Nonlinear Free Surface Flow past a Wedge in Channel

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Abstract: *In this paper, the two-dimensional problem of irrotational flow past a wedge located in the center of the channel is considered. Assuming that the fluid is incompressible and non-viscous, the influence of gravity is ignored but the surface tension is considered. The problem which is characterized by the nonlinear boundary conditions on the free surface of the unknown equation is solved numerically by the series truncation technique. The results show that for all given wedge configurations, there is a critical value for the Weber number, for which there is no solution for every Weber number value smaller than this. In addition, the obtained results extend the work done by Gasmi and Mekias [2].*

Key Words: *Free surface, Incompressible, Inviscid flow, Capillarity, Surface tension, Series truncation*

1. INTRODUCTION

In some physical phenomena, potential free surface flows of fluid past an obstacle can be observed, and due to its importance, it becomes the subject of more research. In this paper the steady two-dimensional irrotational flow of an incompressible and inviscid fluid past a wedge located in the center of a channel with a width of H_0 is considered. The effect of gravity is ignored, but we take into account the surface tension. Far upstream, the flow is uniform and the speed is U_0. Due to the existence of the wedge, the flow is divided into two parts, both of which extend to infinity with depths H and H' and with a constant velocities U and U', respectively, see Fig. 1.

For each value of the wedge angle, the problem is characterized by the tree parameters, the two angles γ and γ' at the separations points between the wedge walls and the two free streamlines and the Weber number α defined by the following formula:

$$\alpha = \frac{T}{\rho U_0 H_0}, \tag{1}$$

where T is the surface tension and ρ is the density of the fluid.

When the influence of surface tension T and the gravity effects are neglected, the mathematical solution can be obtained exactly by hodograph transformation, for example see Birkhof and Zarantonello [7] and Batchelor [8]. Under the same conditions, Dormiani, Bruch and Sloss [9] used the Schwarz alternate procedure to solve the problem of flow past truncated convex shaped profiles between walls in logarithmic plane. Gasmi and Mekias [1, 2], Bounif and Gasmi [3], Gasmi [4, 5], Gasmi and Amara [6], studied the problems of flow over an obstruction in a channel. Asavanant and Vanden-Breock [10] Vanden-Breock [11] and Naghdi and Vangsrnpigoon [12] have investigated the problem of flow under gate and a jet flow in the case when the surface tension effects are neglected and considering the effect of gravity.

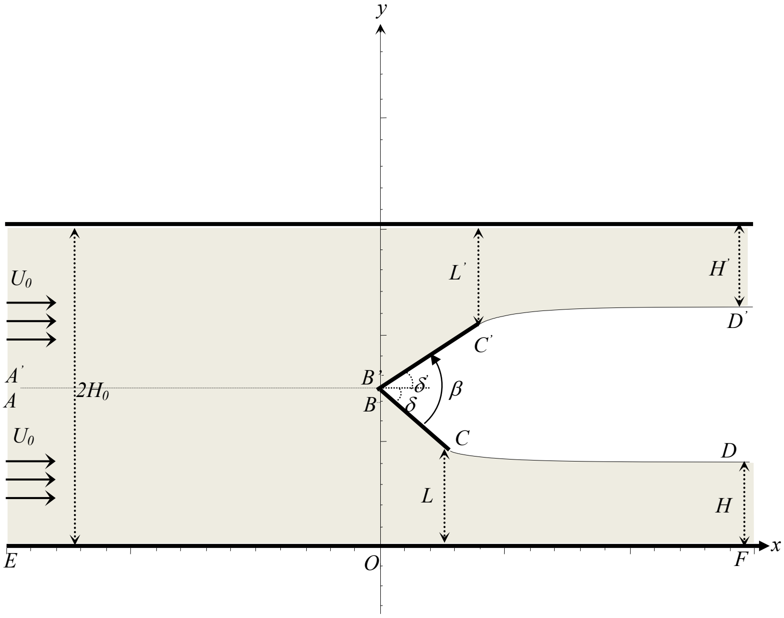

Fig. 1 – Sketch of the flow past a wedge and of the coordinates. The wedge angle is β, the width of the channel is $2H_0$ and the depth of the flow at infinity is $H + H'$. The *x*-axis is along the streamline EOF and the *y* vertically through the point

