First order perturbation approach for the free surface flow over a step with large Weber number

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Abstract: *The problem of two-dimensional free surface flow of inviscid and incompressible fluid over a step is considered. The flow is assumed to be as steady and irrotational, the effect of the surface tension is considered, but the gravity force is neglected. This problem is characterized by the nonlinear condition given by Bernoulli's equation on the unknown free surface, which can be considered as part of the solution. The main purpose of this work is to give an approximate solution of this problem, by using the Hilbert transformation and the perturbation technique; the results are calculated for a large values of the Weber number and small inclination angle of the step values. These results demonstrate that the used method is easily implemented, and provides approximate solutions to these kinds of problems.*

Key Words: *Free surface flow, surface tension, incompressible flow, Hilbert method, perturbation technique.*

1. INTRODUCTION

The problem of potential free surface flow over an obstacle is more common in fluid mechanics, noting that these problems are known as nonlinear, and characterized by the nonlinear condition on the free surface given by the Bernoulli equation, and as the shape of the free surface is unknown, it should be considered as part of the problem solution; it can also be said that it is a boundary value problem with mixed boundary conditions. In this paper, the two-dimensional free-surface flow of a fluid over a step is considered, the flow is assumed to be stationary, and the effect of the superficial tension is considered, but the gravity effects are neglected. The fluid is taken as incompressible and inviscid. Given the above-mentioned features, which make it difficult to solve this problem analytically, it is necessary, in this case, to use a method of approximation in order to obtain solutions.

Several authors have studied numerically the problems of the free-surface flow using various techniques and methods, such as the series truncation technique and the boundary-integral method, which consists in determining the shape of the free surface for some potential flows passing a given obstacle. For example Forbes and Schwartz [1], determined the non-linear solutions of subcritical and supercritical flows over a semi-circular obstacle, Gasmi and Mekias [2], Gasmi and Amara [3] and Vanden-Broeck [4], studied the problems of flow over an obstruction in a channel, Merzougui and Laiadi [5] solved the problem of the flow over a triangular depression, Sekhri, Guechi and Mekias [6] addressed the problem of flow past a submerged triangular obstacle, whilst Dias, Killer and Vanden-Broeck [7], obtained solutions to both subcritical and supercritical free-surface flows past a triangular obstacle; Wiryanto [8] considered the problem of the flow under a sluice gate, M. B. Abd-el-Malek and S. Z. Masoud [9] obtained the linear solution of the flow over a ramp, by representing the bottom in integral form using Fourier's double-integral theorem.

M. B. Abd-el-Malek and S. N. Hanna [10] solved numerically the problem of the flow over a ramp with gravity effect by the Hilbert Method and the perturbation technique.

M. B. Abd-el-Malek, S. N. Hanna and M. T. Kamel [11] investigated the flow over triangular bottom.

The principal goal of this work is to solve the considered problem approximately by following the next three steps; first, the Schwartz-Christoffel transformation was employed to map the flow region of the potential plane, onto the upper half of another plane; then, we applied the Hilbert method for a mixed boundary value problem in this new plane, thus finding a system of nonlinear integral equations and finally we used the perturbation technique to solve the obtained system for some large values of the Weber number and different small value of the angle α of the step.

The obtained results show that the used method is easily implemented, and provides approximate solutions to these kinds of problems.

This paper is organized as follows. Section 2 presents the mathematical formulation of the problem.

In section 3 we derive the approximate equations of the problem. Section 4 describes how to apply the perturbation technique to obtain solutions. Section 5 shows the results and conclusions together with some free streamline profiles.

2. PROBLEM FORMULATION

Let us consider the motion of a two-dimensional flow of a fluid over a step placed in the bottom of a channel. The fluid is assumed to be incompressible, irrotational and inviscid. The effect of gravity is neglected but we take into account the superficial tension effect with small values. Far upstream and downstream, the flow is proposed as uniform with a constant discharge $U_1 h_1 = U_2 h_2$, where $U_i,\ i = 1,\ 2$ design the velocities and $h_i,\ i = 1,\ 2$ the depths of the flow at upstream and downstream respectively, so that the bottom consists of a horizontal walls AB, CD and the inclined wall BC; we note by α and L the inclination angle and the length of the step inclined wall BC and as reference we choose the Cartesian coordinates with the origin placed in the stagnation point B, the $\tilde{x}$ axis in the direction of the wall AB and $\tilde{y}$ axis directed vertically upward (see Figure 1).

