The mathematical description of the bulk fluid flow and that of the content impurity dispersion, obtained by replacing integer order temporal derivatives with general temporal Caputo or general temporal Riemann-Liouville fractional order derivatives, are objective

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Abstract: *In the field of fractional calculus applications, there is a tendency to admit that "integer-order derivatives cannot simply be replaced by fractional-order derivatives to develop fractional-order theories". There are different arguments for that: initialization problem, inconsistency, use of nonsingular or singular kernels, loss of objectivity. In this paper it is shown that the mathematical description of the bulk fluid flow and that of the content impurity spread replacing integer order temporal derivatives with general temporal Caputo or general temporal Riemann-Liouville fractional order derivatives, are objective. More precisely, it is proven that, the mathematical description of the bulk fluid 2D flow and that of the content impurity spread, in a horizontal unconfined aquifer, obtained replacing integer order temporal derivatives with general temporal Caputo or general temporal Riemann-Liouville fractional order derivatives, are objective. It is also proven that, the mathematical description of a Newtonian, incompressible, viscous bulk fluid 3D flow and that of the contained impurity dispersion, obtained replacing integer order temporal derivatives with general temporal Caputo or general temporal Riemann-Liouville fractional order derivatives, are objective. The obtained results show the compatibility of the general temporal Caputo and general temporal Riemann-Liouville fractional order derivatives with the understanding of the "measured time" evolution. In the same time these results reveal that, the objectivity violation, when integer order temporal derivatives are replaced by classic temporal Caputo or classic temporal Riemann-Liouville fractional order derivatives, is originated in the incompatibility of the classic fractional order derivatives, with the understanding of the "measured time" evolution.*

Key Words: *mathematical description, groundwater flow, impurity spread, fractional order derivative, "measured time" evolution*

1. INTRODUCTION

In the early 2000s a discussion has started on the initialization problems when classic temporal Caputo and classic temporal Riemann-Liouville fractional order derivatives are used [1, 2, 3, 4, 5, 6]. Some published results in [7] and [8] concluded to the inconsistency of classic temporal Caputo and classic temporal Riemann-Liouville fractional order derivatives definition to take into account initial conditions if these definitions are used in fractional order partial differential equations or in fractional order ordinary differential equations. In [7, 9, 10] a time shift was used to highlight the above mentioned problem. In [11] the use of non-singular kernels versus the use of singular kernels in fractional calculus is discussed in a similar context. The approach, developed in [12] and [13] to the question: "*why integer-order derivatives cannot be simply replaced by fractional-order derivatives to develop the fractional-order theories*?", is different. In [12] an answer is given to this question, in the case of the 2D-flow of a fluid to the well in an unconfined horizontal aquifer and the spread of the contained impurity. In [13] an answer is given to the question in the case of a Newtonian incompressible, viscous bulk fluid 3D-flow and the dispersion of the contained impurity. In [12] and [13] the description is made replacing directly integer order temporal derivatives with fractional order classic temporal Caputo or classic temporal Riemann-Liouville derivatives. It is proven that the so-obtained mathematical descriptions are non-objective. The objectivity violation is originated in the incompatibility of the classic Caputo and classic Riemann-Liouville fractional order derivatives definition with the understanding of the "measured time" evolution. Remember that in the classical theory of the 2D-flow to the well, in a horizontal, unconfined, homogeneous, isotropic, aquifer [12], [14], [15] the piezometric head $h_O = h_O(t_M, x_1, x_2)$ and the vector valued function $\vec{U}_O = \vec{U}_O(t_M, x_1, x_2)$ that describe the 2D flow velocity, in terms of the observer O, verify equations:

$$S \cdot \frac{\partial h_O}{\partial t_M} + T \cdot \left(\frac{\partial^2 h_O}{\partial x_1^2} + \frac{\partial^2 h_O}{\partial x_2^2}\right) = Q_S \tag{1}$$

$$\vec{U}_O(t_M, x_1, x_2) = -\frac{K}{\phi} \cdot \left(\frac{\partial h_O}{\partial x_1} \cdot \vec{e}_1 + \frac{\partial h_O}{\partial x_2} \cdot \vec{e}_2\right) \tag{2}$$

here: S - is the storage coefficient; T - is the transmissivity of the aquifer; Q_S - is the leakage rate; K - is the hydraulic conductivity and φ - is the porosity. The piezometric head $h_{O^*} = h_{O^*}(t_M^*, x_1^*, x_2^*)$ and the vector valued function $\vec{U}_{O^*} = \vec{U}_{O^*}(t_M^*, x_1^*, x_2^*)$, that describe the 2D-flow in terms of the observer O^* verify equations:

$$S \cdot \frac{\partial h_{O^*}}{\partial t_M^*} + T \cdot (\frac{\partial^2 h_{O^*}}{\partial x_1^{*2}} + \frac{\partial^2 h_{O^*}}{\partial x_2^{*2}}) = Q_S \tag{3}$$

$$\vec{U}_{O^*}(t_M^*, x_1^*, x_2^*) = -\frac{K}{\phi} \cdot (\frac{\partial h_{O^*}}{\partial x_1^*} \cdot \vec{e}_1^* + \frac{\partial h_{O^*}}{\partial x_2^*} \cdot \vec{e}_2^*) \tag{4}$$

In the above equations

$$t_M = t_M^* + t_{M_{O^*}}\,;\ x_k = x_{kO^*} + \sum_{i=1}^{i=2} a_{ik} \cdot x_i^*\,, k = 1{,}2\ \ \vec{e}_i = \sum_{k=1}^{k=2} a_{ki}\vec{e}_k^*\,, i = 1{,}2; \tag{5}$$

$$t_M^* = t_M + t_{M_O}^*;\ \ x_k^* = x_{kO}^* + \sum_{i=1}^{i=2} a_{ki}x_i\,, k = 1{,}2;\ \ \vec{e}_i^* = \sum_{k=1}^{k=2} a_{ik}\vec{e}_k\,, i = 1{,}2 \tag{6}$$

t_M and $t_{M_{O^*}}$ are the real numbers, that represent the moments M and M_{O^*} in the system of time measurement of the observer O, respectively; t_M^* and $t_{M_O}^*$ are the real numbers, that represent the moments M and M_O in the system of time measurement of the observer O^*, respectively; x_k and x_{kO^*} are real numbers that represent the coordinate of the fluid particle P and the point O^* in the reference frame $R_O = (O; \vec{e}_1, \vec{e}_2)$ of observer O, respectively. x_k^* and x_{kO}^* are real numbers that represent the coordinate of the fluid particle P and the point O in the reference frame $R_{O^*} = (O^*; \vec{e}_1^*, \vec{e}_2^*)$ of observer O^*, respectively.