In the present paper, we solved the fully nonlinear problem numerically and the mesh points were only on the free surface. For each value of the wedge angle, we found that there is a value of the Weber number α^* such that there exists a unique solution, for all $\alpha \geq \alpha^*$. If $\alpha < \alpha^*$the numerical scheme diverges, in the particular case when the wedge angle is equal to $\frac{\pi}{2}$. In this work we extend the calculations of Gasmi and Mekias [2].

The problem is formulated in section 2, the numerical procedure is described in section 3 and in section 4 we present some results and discussion. Finally, this work is concluded in section 5.

2. PROBLEM FORMULATION

Let us consider the steady two-dimensional potential flow of fluid past a wedge CBC' with an apes angle $\beta = \delta + \delta'$, which is placed in the center of channel of width $2H_0$ and infinitively length, see Fig. 1. The flow is considered to be steady, inviscid and incompressible. Gravity effects is neglected but the surface tension forces are taken into account. We introduce Cartesian coordinates with the streamline EOF on the $\bar{x}$ axis and the $\bar{y}$ axis is vertically through the point B. Far upstream the flow is uniform with a constant velocity U_0. Far downstream, we assume that the velocity approaches a constant U and the depth of the fluid tends to a constant $H + H'$.

The dimensionless variables are defined by choosing U_0 as the unit velocity and H_0 as the unit length. We introduce the potential function φ and the stream function ψ. Without loss of generality we choose $\varphi = 0$ at $(x, y) = (0,0)$ and $\psi = 0$ on the streamline EOF. It follows from the choice of the dimensionless variables that $\psi = \frac{-1}{2}$ on the streamline $A'B'C'D'$ and $\psi = \frac{1}{2}$ on the streamline $ABCD$. In order to use the conformal mapping techniques, we consider the flow in the complex plane $z = x + iy$ and the complex potential function $f = \phi + i\psi$. The half of the flow region considered in the z-plane will be mapped via the potential function f onto the semi-infinite strip $(-\infty < \varphi < +\infty, \frac{-1}{2} < \psi < \frac{1}{2})$, see Fig. 2.

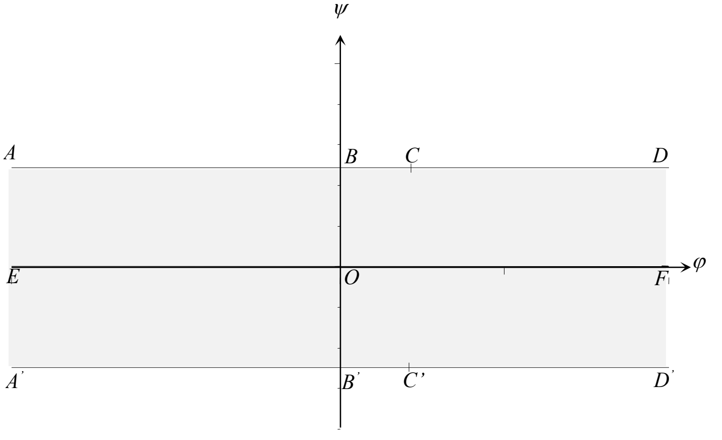

Fig. 2 – Flow configuration in the complex potential $f -$ plane

We introduce the complex velocity $\zeta = u - iv = \frac{df}{dz}$.
Now our task is to solve the boundary value problem

$$\Delta\phi = 0, \qquad \text{in the flow domain.} \tag{2}$$

The dynamic condition on the free surfaces CD and C′D′ is expressed by Bernoulli equation

$$\frac{1}{2}|\nabla\phi|^2 - \frac{\overline{P}}{\rho} = \text{cts}, \tag{3}$$

where $\overline{P}$ is the fluid pressure.
The other condition is kinematic along the solid and the free boundaries satisfying

