Mathematical description of the bulk fluid flow and that of the contained impurity dispersion which uses Caputo or Riemann-Liouville fractional order partial derivatives is nonobjective

Agneta M. BALINT[1], Stefan BALINT*[,2]

*Corresponding author
[1]Department of Physics, West University of Timisoara,
Bulv. V. Parvan 4, 300223 Timisoara, Romania,
agneta.balint@e-uvt.ro
*[,2]Department of Computer Science, West University of Timisoara,
Bulv. V. Parvan 4, 300223 Timisoara, Romania,
stefan.balint@e-uvt.ro

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Abstract: *In this paper it is shown that the mathematical description of a Newtonian, incompressible, viscous bulk fluid flow and that of the contained impurity dispersion which uses Caputo or Riemann-Liouville fractional order derivative, having integral representation on finite interval, is nonobjective. This means that, two different observers describing the flow or the contained impurity dispersion with these tools obtain two different results which cannot be reconciled i.e. transformed into each other using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment of time in two different choices of the origin of time measuring. This is not an academic curiosity! It is rather a problem: which of the obtained results is correct?*

Key Words: *objectivity of a mathematical description, bulk fluid flow, impurity dispersion, fractional order partial derivative*

1. INTRODUCTION

The mathematical description of a Real World phenomena is objective if it is independent on the observer. In other words, it is possible to reconcile the observation of phenomena into a single coherent description of it. This requirement was pointed out by Galileo Galilee (1564-1642), Isaac Newton (1643-1727), Albert Einstein (1879-1955) in the context of mathematical description of mechanical movements: “The mechanical event is independent of the observer”. A possible and elementary understanding of the independence of the mechanical event from the observer is the independence of the event of the choice of the reference frame and of the choice of the moment as origin for time measuring made by observer. In the following we will detail what this means. To describe mathematically the evolution of a mechanical event, an observer chooses a fixed orthogonal reference frame in the affine Euclidean space, a fixed moment of time (called origin for time measuring), and a unit for time measuring [second].

For different observers this choice can be different. In this paper the “objectivity of a mathematical description” means that the description is independent of the choice of the fixed orthogonal reference frame and of the choice of origin for time measuring. This means that the results obtained by two different observers can be reconciled, i.e. transformed into each other using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment of time in two different choices of the origin of time measuring. This concept of “objectivity of a mathematical description” is different from the concept of “objectivity in physics” presented in [1]. The advantage of our concept of “objectivity of a mathematical description” is that it can be easily applied in a specific case and the reader does not need prior knowledge of Galilean invariance, Lorentz invariance or Einstein covariance.

Mathematical descriptions which depend on the choice of the fixed orthogonal reference frame or on the choice of the origin of time measuring are nonobjective in the sense of this paper. In case of descriptions which are nonobjective, two observers who describe the same mechanical event obtain two different results that cannot be reconciled, i.e. cannot be transformed into each other using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing the same moment of time in two different choices of the origin of time measuring. This concept of nonobjective description can be easily applied in a specific case and the reader does not need prior knowledge of Galilean invariance, Lorentz invariance or Einstein covariance.The majority of mathematical descriptions formulated in terms of integer order derivatives or integer order partial derivatives reported in the literature (books of Differential Equations of Mathematical Physics), are objectives in the sense of this work. In accordance with the vision of this work, the following illustrates the objectivity of the descriptions, of some phenomena which occur in fluid mechanics.

In classical theory of fluid mechanics [2], [3] the inside of the container with the bulk fluid, is represented as a connected subset Ω of points of the affine Euclidean space E_3. A point P of Ω is called position. At a moment of time M a particle Q of the bulk fluid is represented by position P that particle Q occupies at the moment of time M.

To describe the P position, observer O chooses a fixed orthogonal reference frame $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$ in E_3 and describes position P with the coordinate (x_1, x_2, x_3) of P with respect to the reference frame R_O. To describe the time evolution observer O chooses a moment of time M_O for fixing the origin for time measuring (the moment, when his stopwatch for measuring time, starts) and a unit for time measuring [second]. A moment of time M which is earlier than M_O is represented by a negative real number $t_M < 0$ (representing the units of time between moment M and moment M_O), a moment of time Mwhich is later than M_O is represented by a positive real number $t_M > 0$ (representing the units of time between moment M_O and moment M), the moment of time M_O is represented by the real number $t_{M_O} = 0$.

Observer O describes the flow of the Newtonian incompressible viscous bulk fluid with a vector valued function $\vec{U}_O = \vec{U}_O(t_M, x_1, x_2, x_3)$ and a real valued function $p_O = p_O(t_M, x_1, x_2, x_3)$. Function $\vec{U}_O$ is called the velocity field and function p_O is called the pressure field. Vector $\vec{U}_O(t_M, x_1, x_2, x_3)$ represents the velocity of the fluid particle Q which at the moment of time t_M is in position P of coordinates (x_1, x_2, x_3) and number $p_O(t_M, x_1, x_2, x_3)$ represents the pressure at the moment of time t_M at position P of coordinates (x_1, x_2, x_3). To describe the P position, observer O^*chooses a fixed orthogonal reference frame $R_{O^*} = (O^*; \vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*)$ in E_3 and describes the P position with the coordinate (x_1^*, x_2^*, x_3^*) of P with respect to the reference frame $R_{O^*} = (O^*; \vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*)$. To describe the time evolution observer O^* chooses a moment of time M_{O^*} for fixing the origin for time measuring (the moment, when his stopwatch for measuring time, starts) and a unit for time measuring [second]. A moment of time M which is earlier than M_{O^*} is represented by a negative real number $t_M^* < 0$ (representing the units of time between moment M and moment M_{O^*}), a moment of time M which is later than M_{O*} is represented by a positive real number $t_M^* > 0$ (representing the units of time between moment M_{O^*} and moment M), the moment of time M_{O^*} is represented by the real number $t_{M_{O^*}}^* = 0$.