The spread of an impurity, contained in the bulk fluid, flowing to the well in case of the 2D flow, in a horizontal unconfined aquifer, for observer O is described by equation:

$$\frac{\partial C_O}{\partial t_M} = \sum_{i=1}^{i=2} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{j=2} D_O^{ij} \cdot \frac{\partial C_O}{\partial x_j}\right) - \sum_{i=1}^{i=2} \frac{\partial}{\partial x_i}\left(U_{Oi}(t_M, x_1, x_2) \cdot C_O\right) + S_O \tag{7}$$

Under the same hypothesis for observer O^* the spread of the impurity is described by equation:

$$\frac{\partial C_{O^*}}{\partial t_M^*} = \sum_{i=1}^{i=2} \frac{\partial}{\partial x_i^*}\left(\sum_{j=1}^{j=2} D_{O^*}^{ij} \cdot \frac{\partial C_{O^*}}{\partial x_j^*}\right) - \sum_{i=1}^{i=2} \frac{\partial}{\partial x_i}\left(U_{O^*i}(t_M^*, x_1^*, x_2^*) \cdot C_{O^*}\right) + S_{O^*} \tag{8}$$

For a continuously differentiable function $f: (-\infty, \infty) \times (-\infty, \infty) \to R$ the general temporal Caputo fractional derivative of order α, $0 < \alpha$ is defined with the following integral representation (see [16]):

$${}_{-\infty}^{\;\;C}D_t^{\alpha} f(t, x) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_{-\infty}^{t} \frac{\frac{\partial^n f}{\partial \tau^n}(\tau, x)}{(t-\tau)^{\alpha+1-n}} d\tau \tag{9}$$

The general temporal Riemann-Liouville fractional derivative of order α, $0 < \alpha$, is defined with the following integral representation (see [16]):

$${}_{-\infty}^{R-L}D_t^{\alpha} f(t, x) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{\partial^n}{\partial t^n} \int_{-\infty}^{t} \frac{f(\tau, x)}{(t-\tau)^{\alpha+1-n}} d\xi \tag{10}$$

In formulas (9) and (10), Γ is the Euler gamma function and $n = [\alpha] + 1$, $[\alpha]$ being the integer part of α

2. IN CASE OF THE 2D FLOW TO THE WELL, IN AN UNCONFINED HORIZONTAL HOMOGENEOUS AND ISOTROPIC AQUIFER, THE PIEZOMETRIC HEAD DYNAMICS DESCRIPTION, WHICH USES GENERAL TEMPORAL CAPUTO FRACTIONAL ORDER DERIVATIVE, IS OBJECTIVE

Assume that, in case of the 2D flow to a well in a horizontal unconfined isotropic homogeneous aquifer, in the piezo metric head dynamics description, the general temporal Caputo fractional derivative of order α, $0 < \alpha < 1$ is used. In this case, Eq. (1) for observer O and Eq. (2) for observer O^* become:

$$S \cdot {}_{-\infty}^{\;\;C}D_{t_M}^{\alpha} h_O + T \cdot \left(\frac{\partial^2 h_O}{\partial x_1^2} + \frac{\partial^2 h_O}{\partial x_2^2}\right) = Q_S \tag{11}$$

$$S\cdot {}_{-\infty}^{\;\;C}D_{t_M^*}^{\alpha}h_{O^*}+T\cdot(\frac{\partial^2 h_{O^*}}{\partial x_1^{*2}}+\frac{\partial^2 h_{O^*}}{\partial x_2^{*2}})=Q_S \tag{12}$$

The piezometric head dynamics description with Eq. (11) and Eq. (12) is objective if the solutions of the fractional partial differential equations (11) and (12) describe the same dynamics. This means that the following statements hold:

i). if $h_O(t_M,x_1,x_2)$ verifies Eq. (11), then function $h_{O^*}(t_M^*,x_1^*,x_2^*)$, defined by

$$h_{O^*}(t_M^*,x_1^*,x_2^*)=h_O(t_M^*+t_{M_{O^*}},x_{1O^*}+\sum_{i=1}^{i=2}a_{i1}\cdot x_i^*,x_{2O^*}+\sum_{i=1}^{i=2}a_{i2}\cdot x_i^*)$$

verifies Eq. (12), and

ii). if $h_{O^*}(t_M^*,x_1^*,x_2^*)$ verifies Eq. (12), then function $h_O(t_M,x_1,x_2)$, defined by

$$h_O(t_M,x_1,x_2)=h_{O^*}\left(t_M+t_{M_O}^*,x_{1O}^*+\sum_{i=1}^{i=2}a_{1i}\cdot x_i,x_{2O}^*+\sum_{i=1}^{i=2}a_{2i}\cdot x_i\right),$$

verifies the Eq. (11). The proof of the statement i) is the following: start with a solution $h_O(t_M,x_1,x_2)$ of the Eq. (11) and consider function $h_{O^*}(t_M^*,x_1^*,x_2^*)$, defined by

$$h_{O^*}(t_M^*,x_1^*,x_2^*)=h_O\left(t_M^*+t_{M_{O^*}},x_{1O^*}+\sum_{i=1}^{i=2}a_{i1}\cdot x_i^*,x_{2O^*}+\sum_{i=1}^{i=2}a_{i2}\cdot x_i^*\right)$$

Remark that, the following equalities hold:

$$ {}_{-\infty}^{\;\;C}D_{t_M^*}^{\alpha}h_{O^*}(t_M^*,x_1^*,x_2^*)={}_{-\infty}^{\;\;C}D_{t_M}^{\alpha}h_O(t_M,x_1,x_2) \tag{13}$$

$$T(\frac{\partial^2 h_O}{\partial x_1^2}+\frac{\partial^2 h_O}{\partial x_2^2})=T\cdot(\frac{\partial^2 h_{O^*}}{\partial x_1^{*2}}+\frac{\partial^2 h_{O^*}}{\partial x_2^{*2}})$$

Using equalities (13) and replacing the terms in (11), it follows that: if function $h_O(t_M,x_1,x_2)$ verifies Eq. (11), then function $h_{O^*}(t_M^*,x_1^*,x_2^*)$ defined by

$$h_{O^*}(t_M^*,x_1^*,x_2^*)=h_O\left(t_M^*+t_{M_{O^*}},x_{1O^*}+\sum_{i=1}^{i=2}a_{i1}\cdot x_i^*,x_{2O^*}+\sum_{i=1}^{i=2}a_{i2}\cdot x_i^*\right),$$

verifies Eq. (12).