$$\frac{\partial\phi}{\partial\eta} = 0;\ \text{on the walls,} \tag{4}$$

where η is the normal vector of the boundaries.
In dimensionless variables (3) becomes:

$$\frac{1}{2}|\nabla\phi|^2 - \frac{2}{\alpha}K = 1. \tag{5}$$

Here K is the curvature of the free surface and α is the Weber number defined by (1).
In order that the curvature will be well defined we introduce the function $\tau - i\theta$ as:

$$\zeta = u - iv = e^{\tau - i\theta}. \tag{6}$$

We shall seek $\tau - i\theta$ as an analytic function of $f = \phi + i\psi$ in the strip, $-1 < \psi < 0$.
From the identity $|\zeta| = e^{\tau}$, we rewrite (5) as

$$\frac{1}{2}e^{2\tau} - \frac{e^{\tau}}{\alpha}\frac{\partial\theta}{\partial\varphi} = \frac{1}{2}. \tag{7}$$

The kinematic condition on AB, BC and EOF can be expressed as:

$$\mathrm{Im}\zeta = 0 \quad \text{on } \psi = \frac{1}{2},\ \psi = -\frac{1}{2} \text{ and } -\infty < \phi < \phi_B. \tag{8}$$

$$\frac{Im\zeta}{Re\zeta} = \tan\delta \quad on\ \psi = 0 \quad and\ 0 < \phi < \phi_C, \tag{9}$$

$$\frac{\mathrm{Im}\zeta}{\mathrm{Re}\zeta} = \tan\delta' \quad \text{on } \psi = 0 \quad \text{and } 0 < \phi < \phi_{C'}. \tag{10}$$

This completes the formulation of the problem of determining $\tau - i\theta$. For given values of α, β and H, this function must be analytic in the strip , $-\frac{1}{2} < \psi < \frac{1}{2}$ and satisfy conditions (7), (8), (9) and (10).

3. NUMERICAL PROCEDURE

Using the same scheme introduced by Vanden-broeck [11] and Gasmi and Mekias [1, 2], first we map the flow domain into the half of the unit disk in the complex t-plane by the transformation

$$f = \frac{2}{\pi}\log\left(\frac{-2it}{1-t^2}\right). \tag{11}$$

The walls $A'B'$, $B'C'$, AB and BC go onto the imaginer interval $(-i, i)$, the wall EOF onto the real interval $(0,1)$ and the free surfaces CD and $C'D'$ onto the circumference, see Fig. 3.

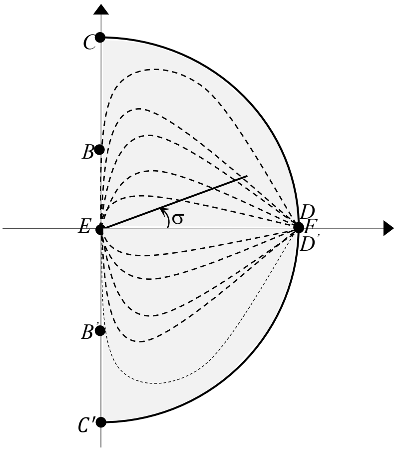

Fig. 3 – The image of the Flow in the t-plane

Table 1 – The positioning of the major points of the flow domain problem in the f-plane and the t-plane

Cartesian plane	f-plane	t-plane
A	$\psi = \frac{1}{2}, \quad \phi = \phi_A = -\infty$	$t = 0$
B	$\psi = \frac{1}{2}, \quad \phi = \phi_B = 0$	$t = ib$
C	$\psi = \frac{1}{2}, \quad \phi = \phi_C > 0$	$t = i$
D	$\psi = \frac{1}{2}, \quad \phi = \phi_D = +\infty$	$t = 1$

A'	$\psi=-\frac{1}{2}, \quad \phi=\phi_{A'}=-\infty$	$t=0$
B'	$\psi=-\frac{1}{2}, \ \phi=\phi_{B'}=0$	$t=-ib'$
C'	$\psi=-\frac{1}{2}, \ \phi=\phi_{C'}>0$	$t=-i$
D'	$\psi=-\frac{1}{2}, \quad \phi=\phi_{D'}=+\infty$	$t=1$
E	$\psi=0, \quad \phi=\phi_E=-\infty$	$t=0$
O	$\psi=0, \quad \phi=\phi_O=0$	$0<t<1$
F	$\psi=0, \quad \phi=\phi_F=+\infty$	$t=1$

