Bending Vibration Analysis of Nanobeams using the Nonlocal Motion Equations Solved by an Integral Approach

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DOI: 10.13111/2066-8201.2021.13.1.2

Received: 25 August 2020/ Accepted: 15 December 2020/ Published: March 2021

International Conference of Aerospace Sciences "AEROSPATIAL 2020", Virtual Conference
15-16 October 2020, Bucharest, Romania,
Section 4 - Materials and Structures

Abstract: *This paper deals with the dynamic characteristics for bending vibrations of Euler-Bernoulli type nanobeams taking into account the scale effects via the nonlocal motion equations. An integral method, based on the use of Green's functions, has been used in order to obtain the corresponding eigenvalue problem. The proposed integral approach is an approximate matrix method. Effects of different boundary conditions and of an elastic foundation have been also included. The presented numerical examples show good agreement when compared to results from literature. The proposed method can be used in the case of nanodevices analysis modeled as beams (MEMS, NEMS).*

Key Words*: Nanobeams, Scale Effects, Vibrations, Green's Functions, Winkler Foundation*

1. INTRODUCTION

A current topic related especially to the design of nanodevices, is the mechanical behavior analysis of nanostructures such as nanorods, nanobeams, nanoplates, etc. As experimental measurements are difficult at the nanoscale levels, molecular dynamic simulations are available that are expensive to calculate or there is also the use of continuum modelling of nanostructures modified in order to capture the size-effects at nano or micro-scale [1]. Scale effects play an important role for nanostructures, less for microstructures, while for the macrostructures one can use the classical continuum-based theories which are scale-free. In the classical elasticity theory the local stress in a point of a structure depends on the local strain. Different so-called nonlocal elasticity theories have been developed based on the idea that the strains at all locations of a structure affect the stress in a given point. A nonlocal coefficient μ has been defined as:

$$\mu = \frac{e_0 \cdot a}{L}, \tag{1}$$

where e_0 is a calibration coefficient, a is an internal characteristic length and L is the external characteristic length. Fig. 1 shows such a characteristic lengths for the case of Carbon Nanotubes (CNs).

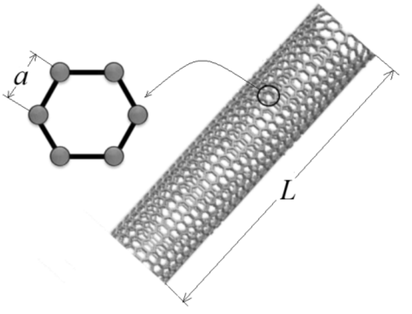

Fig. 1 – Characteristic lengths

One of the first and simplest nonlocal elasticity theory, which considers the size-dependent effects, was proposed by Eringen [2]. His theory was also proved to be in accordance with some experimental conclusions on phonon dispersion [3]. A nonlocal integral constitutive relation is obtained in the following form [1]:

$$\sigma_{ij}^{nl} = \iiint_V \phi\left(\left|x - x'\right|, \mu\right) \sigma_{ij}^{l} dV, \tag{2}$$

where $\sigma_{ij}^{nl}, \sigma_{ij}^{l}, \phi, \mu$ are the nonlocal stress, the local stress, a kernel function (or nonlocal modulus) and the nonlocal scale coefficient, respectively. In the above relation V is the volume and $\left|x - x'\right|$ is the Euclidean distance between two location points. Eringen obtained the following relation for the nonlocal operator:

$$L_{nl}(*) = [1 - (e_0 a)^2 \nabla^2](*) \tag{3}$$

Another theory named Nonlocal Strain Gradient Theory (NGST) was presented in [4]. After the appearance of carbon nanotubes these theories were used to obtain the size dependent differential equations for buckling, bending, vibration and wave propagation for nanostructures as beams, rods, plates and shells.

When one applies the nonlocal theory in the case of a nanoscale beam, the constitutive equation relating the axial stress σ_x with the corresponding strain ε_x, can be written as:

$$[1 - (e_0 a)^2 \nabla^2]\sigma_x = E\varepsilon_x \tag{4}$$

For Euler-Bernoulli beams, the axial strain depending on axial displacement u, the transverse displacement w and z coordinate, takes the form:

$$\varepsilon_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \tag{5}$$

The axial force and bending moment are defined as:

$$N = \iint_A \sigma_x dA, \quad M_y = \iint_A z\sigma_x dA, \tag{6}$$

where A is the area of the beam cross-section. The following equation describing the nanobeams transverse vibration behavior is obtained in [1]:

$$EI\frac{\partial^4 w}{\partial x^4} + (e_0 \cdot a)^2 \left[\frac{\partial^3}{\partial x^3}\left(N\frac{\partial w}{\partial x}\right) - m\frac{\partial^4 w}{\partial x^2 \partial t^2}\right] - \frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right) + m\frac{\partial^2 w}{\partial t^2} = 0, \tag{7}$$

where E is the Young modulus, I is the moment of inertia of the cross-section and $m = \rho A$ is the mass of the nanobeam unit length.

