. Indeed, let us substitute the optimal control law (8) into the equation (4")

$$\dot{\boldsymbol{x}}_s(t)=\boldsymbol{A}_s\boldsymbol{x}_s(t)+\boldsymbol{B}_s R_j^{-1}\boldsymbol{M}^T(t-\tau)\boldsymbol{B}_s^T\boldsymbol{P}\boldsymbol{x}_s(t-\tau) \tag{16}$$

The solution of the equation (16) is given by the Cauchy formula

$$\boldsymbol{x}_s(t)=\Phi(t,r)\boldsymbol{x}_s(r)+\int_r^t\Phi(t,\mu)\boldsymbol{B}_s R_J^{-1}\boldsymbol{M}^{\mathrm{T}}(\mu-\tau)\boldsymbol{P}\boldsymbol{x}_s(\mu-\tau)d\mu \tag{17}$$

where $t,s\geq t_0$ and $\Phi(t,s)$ is the matrix of fundamental solution of the homogeneous equation (4")

$$d(\Phi(t,s))/dt=\mathbf{A}_s\Phi(t,s),\Phi(t,t)=I \tag{18}$$

from where $\Phi(t-\tau,t)=\exp\left(-\int_{t-\tau}^{t}\mathbf{A}_s dr\right)=\exp(-\mathbf{A}_s\tau)$. The integral terms in (17) do not explicitly depend on $u(t)$, so that

$$\partial\boldsymbol{x}_s(t)/\partial u(t)=\Phi(t,r)\partial\boldsymbol{x}_s(r)/\partial u(t)$$

relation that, inverted, gives

$$\partial u(t)/\partial \boldsymbol{x}_s(t) = \left(\partial u(t)/\partial \boldsymbol{x}_s(r)\right)\Phi(r,t)$$

A relation $\boldsymbol{B}_s u(t) = \boldsymbol{K}_1\Phi(t,r)\boldsymbol{K}_2\boldsymbol{x}_s(r)$ holds, with the matrices $\boldsymbol{K}_1, \boldsymbol{K}_2$ considered as dimension connectors between vectors $x \in R^n$ and $u \in R^m$. Simple calculations show that

$$\boldsymbol{B}_s u(t+\tau) = \boldsymbol{K}_1\Phi(t+\tau, r)\boldsymbol{K}_2 x(r)$$

$$\partial\left(\boldsymbol{B}_s u(t)\right)/\partial\left(\boldsymbol{B}_s u(t+\tau)\right) = \Phi(t, r)\left(\Phi(t+\tau, r)^{-1} = \Phi(t, t+\tau)\right)$$

and further on

$$\partial\left(\boldsymbol{B}_s u(t)\right)/\partial u(t+\tau) = \Phi(t, t+\tau)\boldsymbol{B}_s$$

$$\boldsymbol{B}_s\left(\partial u(t-\tau)\right)/\partial u(t) = \boldsymbol{B}_s\boldsymbol{M}(t) = \Phi(t-\tau, t)\boldsymbol{B}_s = \exp\left(-\int_{t-\tau}^{t}\boldsymbol{A}_s ds\right)\boldsymbol{B}_s, \text{for } t \ge t_0 + \tau$$

(by substituting t with $t-\tau$). Now, by substituting

$$\boldsymbol{B}_s\boldsymbol{M}(t) = \exp\left(-\int_{t-\tau}^{t}\boldsymbol{A}_s ds\right)\boldsymbol{B}_s \tag{19}$$

in (15'), the second equation (8) holds.

It remains to show that equation (11) can be brought to the form given by the first equation (8). Since

$$\partial\left(\boldsymbol{B}_s u(t)\right)/\partial\left(\boldsymbol{B}_s u(t+\tau)\right) = \partial\boldsymbol{x}_s(t)/\partial\boldsymbol{x}_s(t+\tau) = \Phi(t,t+\tau)$$