Since there are stagnation points at B and B', and discontinuity in the derivative $\frac{\partial\theta}{\partial\varphi}$ at the separation points C and C', ζ must have zeros and poles at these points. Local analysis shows that appropriate zeros and singularities are

$$\zeta \sim (b^2-t^2)^{\frac{\delta}{\pi}} \ as, \qquad t\to ib \tag{12}$$

$$\zeta \sim \left(t^2+b'^2\right)^{\frac{\delta'}{\pi}} \ as, \qquad t\to -ib' \tag{13}$$

$$\zeta \sim (t^2-1)^{1-\frac{\gamma}{\pi}} \ as, \qquad t\to i \tag{14}$$

$$\zeta \sim (t^2+1)^{1-\frac{\gamma'}{\pi}} \ as, \qquad t\to -i \tag{15}$$

Here b and b' are the images of the corners B and B$'$ in the t-plane.

We now define the function $\zeta(t)$ by the relation:

$$\zeta(t)=\left(t^2+b'^2\right)^{\frac{\delta'}{\pi}}(b^2-t^2)^{\frac{\delta}{\pi}}(t^2-1)^{1-\frac{\gamma}{\pi}}(t^2+1)^{1-\frac{\gamma'}{\pi}}\Omega(t). \tag{16}$$

The ζ singularities are removed in equation (16) by the factors $\left(t^2+b'^2\right)^{\frac{\delta'}{\pi}}$, $(b^2-t^2)^{\frac{\delta}{\pi}}$, $(t^2+1)^{1-\frac{\gamma'}{\pi}}$ and $(t^2-1)^{1-\frac{\gamma}{\pi}}$. It follows that Ω can be represented by Taylor expansion in power of t.

Furthermore, the kinematic conditions (8), (9) and (10) imply that the expansion for Ω has real coefficients a_n, and involves only even powers of t. Hence,

$$\zeta(t)=\left(t^2+b'^2\right)^{\frac{\delta'}{\pi}}(b^2-t^2)^{\frac{\delta}{\pi}}(t^2-1)^{1-\frac{\gamma}{\pi}}(t^2+1)^{1-\frac{\gamma'}{\pi}}\sum_{k=0}^{+\infty}a_n t^{2n}. \tag{17}$$

We use the notation $t=|t|e^{i\sigma}$ so that points on BC are given by $t=|t|e^{i\sigma}$, $\frac{-\pi}{2}<\sigma<\frac{\pi}{2}$. Using (17) we rewrite (7) in the form:

$$e^{2\bar{\tau}(\sigma)}-\frac{2\pi}{\alpha}\tan\sigma\frac{\partial\bar{\theta}}{\partial\varphi}(\sigma)e^{\bar{\tau}(\sigma)}=1. \tag{18}$$

Here $\bar{\tau}(\sigma)$ and $\bar{\theta}(\sigma)$ denote the values of τ and θ on the free surfaces BC and $B'C'$. We solve the problem approximately by truncating the infinite series in (17) after $N+2$ terms. We find the N coefficients a_n and the separation angles γ and γ' by collocation. Thus we introduce the N mesh points.

$$\sigma(I) = -\frac{\pi}{2} + \frac{\pi}{(N+2)}\left(I - \frac{1}{2}\right), I = 1, \cdots, N+2. \tag{19}$$

Using (19) we obtain $[\bar{\tau}(\sigma)]_{\sigma=\sigma_I}$, $[\bar{\theta}(\sigma)]_{\sigma=\sigma_I}$ and $\left[\frac{\partial\bar{\theta}}{\partial\varphi}(\sigma)\right]_{\sigma=\sigma_I}$ in terms of coefficients a_n and the separation angles γ and γ'.

