The Curie effect incorporation in the monodomain equation describing the action potential dynamics in cardiac tissue

Agneta M. BALINT[1], Stefan BALINT[2], Adrian NECULAE*,[1]

*Corresponding author
[1] Department of Physics, West University of Timisoara,
Bulv. V. Parvan 4, 300223 Timisoara, Romania,
agneta.balint@e-uvt.ro, adrian.neculae@e-uvt.ro*
[2] Department of Computer Science, West University of Timisoara,
Bulv. V. Parvan 4, 300223 Timisoara, Romania,
stefan.balint@e-uvt.ro

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Abstract: *In their in-depth study on cardiac tissue modeling, Clayton and Panfilov [1] present several monodomain or bidomain approaches for the mathematical description of the cardiac tissue action potential dynamics. For simulation of wave propagation in the cardiac tissue, the monodomain descriptions which use integer order derivatives reproduce many of the phenomena that are observed experimentally and are thus an appropriate analysis tool. The main objection concerning the monodomain approaches is that the electrical circuit capacitor, appearing in these descriptions, is considered ideal (the space between the capacitor plates is vacuum) and the Curie effect is ignored. The Curie effect consists of the fact that in case of a dielectric material, if at a moment of time a constant external voltage is applied, due to the capacitance of the capacitor and the properties of the dielectric, a supplementary electrical current is produced, besides the ohmic current. This supplementary contribution cannot be neglected in some cases.*
In this paper, the Curie effect, describing the action potential dynamics in cardiac tissue, assumed isotropic, is incorporated in the monodomain equation. The novelty is that this approach does not use fractional order derivatives and the obtained mathematical description with these equations is objective.

Key Words: *Curie effect, monodomain equation, action potential dynamics, cardiac tissue*

1. INTRODUCTION

In their in-depth study on cardiac tissue modeling, Clayton and Panfilov [1] present several mathematical descriptions of the cardiac tissue action potential dynamics.

It is highlighted that the currently used and available tissue level descriptions are monodomain or bidomain approaches. In a monodomain description it is assumed that cardiac tissue behaves as an excitable medium, with diffusion and local excitation of membrane voltage. It provides the simplest description of action potential propagation:

$$\frac{\partial V_m}{\partial t} = \nabla \cdot (D \nabla V_m) - \frac{I_{ion} + I_{applied}}{C_m} \tag{1}$$

Here ∇ is the gradient operator, and D is a coefficient with units of distance^2 time^{-1} that describes the effective diffusion of voltage through the medium.

The currents I_{ion} and $I_{applied}$ are conventional, so they represent the flow of positive ions from inside to outside the cell (i.e. flow of Na+ into the cell during the action potential upstroke is a negative current) through the membrane and any applied stimulus, respectively.

For models describing an isotropic tissue, D is a scalar quantity, and is given by:

$$D = \frac{1}{S_v R_i C_m} = \frac{G_i}{S_v C_m} \tag{2}$$

where S_v is the surface volume ratio of cells, R_i the bulk cytoplasmic resistivity of the tissue, C_m the specific capacitance and G_i the bulk intracellular conductivity.

Cardiac tissue anisotropy in monodomain descriptions is given by the diffusion tensor D which can be found from the fiber direction and orientation of the sheet plane.

For simulation of wave propagation in the cardiac tissue, monodomain descriptions, which use integer order derivatives, reproduce many of the phenomena that are observed experimentally, and are thus an appropriate tool. This description is objective. For more details see [2], [3]. The main objection concerning the monodomain approaches is that the electrical circuit capacitor, appearing in these descriptions, is ideal (the space between the capacitor plates is vacuum) and the Curie effect is ignored.

2. THE CURIE EFFECT

Concerning Curie effect, in [4] it is shown that in case of a dielectric material, if at the moment of time $t = 0$ a constant V_0 external voltage is applied, the current intensity produced in that material is:

$$i_C(t) = \frac{V_0}{ht^\alpha} \tag{3}$$

with $0 < \alpha < 1$, $t > 0$ and h a constant related to the capacitance of the capacitor and the properties of the dielectric.