Observer O^* describes the flow of the bulk fluid with a vector valued function $\vec{U}_{O^*} = \vec{U}_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ and a real valued function $p_{O^*} = p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$. Function $\vec{U}_{O^*}$ is called the velocity field and function p_{O^*} is called the pressure field. Vector $\vec{U}_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ represents the velocity of the fluid particle Q which at the moment of time t_M^* is in position P of coordinates (x_1^*, x_2^*, x_3^*) and number $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ represents the pressure at moment t_M^* at position P of coordinates (x_1^*, x_2^*, x_3^*). Remark that a moment of time M in case of the observer O is described by the real number t_M and in case of the observer O^* by the real number t_M^*. For numbers t_M and t_M^* the following relations hold:

$$t_M = t_M^* + t_{M_{O^*}} \tag{1}$$

$$t_M^* = t_M + t_{M_O}^* \tag{2}$$

In the above relations $t_{M_{O^*}}$ is the real number which represents the moment M_{O^*} in the system of time measuring of the observer O and $t_{M_O}^*$ is the real number which represents the moment M_O in the system of time measuring of the observer O^*.

At any moment of time M, the coordinates (x_1, x_2, x_3) with respect to R_O and the coordinates (x_1^*, x_2^*, x_3^*) with respect to R_{O^*} represent the same position P in the affine Euclidian space E_3. Therefore, for the coordinates the following relations hold:

$$x_k = x_{kO^*} + \sum_{i=1}^{i=3} a_{ik} \cdot x_i^* \quad k = 1,2,3 \tag{3}$$

Or equivalently,

$$x_k^* = x_{kO}^* + \sum_{i=1}^{i=3} a_{ki} x_i \quad k = 1,2,3 \tag{4}$$

The significance of the quantities appearing in the above relations are:
$a_{ij} = \langle \vec{e}_i^*, \vec{e}_j \rangle$ = constant = scalar product of the unit vectors $\vec{e}_1^*$ and $\vec{e}_j$ in E_3 i.e.

$$\vec{e}_i^* = \sum_{k=1}^{k=3} a_{ik} \vec{e}_k \qquad \vec{e}_i = \sum_{k=1}^{k=3} a_{ki} \vec{e}_k^* \tag{5}$$

$(x_{1O^*}, x_{2O^*}, x_{3O^*})$ are the coordinates of point O^* with respect to the reference frame R_O,$(x_{1O}^*, x_{2O}^*, x_{3O}^*)$ are the coordinates of point O with respect to the reference frame R_{O^*}.

At any moment of time M, and at any position P, vectors $\vec{U}_O(t_M, x_1, x_2, x_3)$ and $\vec{U}_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ represent the same velocity as well the scalars $p_O(t_M, x_1, x_2, x_3)$ and $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ represent the same pressure.

Therefore, the following relations hold:

$$U_{O_k}(t_M, x_1, x_2, x_3) = \sum_{i=1}^{i=3} a_{ik} U_{O_i^*}(t_M^*, x_1^*, x_2^*, x_3^*) \qquad \mathrm{k} = 1{,}2{,}3 \tag{6}$$

$$U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*) = \sum_{i=1}^{i=3} a_{ki} U_{O_i}(t_M, x_1, x_2, x_3) \qquad k = 1{,}2{,}3 \tag{7}$$

$$p_O(t_M, x_1, x_2, x_3) = p_{O^*}(t_M + t_{M_O}^*, x_{1_O}^* + \sum_{i=1}^{i=3} a_{1i} x_i\,, x_{2_O}^* + \sum_{i=1}^{i=3} a_{2i} x_i\,, x_{3_O}^* + \sum_{i=1}^{i=3} a_{3i} x_i) \tag{8}$$

$$p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*) = p_O(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{i=1}^{i=3} a_{i1} \cdot x_i^*, x_{2O^*} + \sum_{i=1}^{i=3} a_{i2} \cdot x_i^*\,, x_{3O^*} + \sum_{i=1}^{i=3} a_{i3} \cdot x_i^*) \tag{9}$$

Relations (1)-(4) and (6)-(9) reconcile the mathematical description of the fluid flow made by the two observers, and make possible the flow description by the velocity field $\vec{U}_O$ and pressure field p_O or by the velocity field $\vec{U}_{O^*}$ and pressure field p_{O^*}. This means that the above presented mathematical description of the bulk fluid flow is objective.

In classical theory of fluid mechanics [2], [3] the vector valued function $\vec{U}_O = \vec{U}_O(t_M, x_1, x_2, x_3)$ and the real valued function $p_O = p_O(t_M, x_1, x_2, x_3)$ which describes, in terms of observer O, the flow of a Newtonian incompressible viscous bulk fluid having constant viscosity and density, check the Navier-Stokes equations:

$$\frac{\partial U_{O_i}}{\partial t_M} + \sum_{k=1}^{k=3} U_{O_k} \cdot \frac{\partial U_{O_i}}{\partial x_k} = -\frac{1}{\rho_0} \cdot \frac{\partial p_O}{\partial x_i} + \frac{\mu}{\rho_0} \cdot \sum_{k=1}^{k=3} \frac{\partial^2 U_{O_i}}{\partial {x_k}^2} + f_{O_i} \quad i = 1, 2, 3 \tag{10}$$

$$\sum_{k=1}^{k=3} \frac{\partial U_{O_i}}{\partial x_k} = 0 \tag{11}$$

In the Navier-Stokes equations: ρ_0-constant is the fluid density, μ -constant is the fluid viscosity, f_i are the components of the mass force divided by the fluid density.