So, the statement i) was proven. The proof of the statement ii) is similar. Remark that the objectivity of the 2-D flow description with Eq. 2 and Eq. 4. is a consequence of equality

$$-\frac{K}{\varphi}\cdot\left(\frac{\partial h_O}{\partial x_1}\cdot\vec{e}_1+\frac{\partial h_O}{\partial x_2}\cdot\vec{e}_2\right)=-\frac{K}{\varphi}\cdot(\frac{\partial h_{O^*}}{\partial x_1^*}\cdot\vec{e}_1^{\,*}+\frac{\partial h_{O^*}}{\partial x_2^*}\cdot\vec{e}_2^{\,*})$$

3. IN CASE OF THE 2D FLOW TO THE WELL, IN AN UNCONFINED HORIZONTAL, HOMOGENEOUS AND ISOTROPIC AQUIFER, THE PIEZOMETRIC HEAD DYNAMICS DESCRIPTION, WHICH USES GENERAL TEMPORAL RIEMANN-LIOUVILLE FRACTIONAL ORDER DERIVATIVE, IS OBJECTIVE

Assume that, in case of the 2D flow to the well, in a horizontal, unconfined, isotropic, homogeneous, aquifer, in the piezometric head dynamics description, the general temporal Riemann-Liouville fractional derivative of order α, $0<\alpha<1$ is used. In this case Eq. (1) for observer O and Eq. (2) for observer O^* become:

$$S\cdot {}^{R-L}_{-\infty}D^{\alpha}_{t_M}h_O+T\cdot\left(\frac{\partial^2 h_O}{\partial x_1^2}+\frac{\partial^2 h_O}{\partial x_2^2}\right)=Q_S \tag{14}$$

$$S\cdot {}^{R-L}_{-\infty}D^{\alpha}_{t_M^*}h_{O^*}\ +T\cdot(\frac{\partial^2 h_{O^*}}{\partial x_1^{*2}}+\frac{\partial^2 h_{O^*}}{\partial x_2^{*2}})=Q_S \tag{15}$$

The piezometric head dynamics description is objective if equations (14) and (15) describe the same dynamics. This means that the following statements hold:

i). if $h_O(t_M,x_1,x_2)$ verifies Eq. (14), then function $h_{O^*}(t_M^*,x_1^*,x_2^*)$, defined by

$$h_{O^*}(t_M^*,x_1^*,x_2^*)=h_O(t_M^*+t_{M_{O^*}},x_{1O^*}+\sum_{i=1}^{i=2}a_{i1}\cdot x_i^*,x_{2O^*}+\sum_{i=1}^{i=2}a_{i2}\cdot x_i^*)$$

verifies Eq. (15) and

ii). if $h_{O^*}(t_M^*,x_1^*,x_2^*)$ verifies Eq. (15), then function $h_O(t_M,x_1,x_2)$, defined by

$$h_O(t_M,x_1,x_2)=h_{O^*}\left(t_M+t_{M_O}^*,x_{1O}^*+\sum_{i=1}^{i=2}a_{1i}\cdot x_i,x_{2O}^*+\sum_{i=1}^{i=2}a_{2i}\cdot x_i\right)$$

verifies Eq. (14). The proof of the statement i) is the following: start with a solution $h_O(t_M,x_1,x_2)$ of Eq. (14) and consider function $h_{O^*}(t_M^*,x_1^*,x_2^*)$, defined by

$$h_{O^*}(t_M^*,x_1^*,x_2^*)=h_O\left(t_M^*+t_{M_{O^*}},x_{1O^*}+\sum_{i=1}^{i=2}a_{i1}\cdot x_i^*,x_{2O^*}+\sum_{i=1}^{i=2}a_{i2}\cdot x_i^*\right)$$

Remark that, the following equalities hold:

$${}^{R-L}_{-\infty}D^{\alpha}_{t_M^*}h_{O^*}(t_M^*,x_1^*,x_2^*)={}^{R-L}_{-\infty}D^{\alpha}_{t_M}h_O(t_M,x_1,x_2) \tag{16}$$

$$T\cdot\left(\frac{\partial^2 h_O}{\partial x_1^2}+\frac{\partial^2 h_O}{\partial x_2^2}\right)=T\cdot\left(\frac{\partial^2 h_{O*}}{\partial x_1^{*2}}+\frac{\partial^2 h_{O*}}{\partial x_2^{*2}}\right)$$

Using equalities (16) and replacing the terms in (14), it follows that: if function $h_O(t_M,x_1,x_2)$ verifies Eq. (14), then function $h_{O^*}(t_M^*,x_1^*,x_2^*)$, defined by

$$h_{O^*}(t_M^*,x_1^*,x_2^*)=h_O\left(t_M^*+t_{M_{O^*}},x_{1O^*}+\sum_{i=1}^{i=2}a_{i1}\cdot x_i^*,x_{2O^*}+\sum_{i=1}^{i=2}a_{i2}\cdot x_i^*\right),$$

verifies Eq. (15). So, the statement i) was proven. The proof of the statement ii) is similar. Remark that the objectivity of the 2-D flow description with Eq. 2 and Eq. 4. is a consequence of equality

$$-\frac{K}{\varphi}\cdot\left(\frac{\partial h_O}{\partial x_1}\cdot\vec{e}_1+\frac{\partial h_O}{\partial x_2}\cdot\vec{e}_2\right)=-\frac{K}{\varphi}\cdot\left(\frac{\partial h_{O^*}}{\partial x_1^*}\cdot\vec{e}_1^*+\frac{\partial h_{O^*}}{\partial x_2^*}\cdot\vec{e}_{21}^*\right)$$

4. IN CASE OF THE 2D FLOW TO THE WELL, IN AN UNCONFINED HORIZONTAL, HOMOGENEOUS AND ISOTROPIC AQUIFER, THE IMPURITY SPREAD DESCRIPTION, WHICH USES GENERAL TEMPORAL CAPUTO FRACTIONAL ORDER DERIVATIVE, IS OBJECTIVE

Assume that in the impurity spread dynamics description the general temporal Caputo fractional derivative of order α, $0<\alpha<1$ is used. In this case, Eq. (7) for observer O and Eq. (8) for observer O^* becomes:

$$ {}_{-\infty}^{\ C}D_{t_M}^{\alpha}C_O = D\cdot\sum_{i=1}^{i=2}\frac{\partial^2 C_O}{\partial x_1^2} - \sum_{i=1}^{i=2} U_{Oi}(t_M,x_1,x_2)\cdot\frac{\partial C_O}{\partial x_i} + S \tag{17} $$

$$ {}_{-\infty}^{\ C}D_{t_M^*}^{\alpha}C_{O^*} = D\cdot\sum_{i=1}^{i=2}\frac{\partial^2 C_{O^*}}{\partial {x_1^*}^2} - \sum_{i=1}^{i=2} U_{O^*i}(t_M^*,x_1^*,x_2^*)\cdot\frac{\partial C_{O^*}}{\partial x_i^*} + S \tag{18} $$