In this mathematical description of current intensity, a moment of time M is represented by a real number $t > 0$ (for what this means see [2], [3], [5]).

This representation assumes (implicitly) that Curie, the investigator of this phenomenon (observer O), has chosen the moment of time M_O corresponding to the event "stopwatch start" as origin for the time measurement (see [2], [3], [5]).

Moreover, it is assumed (implicitly) that the event "stopwatch starts" is simultaneous with the event "the constant V_0 external voltage is applied to the circuit".

3. THE CURIE EFFECT INCORPORATION IN THE EQUATION OF THE ACTION POTENTIAL PROPAGATION IN MONODOMAIN MODEL, FOR ISOTROPIC TISSUE AND CONSTANT DIFFUSION

In [6], Schweidler shows that, due to the Curie effect, the current intensity in the electrical circuit is:

$$i(t) = I(t) + i_C(t) \tag{4}$$

where $i_C(t)$ is the Curie current and $I = I(t)$ is the ohmic current.

Extending these considerations to the Eq. (1), which describes the action potential propagation in monodomain model, for isotropic tissue and constant diffusion the following equation is found:

$$C_m \frac{\partial V_m}{\partial t} = \frac{1}{S_v R_i} \Delta V_m - I_{ion} + i_C(t) \tag{5}$$

where $i_C(t)$ is the Curie current given by (3).

Eq. (5) incorporates the Curie effect and describes the action potential propagation in monodomain approach in case of isotropic tissue.

Proposition 1. The mathematical description of the action potential dynamics in isotropic cardiac tissue with constant diffusion coefficient, using formula (5) is objective.

Proof. Consider two observers O and O^* with their fixed orthogonal reference frame $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$ and $R_O = (O^*; \vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*)$, respectively.

The moment of time M_O, corresponding to the event "the stopwatch of observer O starts", coincides with the moment of time M_{O^*}, corresponding to the event "the stopwatch of observer O^* starts". That is because, these events are simultaneous with the event "the constant V_0 external voltage is applied to the circuit".

Therefore, an arbitrary moment of time M for both observers is described mathematically by the same real number $t > 0$ (for details see [2], [3]).

Observer O describes the action potential dynamics by the real valued function $V_m = V_m(t, x_1, x_2, x_3)$ which checks the following partial differential equation:

$$C_m \frac{\partial V_m}{\partial t} = \frac{1}{S_v R_i} \Delta V_m - I(t)_{ion} + i_C(t) \tag{6}$$

Observer O^* describes the action potential dynamics by the real valued function $V_m^* = V_m^*(t, x_1^*, x_2^*, x_3^*)$ which checks the following partial differential equation:

$$C_m \frac{\partial V_m^*}{\partial t} = \frac{1}{S_v R_i} \Delta V_m^* - I(t)_{ion} + i_C^*(t) \tag{7}$$

From this point for the proof of objectivity see [2], [3].

For anisotropic cardiac tissue and variable diffusion extending Schweidler considerations to Eq. (1) for the dynamics of the action potential the following equation is found:

$$C_m \frac{\partial V_m}{\partial t} = C_m \nabla \cdot (D \nabla V_m) - I(t)_{ion} + i_C(t) \tag{8}$$

Proposition 2. The mathematical description of the action potential dynamics in anisotropic cardiac tissue with variable diffusion, using formula (8) is objective.

4. ON THE MATHEMATICAL DESCRIPTION OF THE CARDIAC TISSUE ACTION POTENTIAL DYNAMICS USING CLASSIC FRACTIONAL ORDER DERIVATIVES

In references [7-13], mathematical descriptions of the dynamics of the action potential in cardiac tissue using equations with fractional order partial derivatives, are presented.

The main idea in these papers is that the Curie effect and the cardiac tissue heterogeneities can be incorporated in classic description, directly substituting the integer order temporal derivatives with classic Caputo or classic Riemann-Liouville temporal fractional order derivatives.