Since the flow is potential, its velocity $\tilde{\eta}$ can be described as the gradient of a scalar function $\tilde{\phi}$ called a potential function, and we can write:

$$\tilde{u}(\tilde{x},\tilde{y}) = \frac{\partial \tilde{\phi}}{\partial \tilde{x}}(\tilde{x},\tilde{y}), \quad \tilde{v}(\tilde{x},\tilde{y}) = \frac{\partial \tilde{\phi}}{\partial \tilde{y}}(\tilde{x},\tilde{y}), \tag{1}$$

Here $\tilde{u}$ and $\tilde{v}$are the real and imaginary parts, respectively, of a complex velocity $\tilde{\eta}$. Next, we introduce the complex analytic function $\tilde{f}$ of the variable $\tilde{z} = \tilde{x} + i\tilde{y}$ defined by:

$$\tilde{f} = \tilde{\phi}(\tilde{x},\tilde{y}) + i\tilde{\psi}(\tilde{x},\tilde{y}), \tag{2}$$

where $\tilde{\psi}$ is the stream function which is the conjugate of the potential function.

From (1) and (2), we obtain

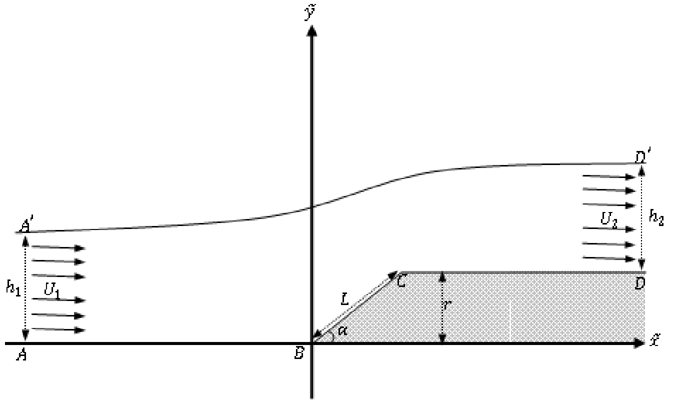

Figure 1: Sketch of the flow and of the coordinates. The length of the step is L and the angle between the inclined wall AB and the horizontal is α

$$\tilde{\eta} = \tilde{u} - i\tilde{v} = \frac{d\tilde{f}}{d\tilde{z}}. \tag{3}$$

For simplicity, we use the dimensionless analysis, then we choose U_1 and h_1 as the unit velocity and the unit length respectively, and the new non-dimensional variables are:

$$x = \frac{\tilde{x}}{h_1}, \quad y = \frac{\tilde{y}}{h_1}, \quad u = \frac{\tilde{u}}{U_1}, \quad v = \frac{\tilde{v}}{U_1}, \quad \phi = \frac{\tilde{\phi}}{U_1 h_1}, \quad \psi = \frac{\tilde{\psi}}{U_1 h_1}, \tag{4}$$

Let us introduce the variable ω as a function of two variables q and θ by the relation:

$$\omega = ln\,\eta = ln\,q - i\theta, \tag{5}$$

here ω is called the logarithmic hodograph variable.
Then, from equations (2) and (5) we get:

$$z = \int e^{-\omega}\, df. \tag{6}$$

Without loss of generality, we choose $\phi = 0$ at the point B, $\psi = 1$ on the streamline $A'D'$, and $\psi = 0$ on the streamline $ABCD$ (see Figure 2). Denote the dimensionless step height by r, where

$$r = L\,sin\,\alpha, \tag{7}$$

From Gasmi [2], on the free-surface, where the pressure is uniform, the Bernoulli's equation in dimensionless is given by:

$$q^2 + \frac{2}{We}\left|\frac{\partial\theta}{\partial\phi}\right| q = 1, \tag{8}$$

where We is the non-dimensional parameter, known as the Weber number and defined by:

$$We = \frac{\rho U_1^2 h_1}{T}, \tag{9}$$

here T is the surface tension, and ρ is the density of the fluid.