(see (17)), the equality $\partial u(t)/\partial\boldsymbol{x}_s(t) = \partial u(t+\tau)/\partial\boldsymbol{x}_s(t+\tau)$ holds for $t \ge t_0$. Thus the expression $R_J^{-1}(t-\tau)\boldsymbol{M}^{\mathrm{T}}(t-\tau)\boldsymbol{B}_s^{\mathrm{T}}\boldsymbol{P}(t-\tau)$ in the relations (11)-(12) for $u(t-\tau)$ can be replaced by $R_J^{-1}\boldsymbol{M}^{\mathrm{T}}(t)\boldsymbol{B}_s^{\mathrm{T}}\boldsymbol{P}(t)$ for any $t \ge t_0 + \tau$, thus yielding the control law (8). Substituting the control law (8) in (4"), the optimally controlled state equation is obtained

$$\begin{gathered}\dot{\boldsymbol{x}}_s(t) = \boldsymbol{A}_s\boldsymbol{x}_s(t) + \boldsymbol{B}_s R_J^{-1}\boldsymbol{B}_s^{\mathrm{T}}\exp(-\boldsymbol{A}_s\tau)\boldsymbol{P}\boldsymbol{x}_s(t-\tau),\ \boldsymbol{x}_s(t_0) = 0\\ \boldsymbol{A}_s^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_s + \boldsymbol{P}\exp(-\boldsymbol{A}_s\tau)\boldsymbol{B}_s R_J^{-1}\boldsymbol{B}_s^{\mathrm{T}}\exp\left(-\boldsymbol{A}_s^{\mathrm{T}}\tau\right)\boldsymbol{P} = \boldsymbol{Q}\end{gathered} \tag{20}$$

The necessity part of the optimal control problem is proved.

The sufficiency part of the proof follows from the satisfaction of the Hamilton-Jacobi-Bellman equation and was proved in old works of Kharatashvili ([21], apud [14]) and Pontryagin [20], in 1960s. □

The second step in the synthesis of the optimal pilot model consists in the statement and the solution of the estimation problem in the context of the system with time delay in control input. So, let us consider the random process $\left(\boldsymbol{x}_s(t), \boldsymbol{y}_o(t)\right)$ described by the equations (4') with the initial condition $\boldsymbol{x}_s(r) = \varphi(r), r \in [t_0-\tau, t_0]$ given by a stochastic process; φ, w_1, v_y are independent white noise stochastic processes.

Thus, the **estimation problem** is the following: based on the observation process

$\boldsymbol{Y}(t)=\{\boldsymbol{y}_O(s), 0 \le s \le t\}$, find the optimal estimate $\hat{\boldsymbol{x}}_s(t)$ of the state $\boldsymbol{x}_s(t)$, which minimizes the Euclidean 2-norm

$$J = \mathrm{E}\left[\left(\boldsymbol{x_s}(t)-\hat{\boldsymbol{x}}_s(t)\right)^{\mathrm{T}}\left(\boldsymbol{x}_s(t)-\hat{\boldsymbol{x}}_s(t)\right)\Big\|\boldsymbol{F}_t^Y\right] \tag{21}$$

at every time moment t. The operator $\mathrm{E}\left[\bullet\|\circ\right]$ in (21) means the conditional expectation of the stochastic process $\bullet$ with respect to the $\sigma-$algebra $\circ$ generated by the observation process $\boldsymbol{Y}(t)=\{\boldsymbol{y}_O(s), 0 \le s \le t\}$ (see [22]-[26] for details). This optimal estimate is given by the conditional expectation

$$\hat{\boldsymbol{x}}_s(t) = \mathrm{E}\left[\boldsymbol{x}_s(t)\Big\|\boldsymbol{F}_t^Y\right] \tag{22}$$

The matrix function

$$\tilde{\boldsymbol{S}}(t)=\mathrm{E}\left[\left(\boldsymbol{x}_s(t)-\hat{\boldsymbol{x}}_s(t)\right)^{\mathrm{T}}\left(\boldsymbol{x}_s(t)-\hat{\boldsymbol{x}}_s(t)\right)\Big\|\boldsymbol{F}_t^Y\right] \tag{23}$$

is the estimation error variance.

The solution to both problems – LQR problem and estimation problem – is the LQG problem solution and is given below.