Thus, we obtain $N+2$ nonlinear algebraic equations of $N+2$ unknowns $(a_{n,\ n=1,\ldots,N}, \gamma, \gamma')$.

For given values of the wedge angle β and Weber number α, this system of equations is solved by Newton's method.

Finally, the shape of the surface is obtained by integrating numerically the relation

$$\begin{cases} \dfrac{\partial x}{\partial \sigma} = \dfrac{2}{\pi} \cot\sigma\, e^{\bar{\tau}(\sigma)} \cos\bar{\theta}(\sigma) \\ \dfrac{\partial y}{\partial \sigma} = \dfrac{2}{\pi} \cot\sigma\, e^{\bar{\tau}(\sigma)} \sin\bar{\theta}(\sigma) \end{cases} \tag{20}$$

4. EXAMPLE RESULTS AND DISCUSSIONS

The numerical schemes of section 3 were used in the symmetrical case when $\delta' = \delta$ and $L = L'$ to compute solutions for different values of the wedge angle β and several values of the Weber number α.

We found that the coefficients a_n decrease rapidly as n increases. Table 2 shows some of the coefficients of the series (17) and the corresponding Weber number for different values of β. Most of the calculations were done and presented with N=60.

Table 2 – Some values of the coefficients a_n of the series (17) for several values of the angle β, $b = b' = 0.5$ and different values of Weber number α.

β	α	a_1	a_{20}	a_{40}	a_{60}
$\frac{\pi}{2}$	1.5	1.445×10^{-1}	1.325×10^{-5}	2.503×10^{-6}	1.377×10^{-7}
	10	5.083×10^{-2}	1.475×10^{-5}	-4.983×10^{-7}	3.151×10^{-7}
	$\alpha \to \infty$	1.554×10^{-9}	3.726×10^{-10}	2.839×10^{-11}	-5.059×10^{-14}
π	1.5	2.194×10^{-1}	2.308×10^{-6}	3.833×10^{-7}	2.049×10^{-8}
	10	1.158×10^{-1}	9.558×10^{-5}	2.078×10^{-5}	1.270×10^{-6}
	$\alpha \to \infty$	1.681×10^{-7}	-4.704×10^{-7}	-1.304×10^{-12}	-4.195×10^{-19}
$\frac{3\pi}{2}$	1.5	4.329×10^{-1}	4.764×10^{-5}	9.542×10^{-6}	5.286×10^{-7}
	10	1.521×10^{-1}	4.516×10^{-5}	-1.178×10^{-6}	9.809×10^{-7}
	$\alpha \to \infty$	4.846×10^{-9}	6.931×10^{-10}	-4.736×10^{-11}	-1.057×10^{-11}

4.1 Flow with surface tension effect

When the effect of the surface tension is included in the dynamical condition on the free surface, the numerical computational shows that there exists a critical value $\alpha = \alpha^*$ for each values of the wedge angle $\beta = 2\delta = 2\delta'$ and $b = b'$, for which there is no solution for $\alpha < \alpha^*$, see table 3.

Table 3 – Some values of the minimal Weber number for and different values of the wedge angle

$\beta = 2\delta = 2\delta'$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{3\pi}{2}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
α^*	1.283	1.211	1.203	1.754	0.942	1.512

Accurate solutions for $\alpha \geq \alpha^*$are obtained. As $b \to 1$, $b' \to 1$, $\delta = \delta'$and for all wedge angle $0 < \beta < \frac{\pi}{2}$ we obtain the same results as Gasmi and Mekias [1-2] and our results also are in agreement with the results of Ackerberg and Liu[13] for different Weber number $\alpha \geq \alpha^*$.

These authors solved the problem via the finite difference method and the mesh point were throughout the fluid domain, they could find solution for all $\alpha \geq \tilde{\alpha} = 6.801483$.

In our procedure mesh points are only needed on the free surface and we computed solutions for $\alpha \geq \tilde{\alpha}$. For $\alpha \geq \tilde{\alpha}$ our results are in agreement with theirs.