2. PROBLEM FORMULATION AND INTEGRAL FORM OF EQUATIONS

The analyzed nanobeam configuration is presented in Fig. 2 where an elastic support of Winkler type was also considered (K_w is the stiffness of the elastic support). In the absence of the axial force *N*, the corresponding equation of motion for bending vibration analysis is of the form, [5]:

$$EI\frac{d^4w}{dx^4} = \rho A\omega^2\left[w - (e_0\cdot a)^2\frac{d^2w}{dx^2}\right] - K_w\left[w - (e_0\cdot a)^2\frac{d^2w}{dx^2}\right], \tag{8}$$

where ω is the natural circular frequency of the beam. This type of equations was analyzed also in references [6, 7] for the case of macrobeams (without nonlocal coefficient: $\mu = 0$).

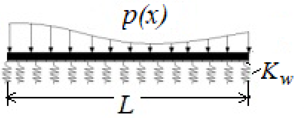

Fig. 2 – Beam on elastic foundation loaded transversally

The differential equation governing the bending behavior of a straight beam, loaded transversally by the distributed force $p(x)$ is given by:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = p(x), \tag{9}$$

It can be written in the integral form, [8-10]:

$$w(x) = \int_0^L G_w(x,\xi)p(\xi)d\xi \tag{10}$$

using the Green function which represents the bending deflection $w(x,\xi)$ at distance *x* due to a unity force applied at distance ξ (Fig. 3).

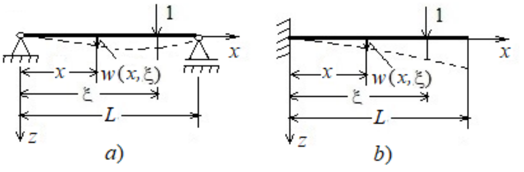

Fig. 3 – Green's functions for simply-supported and cantilever beam

The Green functions and all the integrals (10) are computed by numerical integration, considering *n* sampling (collocation) equally spaced points ξ_i, where $f_i = f(\xi_i)$, with a relation of the form:

$$\int_0^L f(\xi)d\xi = \sum_{i=1}^{n} f_i\cdot W_i, \tag{11}$$

where W_i are weighting coefficients. Relation (10) can be used to compute deflections $w(x)$ arranged in the vector form $\{w\}$, for known distributed load $p(x)$. The matrix form of relation (10) is:

$$\{w\} = [G_w][W]\{p\} \tag{12}$$

where the $(n\times n)$ matrix $[G_w]$ contains the coefficients $G_w(\xi_i, \xi_j)$,

$[W]$ is a weighting matrix corresponding to the Simpson integration rule
and $\{p\}$ is a column vector with the distributed loads in the n sampling points. The equation (8) is considered in the form (9) with $p(x)$ given by the relation:

$$p(x) = \rho A\omega^2 \left[w - (e_0 \cdot a)^2 \frac{d^2w}{dx^2}\right] - K_w \left[w - (e_0 \cdot a)^2 \frac{d^2w}{dx^2}\right] \tag{13}$$

With the notation $\eta = (e_0 a)^2$ the integral form of the equation (8) becomes:

$$\{w\} = \rho A\omega^2[G_w][W](\{w\}[I_n] - \eta[D_2]\{w\}) - [G_w][W][K_w](\{w\}[I_n] - \eta[D_2]\{w\}) \tag{14}$$

where $[I_n]$ is the identity matrix of size n, $[D_2]$ is a differentiation matrix and $[K_w]$ contains the Winkler coefficients (for nonconstant support stiffness). Relation (14) can be arranged in the alternative form:

$$\{w\} = \omega^2[A_1]\{w\} - [A_2]\{w\} \tag{15}$$

where:

$$[A_1] = \rho A\omega^2[G_w][W]([I_n] - \eta[D_2]);\ \ [A_2] = [G_w][W][K_w]([I_n] - \eta[D_2]) \tag{16}$$