The impurity spread description is objective if the fractional partial differential equations (17) and (18) describe the same dynamics. This means that the following two statements hold:
i) if $C_O(t_M,x_1,x_2)$ verifies Eq. (17), then function $C_{O^*}(t_M^*,x_1^*,x_2^*)$, defined by

$$ C_{O^*}(t_M^*,x_1^*,x_2^*) = C_O\left(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{i=1}^{i=2} a_{i1}x_i^*, x_{2O*} + \sum_{i=1}^{i=2} a_{i2}x_i^*\right), $$

verifies Eq. (18) and
ii) if $C_{O^*}(t_M^*,x_1^*,x_2^*)$ verifies Eq. (18), then function $C_O(t_M,x_1,x_2)$, defined by

$$ C_O(t_M,x_1,x_2) = C_{O^*}(t_M + t_{M_O}^*, x_{1O}^* + \sum_{i=1}^{i=2} a_{1i}x_i, x_{2O}^* + \sum_{i=1}^{i=2} a_{2i}x_i), $$

verifies Eq. (17). The proof of the statement i) is the following: start with a solution $C_O(t_M,x_1,x_2)$ of the Eq. (17) and consider function $C_{O^*}(t_M^*,x_1^*,x_2^*)$, defined by

$$ C_{O^*}(t_M^*,x_1^*,x_2^*) = C_O\left(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{i=1}^{i=2} a_{i1}\cdot x_i^*, x_{2O^*} + \sum_{i=1}^{i=2} a_{i2}\cdot x_i^*\right). $$

Remark that, the following equalities hold:

$$ {}_{-\infty}^{\ C}D_{t_M^*}^{\alpha}C_{O^*}(t_M^*,x_1^*,x_2^*), = {}_{-\infty}^{\ C}D_{t_M}^{\alpha}C_O(t_M,x_1,x_2) \tag{19} $$

$$ D\cdot\sum_{i=1}^{i=2}\frac{\partial^2 C_O}{\partial x_i^2} - \sum_{i=1}^{i=2} U_{Oi}(t_M,x_1,x_2)\cdot\frac{\partial C_O}{\partial x_i} + S = D\cdot\sum_{i=1}^{i=2}\frac{\partial^2 C_{O^*}}{\partial {x_i^*}^2} - \sum_{i=1}^{i=2} U_{O^*i}(t_M^*,x_1^*,x_2^*)\cdot\frac{\partial C_{O^*}}{\partial x_i^*} + S $$

Using equalities (19) and replacing the terms in (17) it follows that function $C_{O^*}(t_M^*,x_1^*,x_2^*)$ defined, by

$$ C_{O^*}(t_M^*,x_1^*,x_2^*) = C_O\left(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{i=1}^{i=2} a_{i1}\cdot x_i^*, x_{2O^*} + \sum_{i=1}^{i=2} a_{i2}\cdot x_i^*\right), $$

verifies Eq. (18). The proof of the statement ii) is similar.

5. IN CASE OF THE 2D FLOW TO THE WELL, IN AN UNCONFINED HORIZONTAL, ISOTROPIC AND HOMOGENEOUS AQUIFER, THE IMPURITY SPREAD DESCRIPTION, WHICH USES GENERAL TEMPORAL RIEMANN-LIOUVILLE FRACTIONAL ORDER DERIVATIVE, IS OBJECTIVE

Assume that, in the impurity spread dynamics description, the general temporal Riemann-Liouville fractional partial derivative of order α, $0 < \alpha < 1$ is used. In this case, Eq. (7) for observer O and Eq. (8) for observer O^* become:

$$ {}^{R-L}_{-\infty}D^{\alpha}_{t_M} C_O = D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_O}{\partial x_i^2} - \sum_{i=1}^{i=2} U_{Oi}(t_M, x_1, x_2) \cdot \frac{\partial C_O}{\partial x_i} + S \tag{20} $$

$$ {}^{R-L}_{-\infty}D^{\alpha}_{t_M^*} C_{O^*} = D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_{O^*}}{\partial {x_i^*}^2} - \sum_{i=1}^{i=\#} U_{O^*i}(t_M^*, x_1^*, x_2^*) \cdot \frac{\partial C_{O^*}}{\partial x_i^*} + S \tag{21} $$

The impurity spread description is objective if equations (20) and (21) describe the same dynamics.

This means that the following two statements hold:

i) if $C_O(t_M, x_1, x_2)$ verifies Eq. (20), then function $C_{O^*}(t_M^*, x_1^*, x_2^*)$, defined by

$$ C_{O^*}(t_M^*, x_1^*, x_2^*) = C_O\left(t_M^* + t_{M_{O^*}}, x_{10^*} + \sum_{i=1}^{i=2} a_{i1} x_i^*, x_{20^*} + \sum_{i=1}^{i=2} a_{i2} x_i^* \right), $$

verifies Eq. (21) and

ii) if $C_{O^*}(t_M^*, x_1^*, x_2^*)$verifies equation (21), then function $C_O(t_M, x_1, x_2)$, defined by

$$ C_O(t_M, x_1, x_2) = C_{O^*}(t_M + t_{M_O}^*, x_{1O}^* + \sum_{i=1}^{i=2} a_{1i} x_i, x_{2O}^* + \sum_{i=1}^{i=2} a_{2i} x_i), $$

verifies Eq. (20).

The proof of the above statements is similar to that presented in section 4.

6. IN CASE OF THE 3D FLOW OF A NEWTONIAN INCOMPRESSIBLE VISCOUS FLUID IN A CONTAINER, THE FLOW DESCRIPTION WHICH USES GENERAL TEMPORAL CAPUTO FRACTIONAL ORDER DERIVATIVE, IS OBJECTIVE

Observer O describes the 3D flow of the Newtonian incompressible viscous bulk fluid with a vector valued function $\vec{U}_O = \vec{U}_O(t_M, x_1, x_2, x_3)$, representing the velocity field, and a real valued function $p_O = p_O(t_M, x_1, x_2, x_3)$, representing the pressure field.