We note that the contraction coefficients $= \frac{H}{L}$, $C' = \frac{H'}{L'}$, the angle in the separation points δ and δ' increase as the Weber number α decreases. Numerical values of C and C' versus $\frac{1}{\alpha}$ are presented in Fig. 4.

In Fig. 5 we present values of the angles at the separation point between the wedge walls and the free streamlines γ and γ' versus $\frac{1}{\alpha}$.

It is seen that numerical solutions exist for all $\alpha \geq \alpha^*$.

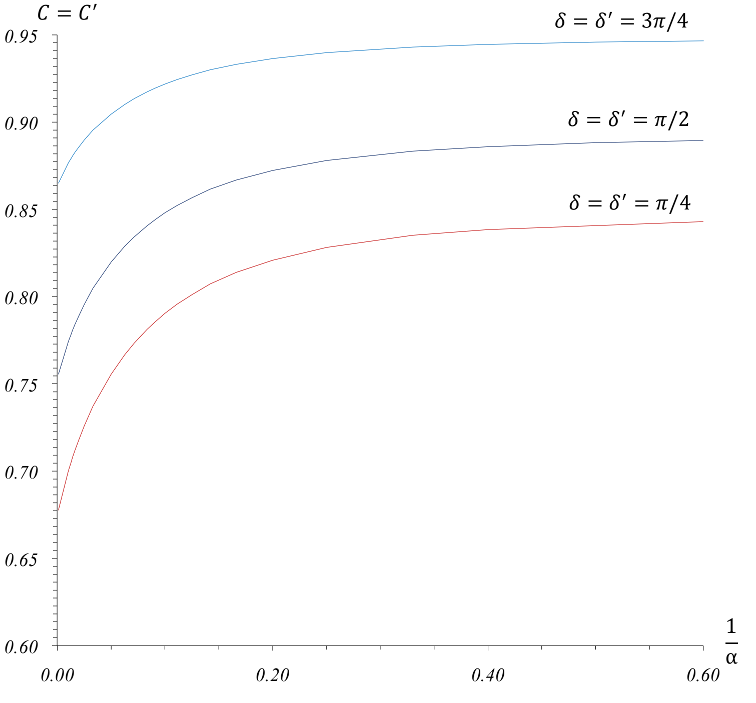

Fig. 4 – Coefficients of contraction $\boldsymbol{C}$ and $\boldsymbol{C'}$ vs $\frac{\mathbf{1}}{\boldsymbol{\alpha}}$

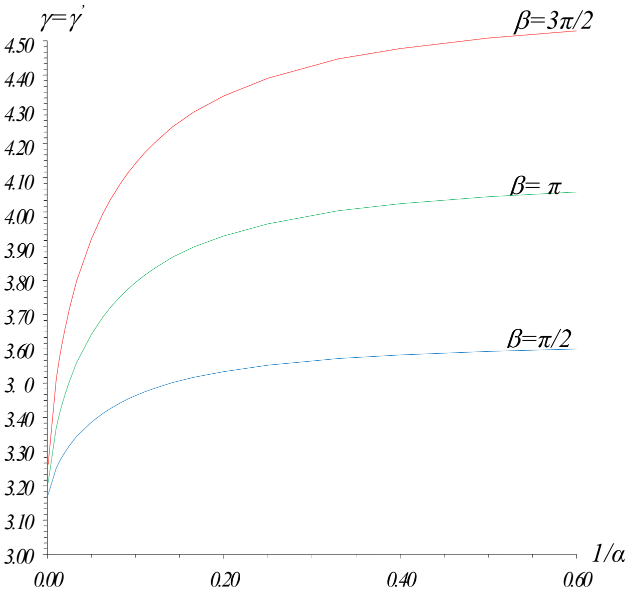

Fig. 5 – The angles of separation $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma'}$ vs $\frac{\mathbf{1}}{\boldsymbol{\alpha}}$

Typical profiles for various Weber numbers of the free surfaces are presented in Fig. 6 for $\beta = \frac{3\pi}{2}$, Fig. 7 for $\beta = \pi$, and Fig. 8 for $\beta = \frac{\pi}{2}$.

