

Unsteady Flow in a Horizontal Double-Sided Symmetric Thin Liquid Films

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Abstract: *In this paper a mathematical model is constructed to describe a two dimensional incompressible flow in a symmetric horizontal thin liquid film for unsteady flow. We apply the Navier-Stokes equations with specified boundary conditions and we obtain the equation of the film thickness by using the similarity method in which we can isolate the explicit time dependence and then the shape of the film will depend on one variable only.*

Key Words: *Unsteady flow; Symmetry; Thin liquid film; Navier-Stokes equations.*

1. INTRODUCTION

Thin liquid films have been widely studied in different areas, such as surface coatings in paints, protective wax and foam development [7]. The derivation of thin film flow equation when the surface tension has a significant effect on the flow was investigated by [8]. Here we consider a class of dynamical system where the surface tension has significant mechanical effect on the system as a whole and no solid boundaries present, though it is perhaps more realistic to express this property by the statement that any solid boundaries which are presented are sufficiently distant to produce no significant direct mechanical effect on that part of the system which is under consideration, for details see [2]. [3] has studied the drainage of thin liquid films on an inclined solid surface for unsteady flow where the gravity and other forces such as viscous and surface tension forces have a significant effect on the flow of the film by using similarity solution. A mathematical model is constructed by [1] to describe a two dimensional flow for an inclined film with an inclination angle α to the horizontal that drains under the action of gravity. An asymptotic analysis is employed with the use of lubrication approximation. The film is assumed to be supported by wire frame elements at the ends. In [5], the stability and dynamics of a free double-sided symmetric thin liquid film are investigated by using the long-wave method. [4] considered the unsteady flow within a horizontal double-sided symmetric thin liquid film with negligible inertia. The similarity and perturbation methods are used to obtain a non-linear differential equation that governs such flow for unsteady state in dimensionless form. Effect of inertia on the rupture process of thin liquid film was studied by [6], and assumed

that the film is thinned enough to neglect the gravity effect and the effect of the Van der Waals potential is considered. He used integral method to study the inertia effect on rupture process of thin liquid film and numerical method to solve nonlinear evaluation equation in order to study the rupture process. The purpose of this work is to study the unsteady flow within a double-sided symmetric horizontal thin liquid film. The non-linear differential equation which governs such flow obtained analytically and the solution of the equation of the liquid film thickness is obtained numerically. The similarity method is used to study the effect of the time on the thickness of the liquid film and it is shown graphically.

2. FORMULATION AND GOVERNING EQUATIONS

To describe the flow of a liquid within a symmetric film in two dimensions, a Cartesian coordinate system (x, t) is chosen so that the x -axis is the axis of symmetry and the z -axis is perpendicular to the plane of the film, as shown in figure 1. The flow is considered to be unsteady incompressible Newtonian fluid, and governed by the Navier-Stokes equations without external forces in x and z directions.

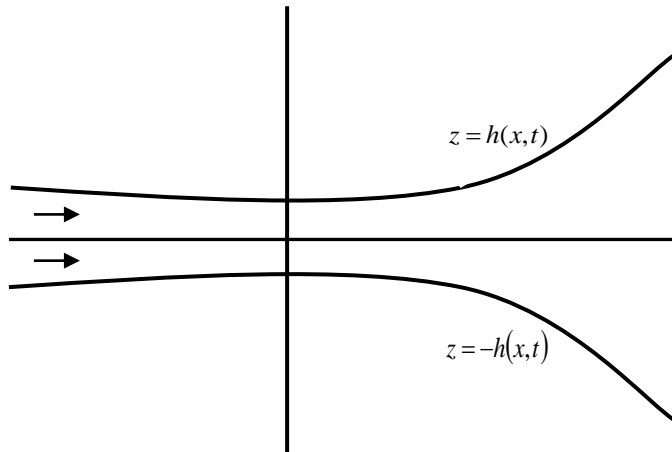


Figure 1. Physical model for symmetric horizontal thin liquid film for unsteady flow

In differential form, the Navier–Stokes equations without external forces in x and z directions for unsteady incompressible flow are given by:

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right] = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right] \quad (1)$$

and

$$\rho \left[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right] = -\frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right] \quad (2)$$

where u and w are the velocity components of the flow field in x and z directions respectively and are functions of x, z and t , where t is the time.