The vector valued function $\vec{U}_{O^*} = \vec{U}_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ and the real valued function $p_{O^*} = p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ which describes, in terms of the observer O^*, the fluid flow check the equations:

$$\frac{\partial U_{O_i^*}}{\partial t_M^*} + \sum_{k=1}^{k=3} U_{O_k^*} \cdot \frac{\partial U_{O_i^*}}{\partial x_k^*} = -\frac{1}{\rho_0} \cdot \frac{\partial p_{O^*}}{\partial x_i^*} + \frac{\mu}{\rho_0} \cdot \sum_{k=1}^{k=3} \frac{\partial^2 U_{O_i^*}}{\partial {x_k^*}^2} + f_{O_i^*} \quad i = 1, 2, 3 \tag{12}$$

$$\sum_{k=1}^{k=3} \frac{\partial U_{O_i^*}}{\partial x_k^*} = 0 \tag{13}$$

Because the functions f_{O_i} and $f_{O_i^*}$ represent the same force field (in two different reference frames) the components verify the following relations:

$$\begin{aligned} f_{O_k^*} &= a_{k1}f_{O1} + a_{k2}f_{O2} + a_{k3}f_{O3} \qquad k = 1,2,3 \\ f_{Ok} &= a_{1k}f_{O_1^*} + a_{2k}f_{O_2^*} + a_{3k}f_{O_3^*} \qquad k = 1,2,3 \end{aligned} \tag{14}$$

Description (10), (11) is objective if and only if it describes the same flow as (12), (13). The objectivity can be proven showing that:
if $U_{O_k}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ check the equations (10), (11) then the functions $U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*)$; $k = 1,2,3$, $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ defined by:

$$\begin{aligned} &U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*) \\ &\quad = \sum_{i=1}^{i=3} a_{ki} U_{O_i}\left(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{j=1}^{j=3} a_{j1}\cdot x_j^*, x_{2O^*} + \sum_{j=1}^{j=3} a_{j2}\cdot x_j^*, x_{3O^*} + \sum_{j=1}^{j=3} a_{j3}\cdot x_j^* \right) \quad k = 1,2,3 \\ &p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*) \\ &\quad = p_O\left(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{i=1}^{i=3} a_{i1}\cdot x_i^*, x_{2O^*} + \sum_{i=1}^{i=3} a_{i2}\cdot x_i^*, x_{3O^*} + \sum_{i=1}^{i=3} a_{i3}\cdot x_i^* \right) \end{aligned} \tag{15}$$

check the equations (12), (13), and
if $U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*)$; $k = 1,2,3$, $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ check the equations (12), (13) then the functions $U_{O_k}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ defined by:

$$\begin{aligned} &U_{O_k}(t_M, x_1, x_2, x_3) \\ &\quad = \sum_{i=1}^{i=3} a_{ik} U_{O_i^*}\left(t_M + t_{M_O}^*, x_{1O}^* + \sum_{j=1}^{j=3} a_{1j}\cdot x_j, x_{2O}^* + \sum_{j=1}^{j=3} a_{2j}\cdot x_j, x_{3O}^* + \sum_{j=1}^{j=3} a_{3j}\cdot x_j \right) \quad k = 1,2,3 \\ &p_O(t_M, x_1, x_2, x_3) \\ &\quad = p_{O^*}(t_M + t_{M_O}^*, x_{1O}^* + \sum_{i=1}^{i=3} a_{1i}x_i, x_{2O}^* + \sum_{i=1}^{i=3} a_{2i}x_i, x_{3O}^* + \sum_{i=1}^{i=3} a_{3i}x_i) \end{aligned} \tag{16}$$

check the equations (10), (11).

A schematic proof of objectivity, is the following: start with functions $U_{O_k}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ that check equations (10), (11) and consider functions $U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*)$; $k = 1,2,3$, $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ defined by (15). Remark that the following equalities hold:

$$\frac{\partial U_{O_i^*}}{\partial t_M^*} = \sum_{k=1}^{k=3} a_{ik} \cdot \frac{\partial U_{O_k}}{\partial t_M}; \sum_{k=1}^{k=3} U_{O_k^*} \cdot \frac{\partial U_{O_i^*}}{\partial x_k^*} = \sum_{k=1}^{k=3} a_{ik} \cdot \sum_{m=1}^{m=3} \frac{\partial U_{O_k}}{\partial x_m} \cdot U_{O_m};$$
$$\frac{\partial p_{O^*}}{\partial x_i^*} = \sum_{k=1}^{k=3} a_{ik} \cdot \frac{\partial p_O}{\partial x_k}; \sum_{k=1}^{k=3} \frac{\partial^2 U_{O_i^*}}{\partial {x_k^*}^2} = \sum_{k=1}^{k=3} a_{ik} \cdot \frac{\partial^2 U_{O_i}}{\partial {x_k}^2}; \sum_{k=1}^{k=3} \frac{\partial U_{O_i^*}}{\partial x_k^*} = \sum_{k=1}^{k=3} \frac{\partial U_{O_i}}{\partial x_k} \tag{17}$$

Using equalities (14), (17) and replacing the terms in (12), (13) it follows that if functions $U_{O_k}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ check equations (10), (11) then functions $U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*)$; $k = 1,2,3$ and $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ defined by (15) check equations (12), (13). The second part of the proof is similar.

So the description of the flow of a Newtonian incompressible viscous bulk fluid having constant density with a vector valued function $\vec{U}_O = \vec{U}_O(t_M, x_1, x_2, x_3)$ and a real valued function $p_O = p_O(t_M, x_1, x_2, x_3)$that check the Navier-Stokes equations (10), (11) is objective. That is, different observers, describing with these tools the fluid flow, get results which can be reconciled i.e. transformed into each other using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment of time in two different choices of the origin of time.