Proposition 3.2. *The solution to the LQG problem (4'), (5), (6'') is given by the following system of equations*

$$\begin{gathered}
u^*(t-\tau) = R_J^{-1}\boldsymbol{B}_s^{\mathrm{T}}\exp\left(-\boldsymbol{A}_s^{\mathrm{T}}\tau\right)\boldsymbol{P}\hat{\boldsymbol{x}}_s(t-\tau)\\
\boldsymbol{A}_s^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A_s} + \boldsymbol{P}\exp\left(-\boldsymbol{A}_s\tau\right)\boldsymbol{B_s}R_J^{-1}\boldsymbol{B}_s^{\mathrm{T}}\exp\left(-\boldsymbol{A}_s^{\mathrm{T}}\tau\right)P = Q\\
\dot{\hat{\boldsymbol{x}}}_{\boldsymbol{s}} = \boldsymbol{A_s}\hat{\boldsymbol{x}}_{\boldsymbol{s}} + \boldsymbol{B}_s u^*(t-\tau) + \boldsymbol{S}\boldsymbol{C}_{\boldsymbol{s}}^{\mathrm{T}}\boldsymbol{V}_y^{-1}\left(\boldsymbol{y_o} - \boldsymbol{C_s}\hat{\boldsymbol{x}}_s\right)\\
\boldsymbol{S}\boldsymbol{A}_s^{\mathrm{T}} + \boldsymbol{A_s}\boldsymbol{S} + \boldsymbol{E_s}\boldsymbol{W}_1\boldsymbol{E}_s^{\mathrm{T}} - \boldsymbol{S}\boldsymbol{C}_s^{\mathrm{T}}\boldsymbol{V}_y^{-1}\boldsymbol{C_s}\boldsymbol{S} = 0
\end{gathered} \tag{24}$$

Proof. The reasoning expands the optimal LQR problem solution in Proposition 3.1 to Kalman filtering solution considering the measuring equation without delay. The separation principle and the duality principle are involved. Due to the separation principle, the original LQG control problem, Section 1, was split into the LQR problem solved in Proposition 3.1 and the optimal estimation problem over linear observations $\boldsymbol{Y}(t)=\{\boldsymbol{y}_O(s), 0 \le s \le t\}$, characterized by the minimization of the index (21) and having as solution Kalman filter defined by the two last equations in (24). Due to the duality principle, if the optimal control exists, then the optimal filter exists for the dual system with Gaussian disturbances and can be found, *mutatis mutandis*, in the case of the presence or absence of the delays, from the optimal control problem solution, using simple algebraic transformations involving the gain matrices, variance equations and system matrices – and vice versa. □

The optimal filtering could be also obtained directly from the Itô formula for the differential of the conditional expectation $\hat{x}_s$ [22], [23], [13]-[16], [18].

Thus, the closed loop pilot-vehicle system, as solution to the pilot optimal control model synthesis in terms of input time delay, takes the form

$$\begin{bmatrix} \dot{\boldsymbol{x}}_s(t) \\ \dot{\hat{\boldsymbol{x}}}_{\boldsymbol{s}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{\boldsymbol{s}} & 0 \\ \boldsymbol{S}\boldsymbol{C}_{\boldsymbol{s}}^{\mathrm{T}}\boldsymbol{V}_y^{-1}\boldsymbol{C}_{\boldsymbol{s}} & \boldsymbol{A}_{\boldsymbol{s}} - \boldsymbol{S}\boldsymbol{C}_{\boldsymbol{s}}^{\mathrm{T}}\boldsymbol{V}_y^{-1}\boldsymbol{C}_{\boldsymbol{s}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_s(t) \\ \hat{\boldsymbol{x}}_{\boldsymbol{s}}(t) \end{bmatrix} +$$

$$\begin{bmatrix} 0 & \boldsymbol{B}_s R_J^{-1}\boldsymbol{B}_s^{\mathrm{T}}\exp\left(-\boldsymbol{A}_s^{\mathrm{T}}\tau\right)\boldsymbol{P} \\ 0 & \boldsymbol{B}_s R_J^{-1}\boldsymbol{B}_s^{\mathrm{T}}\exp\left(-\boldsymbol{A}_s^{\mathrm{T}}\tau\right)\boldsymbol{P} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_s(t-\tau) \\ \hat{\boldsymbol{x}}_{\boldsymbol{s}}(t-\tau) \end{bmatrix} + \begin{bmatrix} \boldsymbol{E}_{\boldsymbol{s}} \\ \boldsymbol{S}\boldsymbol{C}_{\boldsymbol{s}}^{\mathrm{T}}\boldsymbol{V}_y^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_1(t) \\ \boldsymbol{v}_y(t) \end{bmatrix} \tag{25}$$