Let the equation of the liquid film be $z = \pm h(x, t)$, and $\frac{\partial h}{\partial x} \ll 1$ for all x and t . The velocity distributions are

$$\left. \begin{aligned} u &= u(x, t) \\ v &= 0 \\ w &= -z \end{aligned} \right\} \quad (3)$$

and the pressure p is a function of x and t only, that is

$$p = p(x, t) \quad (4)$$

The distribution (3) satisfies the continuity equation for an incompressible flow which is given by:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (5)$$

Since $z = h(x, t)$ is the free surface of the liquid film, then the conservation of mass across the film thickness of the film is therefore given by:

$$\frac{\partial h}{\partial x} + \frac{\partial}{\partial x}(u.h) = 0 \quad (6)$$

At the surface of the thin liquid film, the boundary conditions are:

1. Shear stress condition:

$$\tau = \mu \left(\frac{\partial u}{\partial x} \right)_s = 0 \quad (7)$$

where μ is the viscosity of liquid, $\frac{\partial h}{\partial x}$ is the velocity gradient and the subscribe s denotes the values at the surface of the film.

2. Normal stress condition:

$$T_{zz} = \left(-p + 2\mu \frac{\partial w}{\partial z} \right)_s \quad (8)$$

Now from the continuity equation (5), we have

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} \quad (9)$$

From equation (9), equation (8) gives

$$T_{zz} = \left(-p - 2\mu \frac{\partial u}{\partial x} \right)_s \quad (10)$$

The curvature of the liquid film is given by

$$k = \frac{\partial^2 h}{\partial x^2} \left(1 + \left(\frac{\partial h}{\partial x} \right)^2 \right)^{\frac{3}{2}} \quad (11)$$

Since $\frac{\partial h}{\partial x} \ll 1$ then $\left(\frac{\partial h}{\partial x} \right)^2$ can be neglected and equation (11) reduces to give

$$k = \frac{\partial^2 h}{\partial x^2} \quad (12)$$

Also, on the surface of the liquid film with surface tension σ , the normal component of stress is given by,

$$T_{zz} = \sigma \frac{\partial^2 h}{\partial x^2}. \quad (13)$$

From equations (10) and (13) and on the surface of the film, we have

$$p = -2\mu \frac{\partial h}{\partial x} - \sigma \frac{\partial^2 h}{\partial x^2}, \quad (14)$$

which holds everywhere.

Notice that p, u and h are functions of x and t only. Differentiate equation (14) with respect to x , we get

$$\frac{\partial p}{\partial x} = -2\mu \frac{\partial^2 h}{\partial x^2} - \sigma \frac{\partial^3 h}{\partial x^3} \quad (15)$$

From the velocity distributions, the longitudinal equation of motion (1) for unsteady flow reduces to give

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} \quad (16)$$

Equations (15) and (16), gives

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = \sigma \frac{\partial^3 h}{\partial x^3} + 3\mu \frac{\partial^2 h}{\partial x^2}, \quad (17)$$

Again from the velocity distribution (3), the transverse equation of motion (2) gives

$$\begin{aligned} \frac{\partial p}{\partial z} &= -\rho \left(u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \right) + \mu \frac{\partial^2 w}{\partial x^2}, \text{ or} \\ \frac{\partial p}{\partial z} &= z \left(\rho u \frac{\partial^2 u}{\partial x \partial t} + \rho u \frac{\partial^2 u}{\partial x^2} - \rho \left(\frac{\partial u}{\partial x} \right)^2 - \mu \frac{\partial^3 u}{\partial x^3} \right). \end{aligned} \quad (18)$$

Integrating equation (18) with respect to z , we get

$$p = \frac{z^2}{2} \left(\rho u \frac{\partial^2 u}{\partial x \partial t} + \rho u \frac{\partial^2 u}{\partial x^2} - \rho \left(\frac{\partial u}{\partial x} \right)^2 - \mu \frac{\partial^3 u}{\partial x^3} \right) + f(x, t).$$

That is

$$p = p(x, t) + o(z^2), |z| \leq h. \quad (19)$$

This is relevant only to higher order approximation and thus equations (6) and (17) are the governing equations of and within the liquid film for unsteady flow.