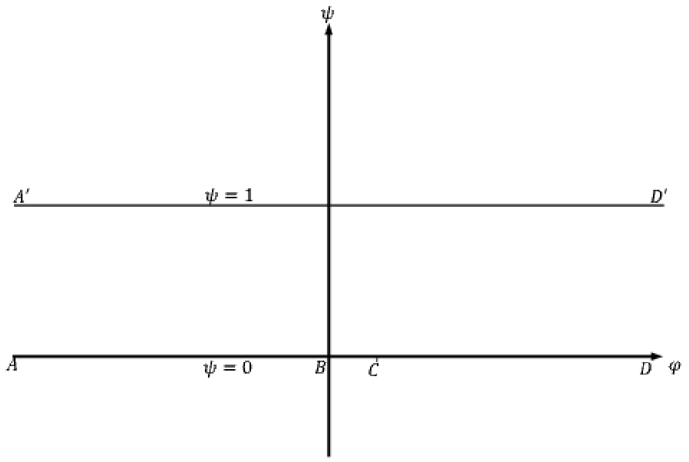

Figure 2: The potential f plane

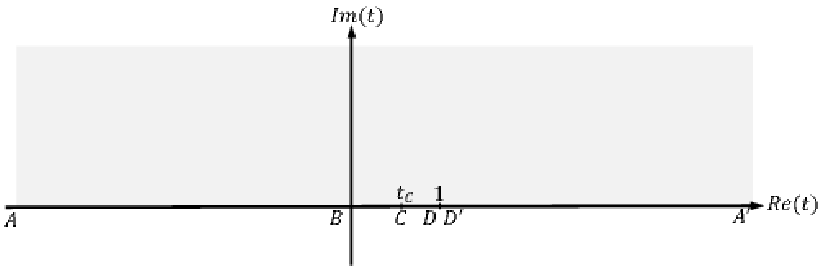

Figure 3: The auxiliary t plane

By using the Schwartz-Christoffel transformation, the strip occupied by the fluid in the potential ff-plane in Figure 2 is mapped onto the upper half of another auxiliary t-plane (see Figure 3), and the obtained transformation is:

$$f(t) = -\frac{1}{\pi} ln(1 - t). \tag{10}$$

After applying the Hilbert method for a mixed boundary-value problem in the t-plane, we have the following system of the nonlinear integral equations:

$$\theta(t) = \frac{2\alpha}{\pi} tan^{-1}\left(\frac{(1-\sqrt{1-t_c})\sqrt{1-t}}{t-1+\sqrt{1-t_c}}\right) + \frac{\sqrt{1-t}}{\pi}\left(p.v.\int_{1}^{+\infty}\frac{ln\, q\,(s)}{(s-t)\sqrt{s-1}}ds\right), t > 1, \tag{11}$$

$$ln\, q_j\,(t) = \frac{\sqrt{1-t}}{\pi}\left(p.v.\int_{1}^{+\infty}\frac{ln\, q\,(s)}{(s-t)\sqrt{s-1}}ds - \alpha\left(p.v.\int_{0}^{t_C}\frac{ds}{(s-t)\sqrt{s-1}}\right)\right). \tag{12}$$

where *p.v.* is the principal value of the integral, and $q_j(t),\ j = 1,2,3$ and $q_1(t)$ is defined in $t < 0$, $q_2(t)$ in $0 < t < t_C$ and $q_3(t)$ in $t_C < t$.
Using (6) and (10) and by separating the real and imaginary part, the shape of the free surface is given by the relations:

$$x(t) = x_\infty - \frac{1}{\pi}\int_{t}^{+\infty}\frac{cos\,\theta\,(s)}{(1-s)q(s)}ds; \quad t > 1, \tag{13}$$

and

$$y(t) = 1 - \frac{1}{\pi}\int_{t}^{+\infty} \frac{\sin\theta\,(s)}{(1-s)q(s)}ds; \quad t > 1. \tag{14}$$

And the form of the bottom by:

$$z(t) = z_0 - \frac{1}{\pi}\int_{t_0}^{t} \frac{e^{-\theta(s)}}{(1-s)q(s)}ds; \quad t < 1, \tag{15}$$