In the classical theory of fluid mechanics [13], [17], [18] the functions $\vec{U}_O = \vec{U}_O(t_M, x_1, x_2, x_3)$, $p_O = p_O(t_M, x_1, x_2, x_3)$ which describe, the 3D flow of a Newtonian incompressible viscous bulk fluid having constant viscosity and density, verify the Navier-Stokes equations:

$$ \begin{gathered} \frac{\partial U_{Oi}}{\partial t_M} + \sum_{k=1}^{k=3} U_{Ok} \cdot \frac{\partial U_{Oi}}{\partial x_k} = -\frac{1}{\rho_0} \cdot \frac{\partial p_O}{\partial x_i} + \frac{\mu}{\rho_0} \cdot \sum_{k=1}^{k=3} \frac{\partial^2 U_{Ok}}{\partial x_k^2} + f_{Oi} \quad \mathrm{i} = 1,2,3 \\ \sum_{k=1}^{k=3} \frac{\partial U_{Oi}}{\partial x_k} = 0 \end{gathered} \tag{22} $$

In the Navier-Stokes equations: ρ_0 - constant is the fluid density, μ - constant is the fluid viscosity, f_{Oi} are the components of the mass force divided by the fluid density.

Observer O^* describes the same 3D flow with functions $\vec{U}_{O^*} = \vec{U}_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$, $p_{O^*} = p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ which verify the equations:

$$\frac{\partial U_{O^*i}}{\partial t_M^*}+\sum_{k=1}^{k=3}U_{O^*i}\cdot\frac{\partial U_{O^*i}}{\partial x_k^*}=-\frac{1}{\rho_0}\cdot\frac{\partial p_{O*}}{\partial x_i^*}+\frac{\mu}{\rho_0}\cdot\sum_{k=1}^{k=3}\frac{\partial^2 U_{O^*k}}{\partial {x_i^*}^2}+f_{O^*i}\quad \mathrm{i}=1,2,3$$
$$\sum_{k=1}^{k=3}\frac{\partial U_{O^*i}}{\partial x_k^*}=0 \tag{23}$$

In (24) a moment of time M is described by the real number t_M and in (25) the same moment M is described by the real number t_M^*. For the numbers t_M and t_M^* the following relations hold: $t_M=t_M^*+t_{M_{O^*}}$, $t_M^*=t_M+t_{M_O}^*$; where $t_{M_{O^*}}$ is the real number which represents the moment M_{O^*} in the system of time measuring of the observer O and $t_{M_O}^*$ is the real number which represents the moment M_O in the system of time measuring of the observer O^*. At any moment of time M, the coordinates (x_1,x_2,x_3) with respect to the reference frame R_O and the coordinates (x_1^*,x_2^*,x_3^*) with respect to the reference frame R_{O^*}, represent the same position P in the affine Euclidian space E_3. Therefore, for the coordinates the following relations hold:

$$x_k=x_{kO^*}+\sum_{i=1}^{i=3}a_{ik}\cdot x_i^*\quad k=1,2,3;\quad x_k^*=x_{kO}^*+\sum_{i=1}^{i=3}a_{ki}x_i\quad k=1,2,3$$

The significance of the quantities appearing in the above relations are:
$a_{ij}=\langle\vec{e}_i^*,\vec{e}_j\rangle=$ constant = scalar product of the unit vectors $\vec{e}_i^*$ and $\vec{e}_j$ in E_3 i.e.

$$\vec{e}_i^*=\sum_{k=1}^{k=3}a_{ik}\vec{e}_k\qquad \vec{e}_i=\sum_{k=1}^{k=3}a_{ki}\vec{e}_k^*$$

$(x_{1O^*},x_{2O^*},x_{3O^*})$ are the coordinates of the point O^* with respect to the reference frame $R_O=(O;\vec{e}_1,\vec{e}_2,\vec{e}_3)$, $(x_{1O}^*,x_{2O}^*,x_{3O}^*)$ are the coordinates of the point O with respect to the reference frame $R_{O^*}=(O^*;\vec{e}_1^*,\vec{e}_2^*,\vec{e}_3^*)$.
Because functions f_{O_i} and $f_{O_i^*}$ represent the same force field (in two different reference frames) the components verify the following relations:

$$f_{O^*k}=a_{k1}f_{O1}+a_{k2}f_{O2}+a_{k3}f_{O3}\quad f_{Ok}=a_{1k}f_{O^*1}+a_{2k}f_{O^*2}+a_{3k}f_{O^*3}\qquad k=1,2,3;$$

Assume that in the 3D flow description the temporal Caputo fractional partial derivative of order α, $0<\alpha<1$ is used. Eq. (22) for observer O and Eq. (23) for observer O^* become:

$${}_{-\infty}^{\;\;C}D_{t_M}^{\alpha}U_{Oi}+\sum_{k=1}^{k=3}U_{Ok}\cdot\frac{\partial U_{Oi}}{\partial x_k}=-\frac{1}{\rho_0}\cdot\frac{\partial p_O}{\partial x_i}+\frac{\mu}{\rho_0}\cdot\sum_{k=1}^{k=3}\frac{\partial^2 U_{Ok}}{\partial x_k^2}+f_{Oi}\quad \mathrm{i}=1,2,3$$
$$\sum_{k=1}^{k=3}\frac{\partial U_{Oi}}{\partial x_k}=0 \tag{24}$$

$${}_{-\infty}^{\;\;C}D_{t_M^*}^{\alpha}U_{O^*i}+\sum_{k=1}^{k=3}U_{O^*k}\cdot\frac{\partial U_{O^*i}}{\partial x_k^*}=-\frac{1}{\rho_0}\cdot\frac{\partial p_{O^*}}{\partial x_i^*}+\frac{\mu}{\rho_0}\cdot\sum_{k=1}^{k=3}\frac{\partial^2 U_{O^*k}}{\partial {x_k^*}^2}+f_{O^*i}\quad \mathrm{i}=1,2,3$$
$$\sum_{k=1}^{k=3}\frac{\partial U_{O^*i}}{\partial x_k^*}=0 \tag{25}$$

The 3D flow description is objective if equations (24) and (25) describe the same dynamics. This means that the following statements hold:

i). if functions $U_{Ok}(t_M, x_1, x_2, x_3); \quad k = 1,2,3,\ p_O(t_M, x_1, x_2, x_3)$ verify equations (24), then functions $U_{O^*k}(t_M^*, x_1^*, x_2^*, x_3^*); \quad k = 1,2,3,\ p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$, defined by:

$$U_{O^*k}(t_M^*, x_1^*, x_2^*, x_3^*) = \sum_{i=1}^{i=3} a_{ki} U_{Oi}\left(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{j=1}^{j=3} a_{j1} \cdot x_j^*, x_{2O*} + \sum_{j=1}^{j=3} a_{j2} \cdot x_j^*, x_{3O*} + \sum_{j=1}^{j=3} a_{j3} \cdot x_j^* \right) \quad k = 1,2,3$$

$$p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*) = p_O(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{i=1}^{i=3} a_{i1} \cdot x_i^*, x_{2O^*} + \sum_{i=1}^{i=3} a_{i2} \cdot x_i^*, x_{3O^*} + \sum_{i=1}^{i=3} a_{i3} \cdot x_i^*)$$

verify equations (25).