3. FLOWS WITH NEGLIGIBLE INERTIA

The governing equation (17) with negligible inertia, reduces to give

$$\frac{\partial^3 h}{\partial x^3} + \frac{3\mu}{\sigma} \frac{\partial^2 h}{\partial x^2} = 0 \quad (20)$$

$$\frac{\partial^3 h}{\partial x^3} + \frac{1}{V} \frac{\partial^2 h}{\partial x^2} = 0 \quad (21)$$

where $V = \frac{3\mu}{\sigma}$ is the only material constant that is relevant to equation (21).

We can determine the value of the parameter V as shown in Table 1.

Table 1. The value of the parameter V for various liquids.

Liquid	Density ρ , kg/cm^3	Surface tension σ , cm/sec^2	Viscosity μ , $kg/cm.sec$	Velocity V , cm/sec
Linseed oil	0.94	33.57	0.4309	25.9698
Olive oil	0.91	33.56	0.8379	13.3508

Integrating equation (21) with respect to x twice, we get

$$u = V \left[G_1(t)x + G_2(t) - \frac{\partial h}{\partial x} \right], \quad (22)$$

where $G_1(t)$ and $G_2(t)$ are arbitrary functions of t and can be determined from asymptotic or initial conditions. From equation (6), we have

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0. \quad (23)$$

From equation (22), equation (25) reduces to give

$$\frac{\partial h}{\partial t} + VG_1(t)x \frac{\partial h}{\partial x} + VG_2(t) \frac{\partial h}{\partial x} - V \left(\frac{\partial h}{\partial x} \right)^2 + VG_1(t)h - Vh \frac{\partial^2 h}{\partial x^2} = 0 \quad (24)$$

Equation (24) is related to lubrication theory. However without surface –active solutes, it seems to be a degenerate relationship; since the shear stress at the edge of the film is then zero, and this ensures that the velocity distribution across the film is uniform, not parabolic.

Now, we have the following two cases:

Case I:

If $G_1(t) \neq 0$, we can write $G(t) = G_1(t)x + G_2(t)$ and thus equation (22), then gives

$$u = V \left[G(t)x - \frac{\partial h}{\partial x} \right], \quad (25)$$

and

$$\frac{\partial u}{\partial x} = V \left[G(t) - \frac{\partial^2 h}{\partial x^2} \right]. \quad (26)$$

By substituting equations (25) and (26) in equation (23), we obtain

$$\frac{\partial h}{\partial t} + V \left[G(t)x \frac{\partial h}{\partial x} - \left(\frac{\partial h}{\partial x} \right)^2 + G(t)h - h \frac{\partial^2 h}{\partial x^2} \right] = 0. \quad (27)$$

Equation (27) is the governing equation when $G_1(t) \neq 0$.

Case II:

If $G_1(t) = 0$, then equation (22), gives

$$u = V \left[G_2(t) - \frac{\partial h}{\partial x} \right] \quad (28)$$

and

$$\frac{\partial u}{\partial x} = -V \frac{\partial^2 h}{\partial x^2}. \quad (29)$$

By substituting equations (28) and (29) in equation (23), we get

$$\frac{\partial h}{\partial x} + V \left[G_2(t) - \left(\frac{\partial h}{\partial x} \right)^2 - h \frac{\partial^2 h}{\partial x^2} \right] = 0 \quad (30)$$

and this is the governing equation when $G_1(t) = 0$.

4. NON-DIMENSIONAL ANALYSIS AND SIMILARITY SOLUTION

For case I, we suppose that the length scale associated with h and x vary with t in a different way.