Left-multiplying relation (15) by the inverse of the matrix $[A_1]$ one obtains:

$$\omega^2\{w\} = [A_1]^{-1}([I_n] + [A_2])\{w\} = [A_3]\{w\} \tag{17}$$

The square of the natural circular frequencies can be obtained as eigenvalues of the matrix $[A_3]$. In order to avoid the differentiation matrix $[D_2]$, one can use the collocation functions approach [9, 11], where for the bending deflection $w(x)$ the following formula is used:

$$w(x) = \sum_{k=1}^{p} C_k\, f_k(x), \tag{18}$$

where $f_k(x)$ are p known functions depending on the boundary conditions and C_k are constant coefficients (vector $\{C\}$). When using n collocation points, and $p < n$ functions, one can obtain the following matrix form relations:

$$\{w\} = [F]\{C\}; \quad \{w''\} = [F_2]\{C\}, \tag{19}$$

where $[F]$ and $[F_2]$, are $(n\times p)$ size matrices containing the values of $f_k(x)$ in collocation points and their corresponding second derivatives, respectively. Using the relations (19) in (14) it becomes:

$$[F]\{C\} = \rho A\omega^2[G_w][W]([F] - \eta[F_2])\{C\} - [G_w][W][K_w]([F] - \eta[F_2])\{C\} \tag{20}$$

Left-multiplying (20) by the transpose of the matrix $[F]$ and making the following notations:

$$\begin{gathered}[A] = [F]^t[F]; [F_\eta] = [F] - \eta[F_2]; [B_1] = \rho A[F]^t[G_w][W][F_\eta]; \\ [B_2] = [F]^t[G_w][W][K_w][F_\eta],\end{gathered} \tag{21}$$

matrix relation (20) can take the form:

$$[A]\{C\} = \omega^2[B_1]\{C\} - [B_2]\{C\}. \tag{22}$$

Left -multiplying relation (22) by the inverse of the matrix $[B_1]$ one obtains:

$$\omega^2\{C\} = [B_1]^{-1}([A] + [B_2])\{C\} = [A_4]\{C\}. \tag{23}$$

The size of the eigenvalue problem in relation (23) is $p < n$; therefore, this form is preferable instead of (17). In this case the squares of the natural circular frequencies are represented by the eigenvalues of the matrix $[A_4]$.

For the simply-supported (S-S) beam case, the collocation functions, according to the Boundary Conditions (BCs), are:

$$w_i(x) = sin\left(\frac{i\pi x}{L}\right), i = 1 \dots p. \tag{24}$$

For other BCs one can use the mode shapes for bending vibrations of uniform Euler-Bernoulli beams from [12]. They are written using the Krylov-Duncan functions:

$$S(x) = \frac{chx+\cos x}{2}, \quad T(x) = \frac{shx+\sin x}{2}, \tag{25}$$

and:

$$U(x) = \frac{chx-\cos x}{2}, \quad V(x) = \frac{shx-\sin x}{2}. \tag{26}$$

In the case of the cantilever beam (C-F) the collocation functions are given by:

$$w_i(x) = T(\beta_i)\cdot U\left(\beta_i\frac{x}{L}\right) - S(\beta_i)\cdot V\left(\beta_i\frac{x}{L}\right) \tag{27}$$

where:

$$\beta_1 = 1.8751, \beta_2 = 4.6941, \beta_i = \frac{2i-1}{2}\pi, \quad i = 3...p. \tag{28}$$

For the beam clamped at both ends (C-C), the collocation functions are of the form:

$$w_i(x) = V(\beta_i)\cdot U\left(\beta_i\frac{x}{L}\right) - U(\beta_i)\cdot V\left(\beta_i\frac{x}{L}\right) \tag{29}$$

where:

$$\beta_1 = 4.73, \beta_2 = 7.8532, \beta_i = \frac{2i+1}{2}\pi, \quad i = 3...p. \tag{30}$$

In the case of the beam clamped at one end and simply-supported at the second end (C-S), the corresponding collocation functions are also of the form, (27) with:

$$\beta_1 = 3.9266, \beta_2 = 7.0685, \beta_i = \frac{4i+1}{4}\pi, \quad i = 3...p. \tag{31}$$

3. NUMERICAL EXAMPLES

In order to check the presented approach, a first comparison with the results presented in [13] was performed. They concern a simply supported beam having the diameter d =1nm (solid nanoshaft), $L = 100d$, $E = 2.1\cdot10^{11}$ Pa, $\rho = 7800$kg/m^3. For different values of the nanoscale parameter μ, the following frequency parameter λ has been computed:

$$\lambda = \sqrt{\frac{\rho A L^4 \omega^2}{EI}}. \tag{32}$$

In reference [13] the results have been obtained using the Rayleigh-Ritz approach. Table 1 presents the results in terms of the square roots of the frequency parameters for the first three modes of vibration for different BCs at ends, namely S-S (simply-supported), C-C (clamped-clamped), C-S (clamped-simply supported).