ii). if functions $U_{O^*k}(t_M^*, x_1^*, x_2^*, x_3^*); k = 1,2,3,\ p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ verify equations (25), then functions $(t_M, x_1, x_2, x_3); \quad k = 1,2,3,\ p_O(t_M, x_1, x_2, x_3)$ defined by:

$$U_{Ok}(t_M, x_1, x_2, x_3) = \sum_{i=1}^{i=3} a_{ik} U_{O^*i}\left(t_M + t_{M_O}^*, x_{1O}^* + \sum_{j=1}^{j=3} a_{1j} \cdot x_j, x_{2O}^* + \sum_{j=1}^{j=3} a_{2j} \cdot x_j, x_{3O}^* + \sum_{j=1}^{j=3} a_{3j} \cdot x_j \right) \quad k = 1,2,3$$

$$p_O(t_M, x_1, x_2, x_3) = p_{O^*}(t_M + t_{M_O}^*, x_{1O}^* + \sum_{i=1}^{i=3} a_{1i} x_i, x_{2O}^* + \sum_{i=1}^{i=3} a_{2i} x_i, x_{3O}^* + \sum_{i=1}^{i=3} a_{3i} x_i)$$

verify equations (24). The proof of the statement i) is the following: start with a solution $U_{Ok}(t_M, x_1, x_2, x_3); \quad k = 1,2,3,\ p_O(t_M, x_1, x_2, x_3)$ of equations (24) and consider functions $U_{O^*k}(t_M^*, x_1^*, x_2^*, x_3^*); k = 1,2,3,\ p_{O^*}(t_M^*, t_M^*, x_1^*, x_2^*, x_3^*)$ defined by:

$$U_{O^*k}(t_M^*, x_1^*, x_2^*, x_3^*) = \sum_{i=1}^{i=3} a_{ki} U_{Oi}\left(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{j=1}^{j=3} a_{j1} \cdot x_j^*, x_{2O^*} + \sum_{j=1}^{j=3} a_{j2} \cdot x_j^*, x_{3O^*} + \sum_{j=1}^{j=3} a_{j3} \cdot x_j^* \right) \quad k = 1,2,3$$

$$p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*) = p_O(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{i=1}^{i=3} a_{i1} \cdot x_i^*, x_{2O^*} + \sum_{i=1}^{i=3} a_{i2} \cdot x_i^*, x_{3O^*} + \sum_{i=1}^{i=3} a_{i3} \cdot x_i^*)$$

Remark that the following equalities hold:

$$ {}_{-\infty}^{\;C}D_{t_M^*}^{\alpha} U_{O^*i} = \sum_{k=1}^{k=3} a_{ik} \cdot {}_{-\infty}^{\;C}D_{t_M}^{\alpha} U_{Oi}\, ; \quad \sum_{k=1}^{k=3} U_{O^*k} \cdot \frac{\partial U_{O^*i}}{\partial x_k^*} = \sum_{k=1}^{k=3} a_{ik} \cdot \sum_{m=1}^{m=3} \frac{\partial U_{Ok}}{\partial x_m} \cdot U_{Om}\, ; \frac{\partial p_{O^*}}{\partial x_i^*} = \sum_{k=1}^{k=3} a_{ik} \cdot \frac{\partial p_O}{\partial x_k} $$

$$ \sum_{k=1}^{k=3} \frac{\partial^2 U_{O^*k}}{\partial {x_k^*}^2} = \sum_{i=1}^{i=3} a_{ik} \cdot \frac{\partial^2 U_{Ok}}{\partial {x_i}^2}; \quad \sum_{k=1}^{k=3} \frac{\partial U_{O^*i}}{\partial x_k^*} = \sum_{k=1}^{k=3} \frac{\partial U_{Oi}}{\partial x_k} $$

Using the above equalities and replacing the terms in (25) via (24) it follows that functions $U_{O^*k}(t_M^*, x_1^*, x_2^*, x_3^*); k = 1,2,3$ and $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ verify the equations (25).
The proof of the statement ii) is similar.

7. IN CASE OF THE 3D FLOW OF A NEWTONIAN INCOMPRESSIBLE VISCOUS FLUID IN A CONTAINER, THE FLOW DESCRIPTION WHICH USES GENERAL TEMPORAL RIEMANN-LIOUVILLE FRACTIONAL ORDER DERIVATIVE, IS OBJECTIVE.

When in the description of the 3D flow, the temporal Riemann-Liouville fractional partial derivative of order α, $0 < \alpha < 1$ is used then Eq. (22) for observer O and Eq. (23) for observer O^* become:

$$ {}_{-\infty}^{R-L}D_{t_M}^{\alpha} U_{Oi} + \sum_{k=1}^{k=3} U_{Ok} \cdot \frac{\partial U_{Oi}}{\partial x_k} = -\frac{1}{\rho_0} \cdot \frac{\partial p_O}{\partial x_i} + \frac{\mu}{\rho_0} \cdot \sum_{k=1}^{k=3} \frac{\partial^2 U_{Ok}}{\partial {x_k}^2} + f_{Oi} \quad \mathrm{i} = 1,2,3 $$
$$ \sum_{k=1}^{k=3} \frac{\partial U_{Oi}}{\partial x_k} = 0 \tag{26} $$

$$ {}_{-\infty}^{R-L}D_{t_M^*}^{\alpha} U_{O^*i} + \sum_{k=1}^{k=3} U_{O^*k} \cdot \frac{\partial U_{O^*i}}{\partial x_k^*} = -\frac{1}{\rho_0} \cdot \frac{\partial p_{O^*}}{\partial x_i^*} + \frac{\mu}{\rho_0} \cdot \sum_{k=1}^{k=3} \frac{\partial^2 U_{O^*k}}{\partial {x_k^*}^2} + f_{O^*i} \quad \mathrm{i} = 1,2,3 $$
$$ \sum_{k=1}^{k=3} \frac{\partial U_{O^*i}}{\partial x_k^*} = 0 \tag{27} $$