The dispersion of an impurity contained in the bulk fluid is described with the concentration of that impurity. Observer O describes the concentration with the real valued function $C_O = C_O(t_M, x_1, x_2, x_3)$ which checks the partial differential equation:

$$\frac{\partial C_O}{\partial t_M} = \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i}(D_O \cdot \frac{\partial C_O}{\partial x_i}) - \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i}(U_{O_i}(t_M, x_1, x_2, x_3) \cdot C_O) + R_O \tag{18}$$

where: $D_O = D_O(x_1, x_2, x_3)$ is the diffusivity (also called diffusion coefficient), $U_{O_i}(t_M, x_1, x_2, x_3)$ are the components of the bulk fluid flow velocity, the term $\sum_{i=1}^{i=3} \frac{\partial}{\partial x_i}(D_O \cdot \frac{\partial C_O}{\partial x_i})$ describes the impurity dispersion by diffusion, the term $-\sum_{i=1}^{i=3} \frac{\partial}{\partial x_i}(U_{O_i}(t_M, x_1, x_2, x_3) \cdot C_O)$ describes the impurity dispersion by convection, $R_O = R_O(t_M, x_1, x_2, x_3)$ describes the source or the sinks of the impurity, see [4] and [5].

In equation (18) $D_O = D_O(x_1, x_2, x_3)$, $R_O = R_O(t_M, x_1, x_2, x_3)$, $U_{O_i}(t_M, x_1, x_2, x_3)$ are assumed to be known and $C_O = C_O(t_M, x_1, x_2, x_3)$ is unknown. Observer O^* describes the dispersion of the impurity by the real valued function $C_{O^*} = C_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ which checks the partial differential equation:

$$\frac{\partial C_{O^*}}{\partial t_M^*} = \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i^*}(D_{O^*} \cdot \frac{\partial C_{O^*}}{\partial x_i^*}) - \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i^*}(U_{O_i^*}(t_M^*, x_1^*, x_2^*, x_3^*) \cdot C_{O^*}) + R_{O^*} \tag{19}$$

where: $t_M^* = t_M + t_{M_O}^*$; $x_k^* = x_{kO}^* + \sum_{i=1}^{i=3} a_{ki} x_i$ $k = 1,2,3$; $D_{O^*} = D_{O^*}(x_1^*, x_2^*, x_3^*) = D_O(x_1, x_2, x_3)$ and $R_{O^*} = R_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*) = R_O(t_M, x_1, x_2, x_3)$. Objectivity of the impurity dispersion description means that the solutions of the partial differential equations (18) and (19) describe the same dispersion. Objectivity can be proven showing that:

if $C_O = C_O(t_M, x_1, x_2, x_3)$ is a solution of the equation (18) then function $C_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ defined by

$$C_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*) = C_O(t_M^* + t_{M_{O^*}}, x_{1O^*} + \sum_{i=1}^{i=3} a_{i1} x_i^*, x_{2O^*} + \sum_{i=1}^{i=3} a_{i2} x_i^*, x_{3O^*} + \sum_{i=1}^{i=3} a_{i3} x_i^*) \tag{20}$$

checks equation (19) and

if $C_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ is a solution of equation (19) then function $C_O(t_M, x_1, x_2, x_3)$ defined by

$$C_O(t_M, x_1, x_2, x_3) = C_{O^*}(t_M + t_{M_O}^*, x_{1O}^* + \sum_{i=1}^{i=3} a_{1i} x_i, x_{2O}^* + \sum_{i=1}^{i=3} a_{2i} x_i, x_{3O}^* + \sum_{i=1}^{i=3} a_{3i} x_i) \tag{21}$$

checks equation (18). We give a short proof of the objectivity of this description in the case $D_O = D_{O^*}, R_O = R_{O^*}$ constant and the bulk fluid is a Newtonian, incompressible, viscous fluid having constant viscosity and density. In this case equations (18) and (19) become:

$$\frac{\partial C_O}{\partial t_M} = D_O \cdot \sum_{i=1}^{i=3} \frac{\partial^2 C_O}{\partial x_i^2} - \sum_{i=1}^{i=3} U_{O_i}(t_M, x_1, x_2, x_3) \cdot \frac{\partial C_O}{\partial x_i} + R_O \tag{22}$$

$$\frac{\partial C_{O^*}}{\partial t_M^*} = D_{O^*} \cdot \sum_{i=1}^{i=3} \frac{\partial^2 C_{O^*}}{\partial x_i^{*2}} - \sum_{i=1}^{i=3} U_{O_i^*}(t_M^*, x_1^*, x_2^*, x_3^*) \cdot \frac{\partial C_{O^*}}{\partial x_i^*} + R_{O^*} \tag{23}$$

We start with solution $C_O = C_O(t_M, x_1, x_2, x_3)$ of equation (22) and consider the function $C_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ defined by (20). For this function the following equalities hold:

$$\frac{\partial C_{O^*}}{\partial t_M^*} = \frac{\partial C_O}{\partial t_M}; \quad \frac{\partial C_{O^*}}{\partial x_i^*} = \sum_{k=1}^{k=3} a_{ik} \cdot \frac{\partial C_O}{\partial x_k} \tag{24}$$

$$\begin{aligned} \sum_{i=1}^{i=3} U_{O_i^*}(t_M^*, x_1^*, x_2^*, x_3^*) \cdot \frac{\partial C_{O^*}}{\partial x_i^*} &= \sum_{i=1}^{i=3} (\sum_{j=1}^{j=3} a_{ij} U_{O_j}(t_M, x_1, x_2, x_3)) \cdot (\sum_{k=1}^{k=3} a_{ik} \cdot \frac{\partial C_O}{\partial x_k}) \\ &= \sum_{j=1}^{j=3} \sum_{k=1}^{k=3} U_{O_j}(t_M, x_1, x_2, x_3) \cdot \frac{\partial C_O}{\partial x_M} \cdot (\sum_{i=1}^{i=3} a_{ij} \cdot a_{ik}) \\ &= \sum_{j=1}^{j=3} \sum_{k=1}^{k=3} U_{O_j}(t_M, x_1, x_2, x_3) \cdot \frac{\partial C_O}{\partial x_k} \cdot \delta_{jk} \\ &= \sum_{k=1}^{k=3} U_{O_k}(t_M, x_1, x_2, x_3) \cdot \frac{\partial C_O}{\partial x_k} \end{aligned} \tag{25}$$