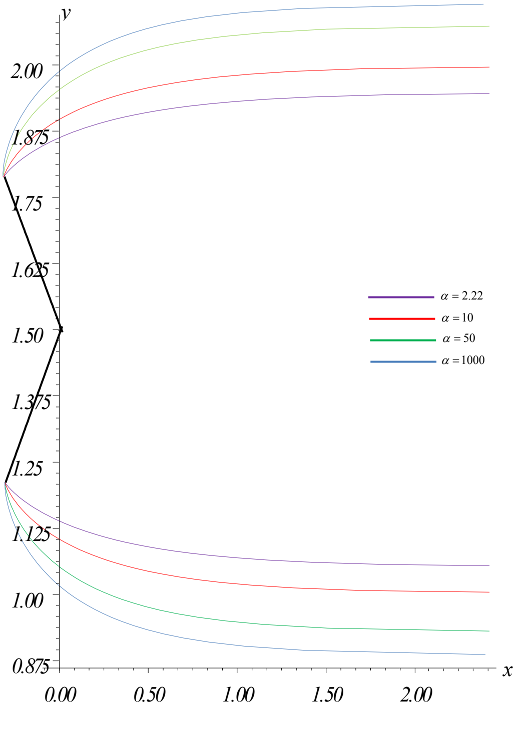

Fig. 6 – Computed free streamline shapes for $\beta = \frac{3\pi}{2}$, $b = b' = 0{,}5$ and various Weber numbers

For $\beta = \pi$ our problem reduces to flow past a vertical flat plate in tunnel treated by Gasmi and

Mekias[2]

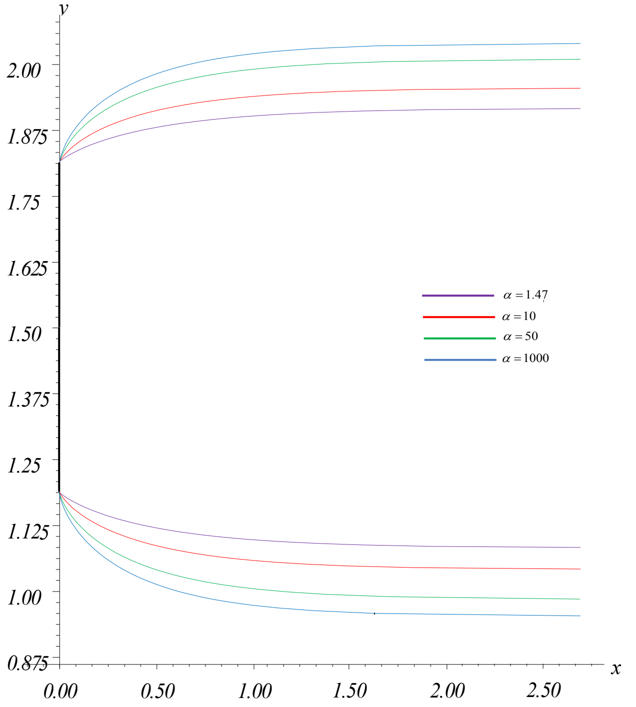

Fig. 7 – Computed free streamline shapes for $\boldsymbol{\beta = \pi}$, $\boldsymbol{b = b' = 0,5}$ and various Weber numbers