The 3D flow description is objective if equations (26) and (27) describe the same dynamics. This means that the following statements hold:
i). if functions $U_{Ok}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ verify equations (26), then functions $U_{O^*k}(t_M^*, x_1^*, x_2^*, x_3^*); k = 1,2,3$, $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$, defined by:

$$ U_{O^*k}(t_M^*, x_1^*, x_2^*, x_3^*) = \sum_{i=1}^{i=3} a_{ki} U_{Oi}\left(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{j=1}^{j=3} a_{j1} \cdot x_j^*, x_{2O^*} + \sum_{j=1}^{j=3} a_{j2} \cdot x_j^*, x_{3O^*} + \sum_{j=1}^{j=3} a_{j3} \cdot x_j^* \right) \quad k = 1,2,3 $$

$$p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*) = p_O(t_M^* + t_{M_{O^*}}, x_{10^*} + \sum_{i=1}^{i=3} a_{i1} \cdot x_i^*, x_{20^*} + \sum_{i=1}^{i=3} a_{i2} \cdot x_i^*, x_{30^*} + \sum_{i=1}^{i=3} a_{i3} \cdot x_i^*)$$

verify equations (27).

ii). if the functions $U_{O^*k}(t_M^*, x_1^*, x_2^*, x_3^*); k = 1,2,3$, $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ verify equations (27), then functions $U_{Ok}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ defined by:

$$U_{Ok}(t_M, x_1, x_2, x_3) = \sum_{i=1}^{i=3} a_{ik} U_{O^*i}\left(t_M + t_{M_O}^*, x_{1O}^* + \sum_{j=1}^{j=3} a_{1j} \cdot x_j, x_{2O}^* + \sum_{j=1}^{j=3} a_{2j} \cdot x_j, x_{3O}^* + \sum_{j=1}^{j=3} a_{3j} \cdot x_j\right) \quad k = 1,2,3$$

$$p_O(t_M, x_1, x_2, x_3) = p_{O^*}(t_M + t_{M_O}^*, x_{1O}^* + \sum_{i=1}^{i=3} a_{1i} x_i, x_{2O}^* + \sum_{i=1}^{i=3} a_{2i} x_i, x_{3O}^* + \sum_{i=1}^{i=3} a_{3i} x_i)$$

verify equations (26).

The proof of the statements i). and ii). is similar with the proof of the same statements when general temporal Caputo derivatives are used in the description.

8. IN CASE OF THE 3D FLOW OF A NEWTONIAN INCOMPRESSIBLE VISCOUS FLUID IN A CONTAINER, THE CONTAINED IMPURITY DISPERSION DESCRIPTION WHICH USES GENERAL TEMPORAL CAPUTO OR GENERAL TEMPORAL RIEMANN-LIOUVILLE FRACTIONAL ORDER DERIVATIVE, IS OBJECTIVE

The dispersion of the impurity contained in the bulk fluid, is described with the concentration of that impurity. Observer O describes the concentration with the real valued function $C_O = C_O(t_M, x_1, x_2, x_3)$ which verifies the partial differential equation:

$$\frac{\partial C_O}{\partial t_M} = \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i}(D_O \cdot \frac{\partial C_O}{\partial x_i}) - \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i}(U_{Oi}(t_M, x_1, x_2, x_3) \cdot C_O) + S_O \tag{28}$$

where: $D_O = D_O(x_1, x_2, x_3)$ is the diffusivity (also called diffusion coefficient), $U_{Oi}(t_M, x_1, x_2, x_3)$ are the components of the bulk fluid flow velocity, the term $\sum_{i=1}^{i=3} \frac{\partial}{\partial x_i}(D_O \cdot \frac{\partial C_O}{\partial x_i})$ describes the impurity dispersion by diffusion, the term $-\sum_{i=1}^{i=3} \frac{\partial}{\partial x_i}(U_{Oi}(t_M, x_1, x_2, x_3) \cdot C_O)$ describes the impurity dispersion by convection, $S_O = S_O(t_M, x_1, x_2, x_3)$ describes the source or the sinks of the impurity. See [19] and [20].

In equation (28) $D_O = D_O(x_1, x_2, x_3)$, $S_O = S_O(t_M, x_1, x_2, x_3)$, $U_{Oi}(t_M, x_1, x_2, x_3)$ are assumed to be known and $C_O = C_O(t_M, x_1, x_2, x_3)$ is unknown.

Observer O^* describes the dispersion of the impurity by the real valued function $C_{O^*} = C_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ which verifies the partial differential equation:

$$\frac{\partial C_{O^*}}{\partial t_M^*}=\sum_{i=1}^{i=3}\frac{\partial}{\partial x_i^*}(D_{O^*}\cdot\frac{\partial C_{O^*}}{\partial x_i^*})-\sum_{i=1}^{i=3}\frac{\partial}{\partial x_i^*}(U_{O^*i}(t_M^*,x_1^*,x_2^*,x_3^*)\cdot C_{O^*})+S_{O^*} \tag{29}$$

where:

$$t_M^*=t_M+t_{M_O}^*;\;\; x_k^*=x_{kO}^*+\sum_{i=1}^{i=3}a_{ki}x_i\quad k=1,2,3;\;\; D_{O^*}=D_{O^*}(x_1^*,x_2^*,x_3^*)$$
$$=D_O(x_1,x_2,x_3);\; S_{O^*}=S_{O^*}(t_M^*,x_1^*,x_2^*,x_3^*)=S_O(t_M,x_1,x_2,x_3)$$

Assume that in the impurity spread dynamics description the general temporal Caputo fractional partial derivative of order α, $0<\alpha<1$ is used. In this case Eq. (28) for observer O and Eq. (29) for observer O^* become:

$$ {}_{-\infty}^{\;C}D_{t_M}^{\alpha}C_O=\sum_{i=1}^{i=3}\frac{\partial}{\partial x_i}(D_O\cdot\frac{\partial C_O}{\partial x_i})-\sum_{i=1}^{i=3}\frac{\partial}{\partial x_i}(U_{Oi}(t_M,x_1,x_2,x_3)\cdot C_O)+S_O \tag{30}$$

$$ {}_{-\infty}^{\;C}D_{t_M^*}^{\alpha}C_{O^*}=\sum_{i=1}^{i=3}\frac{\partial}{\partial x_i^*}(D_{O^*}\cdot\frac{\partial C_{O^*}}{\partial x_i^*})-\sum_{i=1}^{i=3}\frac{\partial}{\partial x_i^*}(U_{O^*i}(t_M^*,x_1^*,x_2^*,x_3^*)\cdot C_{O^*})+S_{O^*} \tag{31}$$