For classic Caputo or classic Riemann-Liouville temporal fractional order derivatives see [14]. In the referred descriptions using fractional order derivatives, the analysis of the objectivity is missing.

In the following it is shown that the mathematical description of the action potential dynamics in isotropic cardiac tissue with constant diffusion coefficient, using classic temporal Caputo or classic temporal Riemann-Liouville fractional order derivatives is not objective.

That is, two observers describing the same dynamics with these tools, obtain two different results.

This is not an academic curiosity! It is rather a problem: which one of the obtained results is correct?

Consider first the use of the temporal classic Caputo fractional partial derivative of order α, $0 < \alpha < 1$ ([2], [3]).

Assume that the reference frames of observers O and O^* coincide and only the moment of times of the start of stopwatches M_O and M_{O^*} are different.

In this case, for isotropic cardiac tissue and constant diffusion coefficient, equation (1) for observers O and O^* becomes:

$$ {}_0^C D_{t_M}^{\alpha} V_m = D \sum_{i=1}^{3} \frac{\partial^2 V_m}{\partial x_i^2} - \frac{I_{ion} + I_{applied}}{C_m} \tag{9} $$

$$ {}_0^C D_{t_M^*}^{\alpha} V_m^* = D \sum_{i=1}^{3} \frac{\partial^2 V_m^*}{\partial x_i^2} - \frac{I_{ion}^* + I_{applied}^*}{C_m} \tag{10} $$

Proposition 3. The mathematical description with equations (9), (10) is not objective.

For details and proof see [2], [3].

Therefore, two observers who describe the same cardiac tissue action potential dynamics with (9) and (10) respectively, obtain two different results. The problem is: which one of the obtained results is correct?

Consider now the use of classic Riemann-Liouville temporal fractional partial derivative of order α, $0 < \alpha < 1$.

Assume that the reference frames of the two observers coincide. In this case, for isotropic cardiac tissue and constant diffusion, equation (1) for observers O and O^* becomes:

$$ {}_0^{R-L} D_{t_M}^{\alpha} V_m = D \sum_{i=1}^{3} \frac{\partial^2 V_m}{\partial x_i^2} - \frac{I_{ion} + I_{applied}}{C_m} \tag{11} $$

$$ {}_0^{R-L} D_{t_M^*}^{\alpha} V_m^* = D \sum_{i=1}^{3} \frac{\partial^2 V_m^*}{\partial x_i^2} - \frac{I_{ion}^* + I_{applied}^*}{C_m} \tag{12} $$

Proposition 4. The mathematical description with equations (11), (12) is not objective.

For details and the proof see [2], [3].

5. CONCLUSIONS AND COMMENTS

1. Simulation of wave propagation in the cardiac tissue using monodomain models reproduce many of the phenomena that are observed experimentally and are thus an appropriate tool. These models are objective. Their main weakness is that the electrical circuit capacitor, appearing in these models, is ideal (the space between the capacitor plates is a vacuum) and the Curie effect is ignored. In this paper, the Curie effect is incorporated in the monodomain cardiac tissue model. For isotropic constant diffusion tissue, the new monodomain model is defined by Eq. (6), and for general monodomain model by Eq. (11). These mathematical descriptions are objective.
2. Mathematical description of the cardiac tissue potential dynamics in case of isotropic and constant diffusion coefficient, using classic temporal Caputo or classic Riemann-Liouville fractional order derivatives, is not objective. The non-objectivity is originated in the incompatibility of the classical temporal Caputo and classical temporal Riemann-Liouville fractional order derivatives with the understanding of the mathematical description of "time" used in case of the mathematical description the evolution of real-world phenomena [5]. Even if, for certain numerical cases, the computed results fit the experimental data well, do not use the classical fractional order model [13] to explain the experimental results. The explanation is questionable because, by changing the origin of the time measurement, the computed results will be different.
3. The relevance of this work to cardiologists performing experiments is: Do not use the fractional order model to explain the experimental results in case of the cardiac tissue potential dynamics.

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