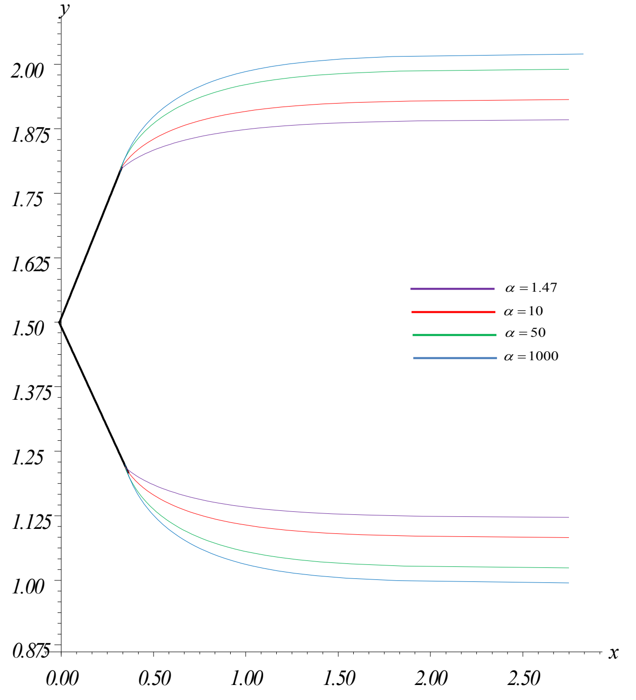

Fig. 8 – Compued free streamline shapes for $\beta = \pi$, $b = b' = 0{,}5$ and various Weber numbers

4.2 Flow without surface tension

When the Weber number is very large $\alpha \geq 10^3$, the exact analytical solutions can be computed via free stream line theory [8]. We computed numerically these solutions using the procedure described above and our results are in agreement with the theoretical and experimental results given in Birkhoff and Zarantonello [7] Fig. 9.
As $0 < b < 1$ and $0 < b' < -1$, the coefficients $a_n \sim 0$ and the angles $\gamma = 3.1415$ $\gamma' = 3.1415$, that is to say that the flow leaves the walls of the wedge tangentially. In other words, there is no singularity at the contact points C and C', hence the solution is

$$\zeta(t) = \left(t^2 + b'^2\right)^{\frac{2\delta'}{\pi}} = (b^2 - t^2)^{\frac{2\delta}{\pi}},$$

which is the classical Kirchhoff solution Batchelor [8].
For $\delta = \delta' = \frac{\pi}{2}$ we obtain $C = C' = 0.611$. The comparison of the free streamline shapes for $\delta = \delta' = \frac{\pi}{2}$ obtained using our methods with the theoretical exact solutions are presented in Fig. 10.

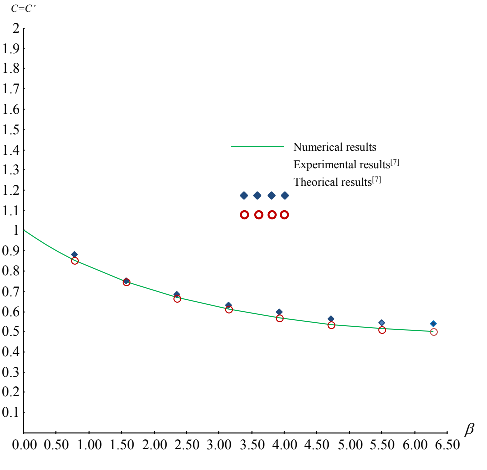

Fig. 9 – Comparison of the obtained results of $\boldsymbol{C = C'}$ for various values of $\boldsymbol{\beta}$ and $\boldsymbol{b = b' = 0,5}$ with the experimental and theatrical results given in [7]

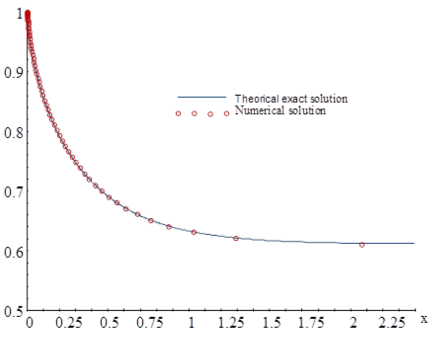

Fig. 10– Comparison of the calculated free surface with the exact solution for $\boldsymbol{\beta = \pi}$, $\boldsymbol{b = b' = 1}$

5. CONCLUSIONS

In this work, the series truncation technique is used to find the solution and shape of the free surface flow. This technique allows us to accurately determine the type of singularity at the connection point. The obtained solution shows that the effect of surface tension is to reduce the curvature of the free surface. The best advantage of this technique is to reduce the dimension of the problem from two-dimensional to one-dimensional, and to find the solution only on the boundary of the flow field.

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