Table 1. Results comparison for a nanoshaft having different BCs ($\sqrt{\lambda}_{1,2,3}$)

Mode/BC	μ=0		μ=0.3		μ=0.5	
	[13]	Present	[13]	Present	[13]	Present
1/S-S	3.1415	3.1415	2.6800	2.6799	2.3022	2.3021
2/S-S	6.2832	6.2822	4.3013	4.3006	3.4604	3.4598
3/S-S	9.4248	9.4213	5.4422	5.4403	4.2941	4.2925
1/C-C	4.7300	4.7300	3.9184	3.9185	3.3153	3.3155
2/C-C	7.8532	7.8532	5.1963	5.1988	4.1561	4.1586
3/C-C	10.9956	10.9956	6.2317	6.2376	4.9328	4.9387
1/C-S	3.9266	3.9266	3.2828	3.2829	2.7899	2.7900
2/C-S	7.0686	7.0686	4.7668	4.7673	3.8325	3.8331
3/C-S	10.2102	10.2102	5.8371	5.8402	4.6105	4.6134

Table 2 presents the results in terms of the square roots of the frequency parameters for the first three modes of vibration for a cantilever beam (C-F) having $L = 10d$. This case has been reported in [14] and was solved using Differential Quadrature Method (DQM).

Table 2. Results comparison for a cantilever nanoshaft ($\sqrt{\lambda}_{1,2,3}$)

Mode/BC	μ=0		μ=0.1		μ=0.3	
	[14]	Present	[14]	Present	[14]	Present
1/C-F	1.8751	1.8751	1.8792	1.8792	1.9154	1.9154
2/C-F	4.6941	4.6941	4.5475	4.5476	3.7665	3.7673
3/C-F	7.8548	7.8548	7.1459	7.1461	5.2988	5.2995

Another example concerns a simply-supported beam configuration from [5], with $L = 10d$ laying on elastic foundation having constant non-dimensional Winkler coefficient:

$$\overline{K_w} = \frac{K_w L^4}{EI}. \tag{33}$$

Table 3 presents the results in terms of the square roots of frequency parameters for the first three modes of vibration in the case of this beam configuration and different constant Winkler coefficients. The results reported in [5] have been obtained also by DQM.

Table 3. Results comparison for different $\overline{K_w}$ parameters (frequency parameters $\sqrt{\lambda}_{1,2,3}$)

$\overline{K_w}$	$\sqrt{\lambda_1}$		$\sqrt{\lambda_2}$		$\sqrt{\lambda_3}$	
	[5]	Present	[5]	Present	[5]	Present
10	3.2192	3.2192	6.2932	6.2922	9.4277	2.3021
50	3.4844	3.4843	6.3329	6.3320	9.4396	3.4598
100	3.7483	3.7483	6.3816	6.3807	9.4544	4.4511
200	4.1527	4.1527	6.4757	6.4748	9.4839	9.4806
500	4.9438	4.9438	6.7358	6.7350	9.5706	9.5674
1000	5.7556	5.7556	7.1121	7.1114	9.7101	9.7070
1500	6.3219	6.3220	7.4366	7.4361	9.8439	9.8409
2000	6.7673	6.7674	7.7235	7.7230	9.9724	9.9695

All the presented results show good agreement in comparison with the available results of other methods. In this paper the calculations have been performed using n = 100 collocation points, and p = 5 collocation functions.

4. CONCLUSIONS

This paper presents the bending free vibrations analysis of nanobeams starting from the equations of motion obtained with the Eringen non-local theory and Euler-Bernoulli beam model. These equations are taken from literature where they have been obtained for example using Hamilton principle. A nonlocal parameter μ makes the difference with respect to the equations used for dynamic analysis of classic (scale-free) beams.
The *integral method* presented in this paper is based on the use of Green's functions. This approach uses collocation (sampling) equally spaces points on the beam and takes a matrix form using *integration* and *differentiation matrices*. Finally, an eigenvalue problem is obtained allowing the calculation of dynamic characteristics. To reduce the dimension of this eigenvalue problem one can also use *collocation functions* depending on the boundary conditions of the beam. Four different boundary conditions types have been considered and the mode shapes for bending vibrations of classic uniform Euler-Bernoulli beams have been used as collocation functions reducing the dimension of the eigenvalue problem from n = 100 to p = 5. For the simply supported beam case the results in the case of different constant stiffness elastic foundation have been also presented.
The results comparison was performed with results from literature obtained by Rayleigh-Ritz and Differential Quadrature Method (DQM). It shows a very good agreement. The small scale plays a significant role and affects the natural frequencies. Depending on nonlocal (small-scale) coefficient μ, the corresponding frequency parameters for nanobeams are smaller than those obtained for the classic beams especially for higher modes. The presented approach is developed in a matrix form, one that is easy to implement and use for parametric studies. Further developments of this method can include the axial and torsion vibrations analysis, buckling and transverse vibration analysis for other nanobeam configurations.

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