For non-trivial solution of equation (27), we define the following non dimensional variables as follows:

$$\left. \begin{aligned} x &= Kt^{p_1}\eta \\ h(x,t) &= \frac{K^2}{V}t^{p_1-1}f(\eta) \\ G(t) &= \frac{1}{Vt} \end{aligned} \right\} \quad (31)$$

where K is of dimension L/t^{p_1-1} .

Thus equation (28), reduces to give

$$f(\eta) \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial f}{\partial \eta} \right)^2 - 2p_1 f(\eta) + (p_1 - 1)\eta f'(\eta) = 0. \quad (32)$$

Equation (32) is the non-dimensional equation of the double-sided symmetric film with negligible inertia in $(\eta, f(\eta))$ plane for unsteady flow. Now equation (32) admits a solution such that

$$f(\eta) = \eta^n \text{ as } \eta \rightarrow \infty, \quad (33)$$

and

$$\lim_{\eta \rightarrow \infty} \eta^{-n} = 0$$

By substituting (33) and its derivative in to equation (32), we get

$$n(n+1)\eta^{-n-2} + n^2\eta^{-n-2} - 2p_1 - n(p_1-1) = 0 \quad (34)$$

As $\eta \rightarrow \infty$, equation (34) gives,

$$n = \frac{-2p_1}{(p_1-1)} \geq 0 \quad (35)$$

And so the condition for semi-infinite extent is

$$0 \leq p_1 < 1 \quad (36)$$

For $p_1 = 0$, some of the solution curves of equation (27) are shown in figure 2 in $(\eta, f(\eta))$ -plane.

For large negative values of η , the liquid film terminates in a sink. for large positive values of η the liquid films reach an asymptotic uniform curvature.

Figure 3 shows the film thickness of linseed oil in $(x, h(x, t))$ - plane y , for different values of time. Furthermore, the thickness decreases when the time increases and this is because the film extracts some liquids as the time increased.

The comparison between the thickness of the liquid films of Linseed oil and Olive oil is shown in figure 4. The thickness of Linseed oil is less than that of Olive oil and this may be related to the viscosity of the Linseed oil which is smaller than that of the Olive oil.

For $p_1 = \frac{1}{2}$, some of the solution curves in $(\eta, f(\eta))$ -plane of equation (27) are presented in figure 5. It is shown that as $\eta \rightarrow \infty$, the film thickness is asymptotic to a parabolic distribution.

In $(x, h(x, t))$ -plane and from figure 6 the film thickness of some liquids namely Linseed oil for the different values of time presented.

Furthermore the comparison between the thickness of liquid films of Olive oil and Linseed oil is presented in figure (7), and the conclusion is the same as that given for $p_1 = 0$.

For $p_1 = \frac{1}{3}$, some of the solution curves of equation (27) in $(\eta, f(\eta))$ - plane are presented in figure (8) and it is shown that the solution has a parabolic distribution as $\eta \rightarrow -\infty$ and it terminates to a sink as $\eta \rightarrow \infty$.

Figure 9 presents some of the solution curves in $(x, h(x, t))$ - plane for Linseed oil and it is shown that the thickness decreases as the time increases and for the large value of η the thickness terminates into a sink.

Moreover, the comparison between the thickness of Linseed oil and Olive oil are presented in figure 10 and it is shown that the thickness of the Linseed oil film is smaller than that of Olive oil.

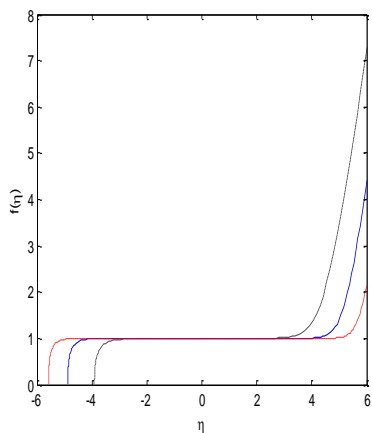


Figure 2. Solution curves of equation (27) in $(\eta, f(\eta))$ -plane when $p_1 = 0$

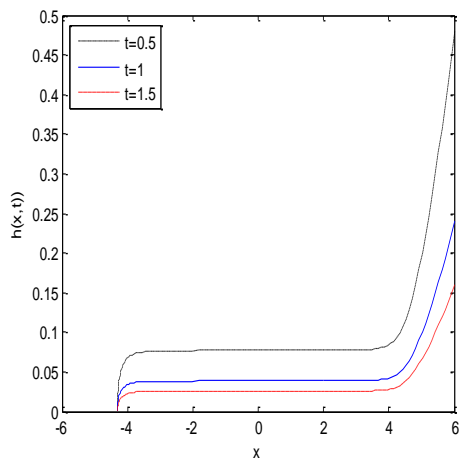


Figure 3. The film thickness of Linseed in $(x, h(x,t))$ -plane for different values of time.