$$\sum_{i=1}^{i=3}\frac{\partial^2 C_{O^*}}{\partial x_i^{*2}}=\sum_{i=1}^{i=3}\frac{\partial}{\partial x_i^*}(\frac{\partial C_O}{\partial x_i^*})$$
$$=\sum_{i=1}^{i=3}\frac{\partial}{\partial x_i^*}(\sum_{k=1}^{k=3}a_{ik}\cdot\frac{\partial C_O}{\partial x_k})=\sum_{i=1}^{i=3}\sum_{k=1}^{k=3}\sum_{l=1}^{l=3}a_{ik}\cdot a_{il}\cdot\frac{\partial^2 C_O}{\partial x_k\partial x_l} \qquad (26)$$
$$=\sum_{k=1}^{k=3}\frac{\partial^2 C_O}{\partial {x_k}^2}$$

Replacing in (19) the terms $\frac{\partial C_{O^*}}{\partial t_M^*}$; $\sum_{i=1}^{i=3}\frac{\partial^2 C_{O^*}}{\partial x_i^{*2}}$ and $\sum_{i=1}^{i=3}U_{O_i^*}(t_M^*,x_1^*,x_2^*,x_3^*)\cdot\frac{\partial C_{O^*}}{\partial x_i^*}$ with those obtained in formulas (24), (25), (26) and taking into account on (18) equality (19) is obtained.

Therefore, function $C_{O^*}(t_M^*,x_1^*,x_2^*,x_3^*)$ defined by (20) is a solution of equation (19). If we start with a solution $C_{O^*}(t_M^*,x_1^*,x_2^*,x_3^*)$ of equation (19) and we consider the function $C_O(t_M,x_1,x_2,x_3)$ given by (21) then in a similar way we can obtain that this function is a solution of equation (18). So the description of the impurity dispersion with a real valued function $C_O=C_O(t_M,x_1,x_2,x_3)$ that checks the partial differential equation (18) is objective. That is, different observers, describing with these tools the impurity dispersion, get results which can be reconciled i.e. transformed into each other using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment of time in two different choices of the origin of time.

Beside the objective mathematical descriptions of the bulk fluid flow and impurity dispersion (see references ([2]-[5]) formulated in terms of integer order derivatives, there are mathematical descriptions of the bulk fluid flow and that of the impurity dispersion formulated in terms of fractional order partial derivatives, see for instance references [6]-[17]. In these descriptions, the analysis of the description objectivity is missing. At first, we thought that in the case of the description with fractional derivatives, the objectivity is fulfilled and therefore it is ignored. But the curiosity pushed us to see how the fulfillment of the objectivity condition (in sense of our paper) can be proven mathematically. We chose for the bulk fluid flow a description similar to that reported in [6], [10] and for the spread of impurities, a description similar to that reported in [16]. Thus were “born” sections 2 and 3 of the paper in which we analyzed the objectivity of the description of the bulk fluid flow and sections 4 and 5 in which we analyzed the objectivity of the description of the impurities spread, using temporal Caputo or Riemann-Liouville fractional partial derivatives, with integral representation on finite interval, instead of the integer order derivatives.

The purpose of the present paper is to show that, the mathematical descriptions which use temporal Caputo or Riemann-Liouville fractional order partial derivatives (having integral representation on finite interval), in the Navier-Stoks equations, or in the impurity dispersion equation, are nonobjective. That is, two observers describing the fluid flow or the impurity dispersion using fractional derivatives, obtain two different results which cannot be reconciled i.e. transformed into each other using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment of time in two different choices of the origin of time.

The problem is: which of the obtained results is correct?

Remember that for a continuously differentiable function $f:[0,\infty)\times[0,\infty)\to R$ the spatial and temporal Caputo fractional partial derivative of order α, $0<\alpha$, is defined with the first and the second of the following integral representation, respectively [18]:

$$
\begin{aligned}
{}_{0}^{C}D_{x}^{\alpha} f(t,x) &= \frac{1}{\Gamma(n-\alpha)} \cdot \int_{0}^{x} \frac{\frac{\partial^{n} f}{\partial \xi^{n}}(t,\xi)}{(x-\xi)^{\alpha+1-n}} d\xi \\
{}_{0}^{C}D_{t}^{\alpha} f(t,x) &= \frac{1}{\Gamma(n-\alpha)} \cdot \int_{0}^{t} \frac{\frac{\partial^{n} f}{\partial \tau^{n}}(\tau,x)}{(t-\tau)^{\alpha+1-n}} d\tau
\end{aligned} \tag{27}
$$

Remark that the derivative defined with (27) was considered by other people before Caputo, like Gherasimov (see [18]). So the name of Caputo used in this paper may be not appropriate. For a continuously differentiable function $f: [0,\infty) \times [0,\infty) \to R$ the spatial and temporal Riemann-Liouville fractional partial derivative of order α, $0 < \alpha$, is defined with the first and the second of the following integral representation, respectively [18]:

$$
\begin{aligned}
{}^{R-L}_{\ \ \ 0}D_{x}^{\alpha} f(t,x) &= \frac{1}{\Gamma(n-\alpha)} \cdot \frac{\partial^{n}}{\partial x^{n}} \int_{0}^{x} \frac{f(t,\xi)}{(x-\xi)^{\alpha+1-n}} d\xi \\
{}^{R-L}_{\ \ \ 0}D_{t}^{\alpha} f(t,x) &= \frac{1}{\Gamma(n-\alpha)} \cdot \frac{\partial^{n}}{\partial t^{n}} \int_{0}^{t} \frac{f(\tau,x)}{(t-\tau)^{\alpha+1-n}} d\tau
\end{aligned} \tag{28}
$$

In formulas (27), (28), Γ is the Euler gamma function and $n = [\alpha] + 1$, $[\alpha]$ being the integer part of α.