$$\boldsymbol{y}_s(t) = \begin{bmatrix} \boldsymbol{y}_o(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}_{\boldsymbol{s}} & 0 \\ 0 & R_J^{-1}\boldsymbol{B}_s^{\mathrm{T}}\exp\left(-\boldsymbol{A}_s^{\mathrm{T}}\tau\right)\boldsymbol{P} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_s(t) \\ \hat{\boldsymbol{x}}_{\boldsymbol{s}}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_y(t) \\ 0 \end{bmatrix}$$

The same system, in matrix form, will be written so

$$\dot{\boldsymbol{X}}_s(t) = \boldsymbol{A}_{\boldsymbol{cl}}\boldsymbol{X}_s(t) + \boldsymbol{A}_{\boldsymbol{cl}\tau}\boldsymbol{X}_s(t-\tau) + \boldsymbol{E}_{cl}\omega(t),\ \omega(t) := \begin{bmatrix} \boldsymbol{w}_1(t) \\ \boldsymbol{v}_y(t) \end{bmatrix}$$

$$\boldsymbol{Y}_{cl}(t) = \boldsymbol{C}_{\boldsymbol{cl}}\boldsymbol{X}_s(t) + \tilde{v}_y(t),\ \tilde{v}_y(t) := \begin{bmatrix} \boldsymbol{v}_y(t) \\ 0 \end{bmatrix} \tag{25'}$$

4. CONCLUDING REMARKS

The present study is part of a series of works by authors with the objective of some contributions in the problem of aircraft susceptibility investigation to PIO phenomenon. In general, as shown in [27], the first step of any such approach is the determination of an analytical model of human pilot, appropriate for the flying task under study. Common models in literature are so-called “crossover” and “optimal control” forms. In the latter category, the most common are the Kleinman’s optimal control model (OCM) [5] and the Davidson and Schmidt’s modified optimal control model (MOCM) [4]. The first model includes a cascade combination of a Kalman state estimator and a predictor, in view of model pilot ability to adapt to his/her inherent time delay. However, the Kleinman’s model is focused mainly on the frequency domain description of the model of the human response. Instead, the Davidson and Schmidt's model, appealing to Padé approximation, provides more detailed and elaborated information to predict closed loop pilot performance for three basic tracking experiments: a velocity command system, an acceleration command system and a position command system. Davidson-Schmidt’s model disadvantage is related to the use of such approximations on the “nervous” reaction component of the pilot.

In this paper, some concepts and results are presented proposing a new approach to the optimal control pilot model synthesis in the framework of the systems with time delay. The approach consists of removing the Padé or Hess [4] approximations for characterizing the “central nervous block” component and of their introduction as a pure delay block, $u(t) \rightarrow u(t-\tau)$.

Two advantages of the method are stressed. On the one hand, the method ensures a better accuracy of the optimal control pilot model and on the other hand the method is advantageous with respect to general results to date for time delay systems since: a) the optimal control law is given explicitly and not as a solution of a system of integro-differential equations generated by the transformation (7) and b) the Riccati equations for the gain matrices do not contain any time advanced or delayed arguments. An analogy with the classical Smith predictor tool [11] in the study of systems with delay is invoked [15], but we believe that the machinery described in this article is completely different.

Future work will finalize and develop this first step in the study of PIO susceptibility, by considering extensions in the case of both input and output delays. Also, the model validation in terms of comparison with experimental results in the literature will be performed.

In the validation step of the pilot optimal control model, the explicit determination of the system matrices (25) supposes an attentive *trial and error* procedure for the selection of the noise intensities W_1, V_y in order to obtain the signal noise ratios $V_{u_i} \big/ \sigma_{u_i}^2 = 0.003$ and $V_{y_i} \big/ \sigma_{y_i}^2 = 0.01$, which correspond to the normalized control noise and normalized observation noise of −25 dB and −20 dB, respectively [4], [5].

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