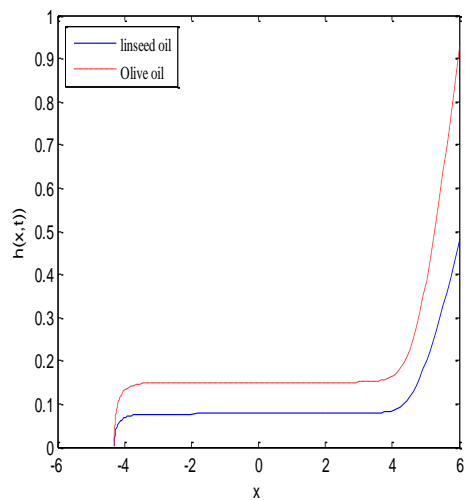


Figure 4. Comparison between the film thickness of Olive oil and Linseed oil when $p_1 = 0$.

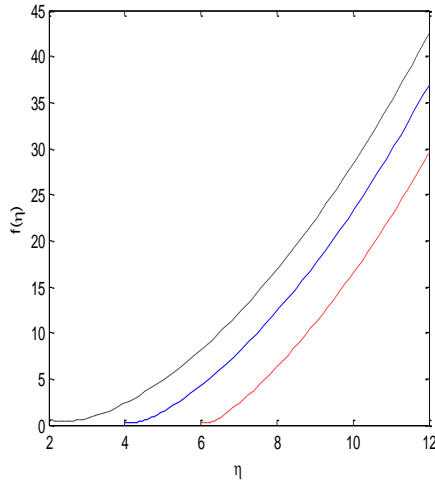


Figure 5. Solution curves of equation (27) in $(\eta, f(\eta))$ -plane when $p_1 = \frac{1}{2}$

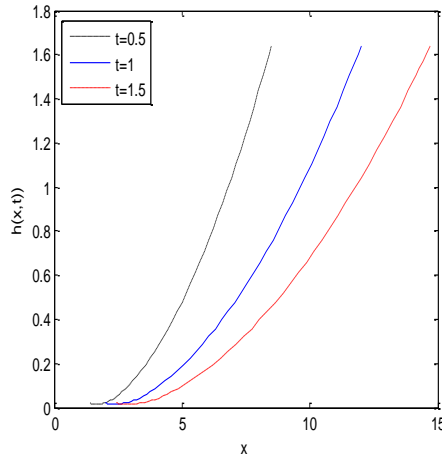


Figure 6. The film thickness of Linseed oil in $(x, h(x, t))$ -plane for different values of time

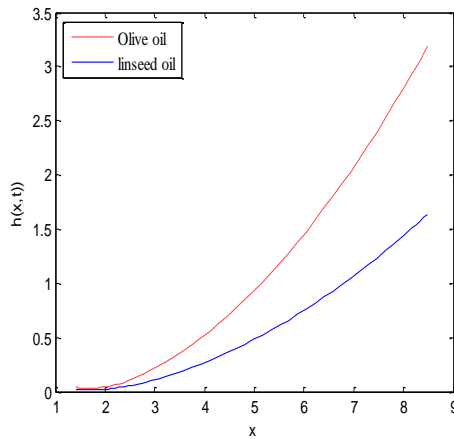


Figure 7. Comparison between the film thickness of Olive oil and Linseed oil when $p_1 = \frac{1}{2}$