Remember that the purpose of the following sections is to show that, if observers O and O^* describe the flow of a Newtonian, incompressible, viscous bulk fluid having constant density, or the impurity dispersion contained in this fluid, using temporal fractional order partial derivative, then O and O^* get different results which cannot be reconciled i.e. transformed into each other using only formulas that link the numbers representing a moment of time in two different choices of the origin of time.

2. THE BULK FLUID FLOW DESCRIPTION WHICH USES TEMPORAL CAPUTO FRACTIONAL ORDER PARTIAL DERIVATIVE, WITH INTEGRAL REPRESENTATION ON FINITE INTERVAL, IS NONOBJECTIVE

Assume that in the bulk fluid flow description the temporal Caputo fractional partial derivative of order α, $0 < \alpha < 1$, with integral representation on finite interval, is used (as in references [6], [10]). In this case, equation (10), (11) for observer O and equation (12), (13) for observer O^* become:

$$
{}_{0}^{C}D_{t_M}^{\alpha} U_{O_i} + \sum_{k=1}^{k=3} U_{O_k} \cdot \frac{\partial U_{O_i}}{\partial x_k} = -\frac{1}{\rho_0} \cdot \frac{\partial p_O}{\partial x_i} + \frac{\mu}{\rho_0} \cdot \sum_{k=1}^{k=3} \frac{\partial^2 U_{O_i}}{\partial {x_k}^2} + f_{O_i} \quad i = 1,2,3 \tag{29}
$$

$$
\sum_{k=1}^{k=3} \frac{\partial U_{O_i}}{\partial x_k} = 0 \tag{30}
$$

$$
{}_{0}^{C}D^{\alpha}{}_{t_M^*} U_{O_i^*} + \sum_{k=1}^{k=3} U_{O_k^*} \cdot \frac{\partial U_{O_i^*}}{\partial x_k^*} = -\frac{1}{\rho_0} \cdot \frac{\partial p_{O^*}}{\partial x_i^*} + \frac{\mu}{\rho_0} \cdot \sum_{k=1}^{k=3} \frac{\partial^2 U_{O_i^*}}{\partial {x_k^*}^2} + f_{O_i^*} \quad i = 1,2,3 \tag{31}
$$

$$\sum_{k=1}^{k=3} \frac{\partial U_{O_i^*}}{\partial x_k^*} = 0 \quad (32)$$

Objectivity of this description means that the solutions of fractional partial differential equations (29), (30) and (31), (32) describe the same flow of the bulk fluid.

That is:

if $U_{O_k}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ check equations (29), (30) then $U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*)$; $k = 1,2,3$, $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ defined by (15) check equations (31), (32)

and

if $U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*)$; $k = 1,2,3$, $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ check equations (31), (32) then $U_{O_k}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ defined by (16) check equations (29), (30).

We assume that the description is objective and start with a solution $U_{O_k}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ of the equations (29), (30). Consider $U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*)$; $k = 1,2,3$, $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ defined by (15), the case when the reference frame R_O, R_{O^*} of the observers O, O^* coincides and $M_O \neq M_{O*}$. Remark that, in this case, the following equalities hold:

$${}_0^C D^{\alpha}{}_{t_M^*} U_{O_i^*} = {}_0^C D_{t_M}^{\alpha} U_{O_i} + \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t_{M_O}^*} \frac{\frac{\partial U_{O_i^*}}{\partial \tau}(\tau, x_1, x_2, x_3)}{(t_M^* - \tau)^{\alpha}} d\tau \quad i = 1,2,3 \quad (33)$$

Using equalities (33) and (17) it follows that: if the functions defined by (15) check the equations (31), (32) then the following equalities hold:

$$\frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t_{M_O}^*} \frac{\frac{\partial U_{O_i^*}}{\partial \tau}(\tau, x_1, x_2, x_3)}{(t_M^* - \tau)^{\alpha}} d\tau = 0; \quad i = 1,2,3 \quad (34)$$

Equalities (34) are consequence of the assumption that the mathematical description (29), (30) is objective. But generally (34) are not verified. So the mathematical description (29), (30) is nonobjective. That is, observers O and O^*, describing the same flow, with (29), (30) and (31), (32) respectively, get different results which cannot be reconciled i.e. transformed into each other using only formulas that link the numbers representing a moment of time in two different choices of the origin of time. The problem is: which of the results is correct? This result may be an additional argument in favor of the need to analyze the objectivity of the mathematical description proposed in [6] - [12].

3. THE BULK FLUID FLOW DESCRIPTION WHICH USES TEMPORAL RIEMANN-LIOUVILLE FRACTIONAL ORDER PARTIAL DERIVATIVE, WITH INTEGRAL REPRESENTATION ON FINITE INTERVAL, IS NONOBJECTIVE

Assume that in the bulk fluid flow description the temporal Riemann-Liouville fractional partial derivative of order α, $0 < \alpha < 1$ with integral representation on finite interval is used. In this case, equations (10), (11) for observer O and equations (12), (13) for observer O^* become:

$$ {}^{R-L}_{0}D^{\alpha}_{t_M} U_{O_i} + \sum_{k=1}^{k=3} U_{O_k} \cdot \frac{\partial U_{O_i}}{\partial x_k} = -\frac{1}{\rho_0} \cdot \frac{\partial p_O}{\partial x_i} + \frac{\mu}{\rho_0} \cdot \sum_{k=1}^{k=3} \frac{\partial^2 U_i}{\partial {x_k}^2} + f_{O_i}; \quad i = 1,2,3 \tag{35} $$