The impurity dispersion description is objective if equations (30) and (31) describe the same dispersion. This means that the following statements hold:

i) if $C_O=C_O(t_M,x_1,x_2,x_3)$ is a solution of equation (30) then function $C_{O^*}(t_M^*,x_1^*,x_2^*,x_3^*)$ defined by:

$$C_{O^*}(t_M^*,x_1^*,x_2^*,x_3^*)=C_O(t_M+t_{M_O}^*,x_{1O^*}+\sum_{i=1}^{i=3}a_{i1}x_i^*,x_{2O^*}+\sum_{i=1}^{i=3}a_{i2}x_i^*,x_{3O^*}+\sum_{i=1}^{i=3}a_{i3}x_i^*)$$

verifies equation (31), and

ii) if $C_{O^*}(t_M^*,x_1^*,x_2^*,x_3^*)$ is a solution of equation (31) then function $C_O(t_M,x_1,x_2,x_3)$ defined by:

$$C_O(t_M,x_1,x_2,x_3)=C_{C_{O^*}}(t_M+t_{M_O}^*,x_{1O}^*+\sum_{i=1}^{i=3}a_{1i}x_i,x_{2O}^*+\sum_{i=1}^{i=3}a_{2i}x_i,x_{3O}^*+\sum_{i=1}^{i=3}a_{3i}x_i)$$

verifies equation (30).

In the following we give a short proof of the objectivity of the description using equations (30) and (31) in the case $D_O=D_{O^*}$, $S_O=S_{O^*}$ constant and the bulk fluid is a Newtonian, incompressible, viscous fluid having constant viscosity and density.

Proof of the statement i). Start with the function $C_O=C_O(t_M,x_1,x_2,x_3)$ solution of the equation (30) and consider function $C_{O^*}(t_M^*,x_1^*,x_2^*,x_3^*)$ defined by:

$$C_{O^*}(t_M^*,x_1^*,x_2^*,x_3^*)=C_O(t_M^*+t_{M_{O^*}},x_{1O^*}+\sum_{i=1}^{i=3}a_{i1}x_i^*,x_{2O^*}+\sum_{i=1}^{i=3}a_{i2}x_i^*,x_{3O^*}+\sum_{i=1}^{i=3}a_{i3}x_i^*)$$

For this function the following equalities hold:

$$ {}_{-\infty}^{\;C}D_{t_M^*}^{\alpha}C_{O^*}={}_{-\infty}^{\;C}D_{t_M}^{\alpha}C_O;\qquad \frac{\partial C_{O^*}}{\partial x_i^*}=\sum_{k=1}^{k=3}a_{ik}\cdot\frac{\partial C_O}{\partial x_k};$$

$$\sum_{i=1}^{i=3} U_{O^*i}(t_M^*, x_1^*, x_2^*, x_3^*) \cdot \frac{\partial C_{O^*}}{\partial x_i^*} = \sum_{i=1}^{i=3} \left(\sum_{j=1}^{j=3} a_{ij} U_{Oj}(t_M, x_1, x_2, x_3) \right) \cdot \left(\sum_{k=1}^{k=3} a_{ik} \cdot \frac{\partial C_O}{\partial x_k} \right)$$
$$= \sum_{j=1}^{j=3} \sum_{k=1}^{k=3} U_{Oj}(t_M, x_1, x_2, x_3) \cdot \frac{\partial C_O}{\partial x_k} \cdot \left(\sum_{i=1}^{i=3} a_{ij} \cdot a_{ik} \right)$$
$$= \sum_{j=1}^{j=3} \sum_{k=1}^{k=3} U_{Oj}(t_M, x_1, x_2, x_3) \cdot \frac{\partial C_O}{\partial x_k} \cdot \delta_{jk} = \sum_{k=1}^{k=3} U_{Ok}(t_M, x_1, x_2, x_3) \cdot \frac{\partial C_O}{\partial x_k}$$
$$\sum_{i=1}^{i=3} \frac{\partial^2 C_{O^*}}{\partial x_1^*} = \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i^*} \left(\frac{\partial C_O}{\partial x_i^*} \right) = \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i^*} \left(\sum_{k=1}^{k=3} a_{ik} \cdot \frac{\partial C_O}{\partial x_k} \right) = \sum_{i=1}^{i=3} \sum_{k=1}^{k=3} \sum_{l=1}^{l=3} a_{ik} \cdot a_{il} \cdot \frac{\partial^2 C_O}{\partial x_k \partial x_l}$$
$$= \sum_{k=1}^{k=3} \frac{\partial^2 C_O}{\partial {x_k}^2}$$

Using the above equalities and replacing the terms in (31) it follows that function $C_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ verifies (31).
The proof of the statement ii). is similar. When in the impurity dispersion description the general temporal Riemann-Liouville fractional partial derivative of order α, $0 < \alpha < 1$ is used then Eq. (28) for observer O and Eq. (29) for observer O^* become:

$$ {}_{-\infty}^{R-L}D_{t_M}^{\alpha} C_O = \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i} \left(D_O \cdot \frac{\partial C_O}{\partial x_i} \right) - \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i} \left(U_{Oi}(t_M, x_1, x_2, x_3) \cdot C_O \right) + S_O \qquad (32)$$

$$ {}_{-\infty}^{R-L}D_{t_M^*}^{\alpha} C_{O^*} = \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i^*} \left(D_{O^*} \cdot \frac{\partial C_{O^*}}{\partial x_i^*} \right) - \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i^*} \left(U_{O^*i}(t_M^*, x_1^*, x_2^*, x_3^*) \cdot C_{O^*} \right) + S_{O^*} \qquad (33)$$

The proof of the objectivity, of the impurity dispersion description, with Eq. 32 and Eq. 33 is similar to that with the proof from the previous section.

9. CONCLUSIONS AND COMMENTS

1. The mathematical descriptions of the bulk groundwater 2D flow to the well, in a horizontal unconfined homogeneous and isotropic aquifer and the description of the spread of the contained impurity, which uses general temporal Caputo or general temporal Riemann-Liouville fractional order derivatives, are objective; i.e. independent on the choice of the origin of time measurement and on the reference frame. Due to that, two observers describing the groundwater flow and the spread of impurity using these tools, obtain results that can be reconciled; i.e. transformed into each other using formulas (5), (6) that link the numbers representing a moment of time in two different choices of the origin of time measurement and coordinates of a point in two different reference frames.
2. The mathematical descriptions of the bulk fluid 3D flow, in container and the description of the dispersion of the contained impurity, which uses general temporal Caputo or general temporal Riemann-Liouville fractional order derivatives, are objective.
3. The obtained results show the compatibility of the general temporal Caputo and general temporal Riemann-Liouville fractional order derivatives with the understanding of the

measured time evolution. At the same time these results support the idea that the objectivity violation is originated in the incompatibility of the definition of the classic temporal Caputo and classic temporal Riemann-Liouville fractional order derivatives with the understanding of the measured time evolution.

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