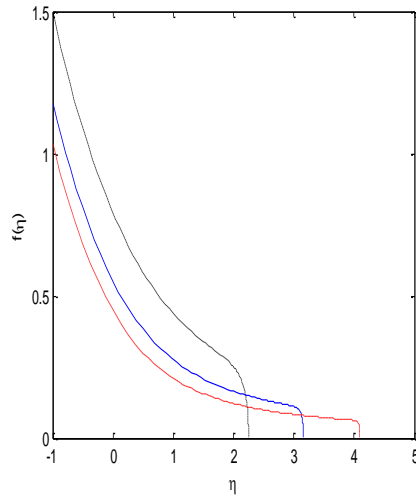


Figure 8. Solution curves of equation (27) in $(\eta, f(\eta))$ -plane where $p_1 = \frac{1}{3}$.

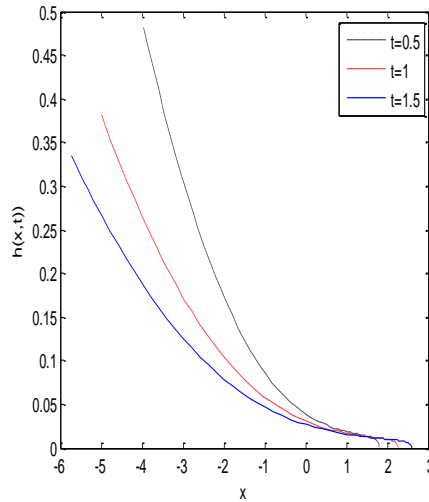


Figure 9. The film thickness of Linseed oil in $(x, h(x, t))$ -plane for different values of time when $p_1 = \frac{1}{3}$

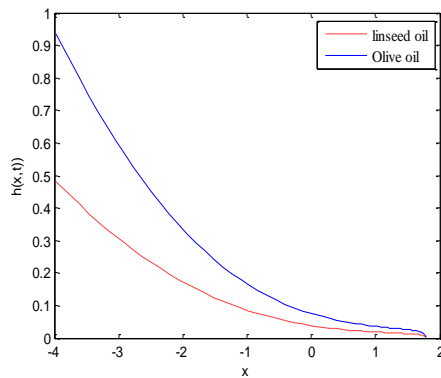


Figure 10. Comparison between the film thicknesses of Linseed oil and Olive oil where $p_1 = \frac{1}{3}$

For case 2, we define the following non-dimensional variables as follows:
Let

$$\left. \begin{aligned} x &= Kt^{q_1}\eta \\ h(x,t) &= \frac{K^2}{V}t^{2q_1-1}f(\eta) \\ G(t) &= \frac{1}{V}t^{q_1-1} \end{aligned} \right\} \quad (37)$$

where K is of dimension L/t^{2q_1-1} .

Thus equation (30), reduces to give

$$f(\eta)\frac{\partial^2 f}{\partial \eta^2} + \left(\frac{\partial f}{\partial \eta}\right)^2 + (1-2q_1)f(\eta) + (p_1\eta-1)\frac{\partial f}{\partial \eta} = 0. \quad (38)$$

Now equation (38) admits a solution of the form given in (33). By substituting (33) and its derivative in to equation (38), we get

$$n(n+1)\eta^{-n-2} + n^2\eta^{-n-2} - (2q_1-1) - n(q_1 - \eta^{-1}) = 0 \quad (39)$$

As $\eta \rightarrow \infty$, equation (39) gives,

$$n = \frac{-(2q_1-1)}{q_1} \geq 0 \quad (40)$$

From inequality (40), the condition for semi-infinite extent is

$$0 < q_1 \leq \frac{1}{2} \quad (41)$$

For $q_1 = \frac{1}{3}$, some of the solution curves are obtained from equation (30) in $(\eta, f(\eta))$ -plane and shown in figure 11 where the thin liquid film moves bodily while it is stretched and thinned.

Figure 12 presents some solution curves in $(x, h(x,t))$ - plane for Linseed oil, for different values of time and it is shown that the thickness decreases as the time increases. The comparison between the thickness of the liquid films of Linseed oil and Olive oil is shown in figure 13.