$$ \sum_{k=1}^{k=3} \frac{\partial U_{O_i}}{\partial x_k} = 0 \tag{36} $$

$$ {}^{R-L}_{0}D^{\alpha}{}_{t_M^*} U_{O_i^*} + \sum_{k=1}^{k=3} U_{O_k^*} \cdot \frac{\partial U_{O_i^*}}{\partial x_k^*} = -\frac{1}{\rho_0} \cdot \frac{\partial p_{O^*}}{\partial x_i^*} + \frac{\mu}{\rho_0} \cdot \sum_{k=1}^{k=3} \frac{\partial^2 U_{O_i^*}}{\partial {x_k^*}^2} + f_{O_i^*}; \quad i = 1,2,3 \tag{37} $$

$$ \sum_{k=1}^{k=3} \frac{\partial U_{O_i^*}}{\partial x_k^*} = 0 \tag{38} $$

Objectivity of this description means that the solutions of the fractional partial differential equations (35), (36) and (37), (38) describe the same flow.
That is:
if $U_{O_k}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ check equations (35), (36) then $U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*)$; $k = 1,2,3$, $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ defined by (15) check equations (37), (38)
and
if $U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*)$; $k = 1,2,3$, $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ check equations (37), (38) then $U_{O_k}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ defined by (16) check equations (35), (36).

We assume that the description is objective and start with a solution $U_{O_k}(t_M, x_1, x_2, x_3)$; $k = 1,2,3$, $p_O(t_M, x_1, x_2, x_3)$ of equations (35), (36). Consider the functions $U_{O_k^*}(t_M^*, x_1^*, x_2^*, x_3^*)$; $k = 1,2,3$, $p_{O^*}(t_M^*, x_1^*, x_2^*, x_3^*)$ defined by (15), the case when the reference frame R_O, R_{O^*} of the observers O, O^* coincides and $M_O \neq M_{O*}$. Remark that in this case the following equalities hold:

$$ {}^{R-L}_{0}D^{\alpha}{}_{t_M^*} U_{O_i^*} = {}^{R-L}_{0}D^{\alpha}_{t_M} U_{O_i} + \frac{1}{\Gamma(1-\alpha)} \cdot \frac{\partial}{\partial t_M^*} \int_0^{t^*_{M_O}} \frac{U_{O_i^*}(\tau, x_1, x_2, x_3)}{(t_M^* - \tau)^{\alpha}} d\tau; \quad i = 1.2.3 \tag{39} $$

Using equalities (39) and (17) it follows that: if the functions defined by (15) check the equations (37), (38) then the following equalities hold:

$$ \frac{1}{\Gamma(1-\alpha)} \cdot \frac{\partial}{\partial t_M^*} \int_0^{t^*_{M_O}} \frac{U_{O_i^*}(\tau, x_1, x_2, x_3)}{(t_M^* - \tau)^{\alpha}} d\tau = 0; \quad i = 1.2.3 \tag{40} $$

Generally, equalities (40) are not verified. So the mathematical description (35), (36) is nonobjective.

That is, observers O and O^*, describing the same flow with (35), (36) and (37), (38) respectively, obtain different results which cannot be reconciled i.e. transformed into each other using only formulas that link the numbers representing a moment of time in two different choices of the origin of time.
The problem is: which of the results is correct? This result may be an additional argument in favor of the need to analyze the objectivity of the mathematical description proposed in [6] - [12].

4. THE IMPURITY DISPERSION DESCRIPTION WHICH USES TEMPORAL CAPUTO FRACTIONAL ORDER PARTIAL DERIVATIVE, WITH INTEGRAL REPRESENTATION ON FINITE INTERVAL, IS NONOBJECTIVE

Assume that in the impurity dispersion description the temporal Caputo fractional partial derivative of order α, $0 < \alpha < 1$, with integral representation on finite interval (formula (27)) is used. Assume that $D_O = D_{O^*} = D,\ R_O = R_{O^*} = R$ are constant, the bulk fluid is incompressible, its velocity is constant, the flow domain is 1D as in [16] and the reference frames of the two observers coincides. In this case, equation (22) for observer O and equation (23) for observer O^* become:

$$ {}_0^C D_{t_M}^{\alpha} C_O = D \cdot \frac{\partial^2 C}{\partial x^2} - U_O \cdot \frac{\partial C}{\partial x} + R \tag{41} $$

$$ {}_0^C D_{t_M^*}^{\alpha} C_{O^*} = D \cdot \frac{\partial^2 C_{O^*}}{\partial x^2} - U_{O^*} \cdot \frac{\partial C_{O^*}}{\partial x} + R \tag{42} $$

with $U_O = U_{O^*}$.

Objectivity of this description means that the solutions of partial differential equations (41) and (42) describe the same dispersion of the impurity.

That is:

if $C_O = C_O(t_M, x)$ is a solution of equation (41) then function $C_{O^*}(t_M^*, x)$ defined by

$$ C_{O^*}(t_M^*, x)\ = C_O(t_M^* + t_{M_{O^*}}, x) \tag{43} $$

checks equation (42)

and

if $C_{O^*} = C_{O^*}(t_M^*, x)$ is a solution of equation (42) then function $C_O(t_M, x)$ defined by

$$ C_O(t_M, x) = C_{O^*}(t_M + t_{M_O}^*, x) \tag{44} $$

checks equation (41).