For $q_1 = \frac{1}{2}$, some of the solution curves are presented in in $(\eta, f(\eta))$ -plane and shown in figure 14.

Figure 15 shows some of the solution curves in $(x, h(x,t))$ -plane for Linseed oil and for different values of time and it is shown that after the formation of the liquid films the thickness decreases as the time increases.

Finally, Figure 16 shows that the thickness of Linseed oil is less than that of Olive oil.

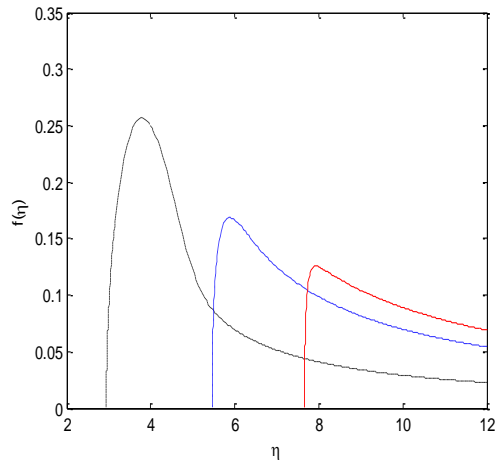


Figure 11. Solution curves of equation (30) in $(\eta, f(\eta))$ -plane when $q_1 = \frac{1}{3}$

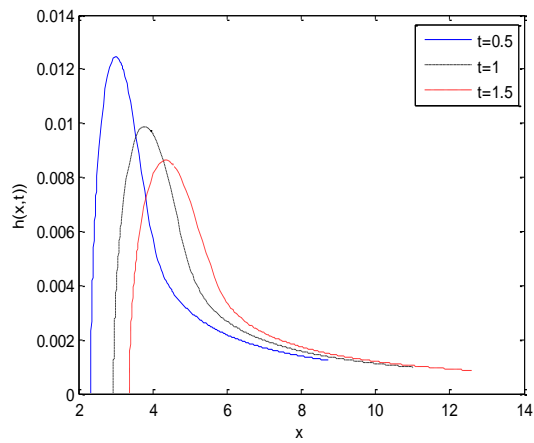


Figure 12. The film thickness of Linseed oil in $(\eta, f(\eta))$ -plane for different values of times when $q_1 = \frac{1}{3}$

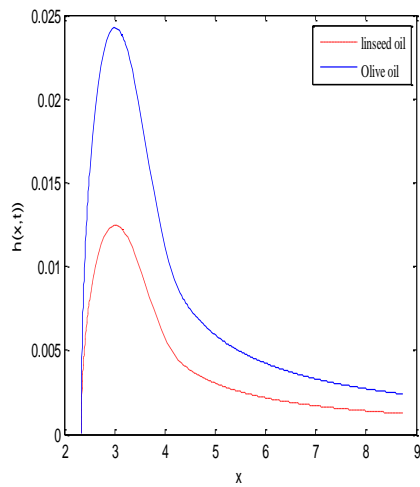


Figure 13. Comparison between the film thickness of Olive oil and Linseed oil when $q_1 = \frac{1}{3}$ for fixed t

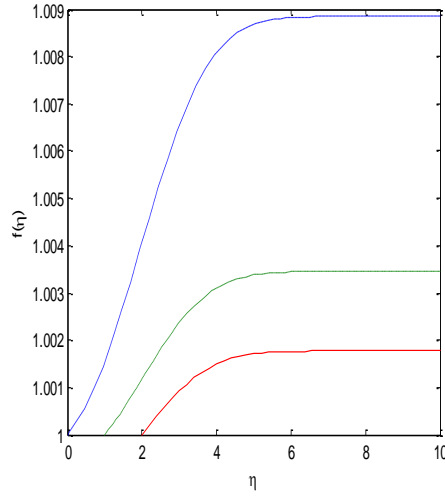


Figure 14. Solution curves of equation (30) in $(\eta, f(\eta))$ - plane, when $q_1 = 1/2$

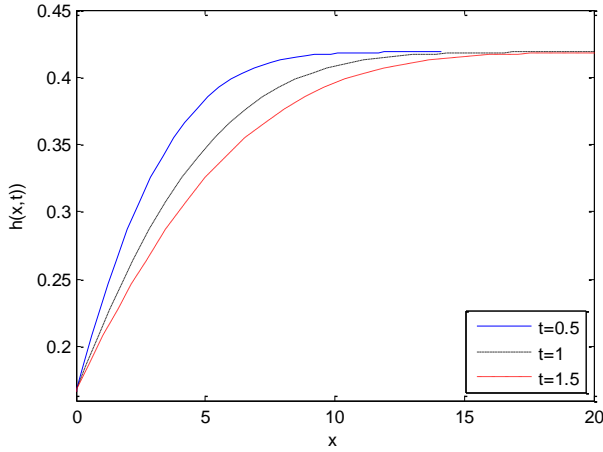


Figure 15. The film thickness of Linseed oil in $(x, h(x,t))$ -plane for different values of time

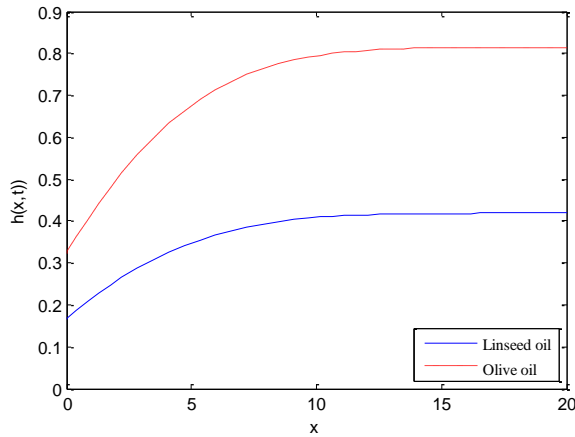


Figure 16. Comparison between the film thickness of Olive oil and Linseed oil when $q_1 = 1/2$.

5. CONCLUSIONS

For unsteady flow within a symmetric double sided film the nonlinear differential equation that governs such flow is obtained. For the derivation, we use the similarity method to isolate the time from the other variables. From macroscopic point of view we assume that the thickness of the film is about 100 molecules and so we can take only the thin liquid films of some liquids namely, linseed oil and olive oil. Some of the solution curves are obtained in $(\eta, f(\eta))$ - plane and shown in figure 2 for $p_1 = 0$. The curves show that the liquid film terminates in a sink for large negative values of η and it has an asymptotic uniform curvature for the large positive values of η . For Linseed oil and Olive oil, the thickness decreases as the time increases and this because the film extracts some of its liquid, as shown in figure 3 in $(x, h(x, t))$ - plane. Figure 4 shows that the thickness of Linseed oil is less than that of the Olive oil at a fixed time and the reason for this may be the viscosity of Linseed oil is smaller than that of the Olive oil. For $p_1 = \frac{1}{2}$, the solution curves in $(\eta, f(\eta))$ -plane are shown in figure (25) and the curves shows that the film thickness is asymptotic to a parabola as $\eta \rightarrow \infty$. Figure 6 shows that the thickness decreases as the time increases $(x, h(x, t))$ for some liquids. Figure 7 shows that the thickness of Olive oil is greater than that of Linseed oil at a fixed time. For $p_1 = \frac{1}{3}$, some of the solution curves in $(\eta, f(\eta))$ - plane are shown in figure 8 and it is shown that the film terminates to a sink as $\eta \rightarrow \infty$. the thickness of the Linseed oil film is smaller than that of Olive oil as it is shown in figure (10). The solution curves in $(\eta, f(\eta))$ - plane for $q_1 = \frac{1}{3}$ are shown in figure 11, and some of the solution curves $(x, h(x, t))$ - plane for some liquid namely (linseed oil, olive oil) are shown in figure 12. Figure 14 shows some of the solution curves in $(\eta, f(\eta))$ -plane for $q_1 = \frac{1}{2}$. Figures 13 and 16 show the comparison of thickness between the liquid film of olive oil and linseed oil at a fixed time for $q_1 = \frac{1}{2}$ and $q_1 = \frac{1}{3}$ respectively.

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