We assume that the description is objective and start with a solution $C_O = C_O(t_M, x)$ of equation (41). Consider the function $C_{O^*}(t_M^*, x)$ defined by (43), the case when the reference frame R_O, R_{O^*} of the observers O, O^* coincides and $M_O \neq M_{O*}$. Remark that for this function the following equalities hold:

$$ {}_0^C D_{t_M}^{\alpha} C_O = {}_0^C D_{t_M^*}^{\alpha} C_{O^*} + \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t_{M_O}^*} \frac{\frac{\partial C_O}{\partial \tau}(\tau, x)}{(t_M - \tau)^{\alpha}} d\tau \tag{45} $$

$$ D \cdot \frac{\partial^2 C_{O^*}}{\partial x^2} - U_{O*} \cdot \frac{\partial C_{O^*}}{\partial x} + R = D \cdot \frac{\partial^2 C_O}{\partial x^2} - U_O \cdot \frac{\partial C_O}{\partial x} + R \tag{46} $$

Using equalities (45) and (46) it follows that: if the function defined by (43) check the equation (42) then the following equality hold:

$$ \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t_{M_O}^*} \frac{\frac{\partial C}{\partial \tau}(\tau, x)}{(t_M - \tau)^{\alpha}} d\tau = 0 \tag{47} $$

Generally, equality (47) is not verified. So the mathematical description (41) is nonobjective.

That is, observers O and O^* describing the same impurity dispersion with (41) and (42) respectively, get different results which cannot be reconciled i.e. transformed into each other using only formulas that link the numbers representing a moment of time in two different choices of the origin of time.

This result raises the question of which of the obtained results is correct?

So the analysis of the objectivity of the mathematical description proposed in [13]- [17] is necessary.

5. THE IMPURITY DISPERSION DESCRIPTION WHICH USES TEMPORAL RIEMANN-LIOUVILLE FRACTIONAL ORDER PARTIAL DERIVATIVE, WITH INTEGRAL REPRESENTATION ON FINITE INTERVAL, IS NONOBJECTIVE

Assume that in the description of the impurity dispersion the temporal Riemann-Liouville fractional partial derivative of order α, $0 < \alpha < 1$, defined with integral representation on a finite interval (formula (27)) is used. Assume that $D_O = D_{O^*} = D, R_O = R_{O^*} = R$ are constant, the bulk fluid is incompressible, its velocity is constant, the flow domain is 1D as in [16] and the reference frames of the two observers coincides. In this case, equation (22) for observer O and equation (23) for observer O^* become:

$$ {}^{R-L}_{\ 0}D^{\alpha}_{t_M} C_O = D \cdot \frac{\partial^2 C_O}{\partial x^2} - U_O \cdot \frac{\partial C_O}{\partial x} + R \tag{48} $$

$$ {}^{R-L}_{\ 0}D^{\ \alpha}_{t^*_M} C_{O^*} = D \cdot \frac{\partial^2 C_{O^*}}{\partial x^2} - U_{O^*} \cdot \frac{\partial C_{O^*}}{\partial x} + R \tag{49} $$

with $U_O = U_{O^*}$.

Objectivity of this description means that the solutions of the partial differential equations (48) and (49) describe the same dispersion of the impurity.

That is:

if $C_O = C_O(t_M, x)$ is a solution of equation (48) then function $C_{O^*}(t^*_M, x)$ defined by

$$ C_{O^*}(t^*_M, x) = C_O(t^*_M + t_{M_{O^*}}, x) \tag{50} $$

checks equation (49)

and

if $C_{O^*} = C_{O^*}(t^*_M, x)$ is a solution of equation (49) then function $C_O(t_M, x)$ defined by

$$ C_O(t_M, x) = C_{O^*}(t_M + t^*_{M_O}, x) \tag{51} $$

checks equation (48).

We assume that the description is objective and start with a solution $C_O = C_O(t_M, x)$ of equation (48).

Consider the function $C_{O^*}(t^*_M, x)$ defined by (50) the case when the reference frame R_O, R_{O^*} of the observers O, O^* coincides and $M_O \neq M_{O*}$. Remark that for this function the following equalities hold:

$$ {}^{R-L}_{\ 0}D^{\alpha}_{t_M} C_O(t_M, x) = {}^{R-L}_{\ 0}D^{\ \alpha}_{t^*_M} C_{O^*}(t^*_M, x) + \frac{1}{\Gamma(1-\alpha)} \cdot \frac{\partial}{\partial t_M} \int_0^{t_{M_{O*}}} \frac{C_O(\tau, x)}{(t_M - \tau)^\alpha} d\tau \tag{52} $$

$$D \cdot \frac{\partial^2 C_{O^*}}{\partial x^2} - U_{O^*} \cdot \frac{\partial C_{O^*}}{\partial x} + R = D \cdot \frac{\partial^2 C_O}{\partial x^2} - U_O \cdot \frac{\partial C_O}{\partial x} + R \quad (53)$$

Using equalities (52) and (53) it follows that: if the function defined by (50) checks the equation (49) then the following equality holds:

$$\frac{1}{\Gamma(1-\alpha)} \cdot \frac{\partial}{\partial t_M} \int_0^{t_{M_{O^*}}} \frac{C_O(\tau, x)}{(t_M - \tau)^\alpha} d\tau = 0 \quad (54)$$

Equality (54) in general is not verified. Therefore, the mathematical description (48) is nonobjective. That is, two observers describing the same impurity dispersion with (48) and (49) respectively get different results. The problem is: which of the obtained results is correct? This result may be an additional argument in favor of the need to analyze the objectivity of the mathematical description proposed in [13] - [17].

6. CONCLUSIONS

1. Mathematical description of the bulk fluid flow and that of the contained impurity dispersion using integer order partial derivatives is objective. That is, different observers, describing with these tools the fluid flow,and the impurity dispersion get results which can be reconciled i.e. transformed into each other using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment of time in two different choices of the origin of time.
2. Mathematical description of the bulk fluid flow and that of the contained impurity dispersion using temporal Caputo or Riemann-Liouwille fractional partial derivative, having integral representation on finite interval, is nonobjective. This means that, observers O and O^* describing the same phenomenon with these tools get different results which cannot be reconciled i.e. transformed into each other using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment of time in two different choices of the origin of time .This conclusion is not an academic curiosity! It is rather a problem: which of the results is correct?.
3. A given mathematical tool is not necessarily apropriate for the mathematical description of a certain Real Word phenomenon.

CONFLICT OF INTERES

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