From (15), the length of the step is:

$$L = \frac{1}{\pi}\int_{0}^{t_C} \frac{1}{(1-s)q_2(s)}ds. \tag{16}$$

3. THE APPROXIMATE EQUATIONS

When the value of the Weber number is large, and by using first-order Taylor development with respect to $\frac{1}{We}\left|\frac{\partial\theta}{\partial\phi}\right|$, the solution of equation (8) is given by:

$$q(t) \approx 1 - \frac{1}{We}\left|\frac{\partial\theta}{\partial\phi}\right|, \tag{17}$$

which yields

$$ln\, q\,(t) \approx -\frac{1}{We}\left|\frac{\partial\theta}{\partial\phi}\right|, \tag{18}$$

and

$$\frac{1}{q(t)} \approx 1 + \frac{1}{We}\left|\frac{\partial\theta}{\partial\phi}\right|. \tag{19}$$

Using the relation (10), we obtain:

$$\frac{\partial\theta}{\partial\phi} = \frac{\partial\theta}{\partial t}\frac{\partial t}{\partial\phi} = \pi(t+1)\frac{\partial\theta}{\partial t}, \quad t > 1, \tag{20}$$

From (20), the equation (17) become

$$q(t) \approx 1 - \frac{1}{We}\pi(t+1)\frac{\partial\theta}{\partial t}. \tag{21}$$

For small inclination angle α, the change in θ will be very small, and we can approximate $sin\,\theta$ by θ and $cos\,\theta$ by one.

In this case, and by using (18), the angle of the free surface with the horizontal (11) is approximated by:

$$\theta(t) \approx \frac{2\alpha}{\pi}tan^{-1}\left(\frac{(1-\sqrt{1-t_c})\sqrt{1-t}}{t-1+\sqrt{1-t_c}}\right) - \frac{\sqrt{1-t}}{We}\left(p.v.\int_{1}^{+\infty}\frac{(s-1)\frac{\partial\theta}{\partial s}(s)}{(s-t)\sqrt{s-1}}ds\right), \quad t > 1, \tag{22}$$

Substituting (19) into (13) and (14) and after simplification the free surface equations take the form

$$x(t) = x_\infty - \frac{\theta(t)}{We} - \frac{1}{\pi}\int_t^{+\infty} \frac{1}{1-s} ds; \quad t > 1, \tag{23}$$

and

$$y(t) \approx 1 - \frac{\theta^2(t)}{2We} - \frac{1}{\pi}\int_t^{+\infty} \frac{\theta(s)}{1-s} ds; \quad t > 1, \tag{24}$$

Then the length of the step (16) become

$$L \approx -\frac{1}{\pi} ln(1 - t_C). \tag{25}$$

To solve the obtained system of the nonlinear integral equations (21)-(24) we apply the Perturbation technique.

4. PERTURBATION TECHNIQUE

The expansion of a given function $X(t)$ in terms of the small parameter α is represented by the series:

$$X(t) = \sum_{k=0}^{+\infty} X_k\,(t)\alpha^k. \tag{26}$$

where $X(t)$ stands for $q(t)$, $\theta(t)$, $\frac{\partial\theta}{\partial t}(t)$, $x(t)$ and $y(t)$.

4.1 Zero-order approximation

This case corresponds to the flow far upstream, which is considered uniform. Then the zero-order approximation of the solution is presented by:
The velocity of the flow

$$q_0(t) \approx 1; \quad t > 1, \tag{27}$$

the velocity direction relative to the horizontal

$$\theta_0(t) \approx 0, \quad t > 1, \tag{28}$$

The free streamline equations

$$x_0(t) \approx x_\infty - \frac{1}{\pi}\int_t^{+\infty} \frac{1}{1-s} ds; \quad t > 1, \tag{29}$$

and

$$y_0(t) \approx 1; \quad t > 1. \tag{30}$$

On the other hand, we have the formula:

$$x_\infty(t) \approx \frac{1}{\pi}\left(v.p.\int_0^{+\infty} \frac{ds}{1-s}\right), \tag{31}$$

hence

$$x_0(t) \approx -\frac{1}{\pi} ln(t - 1); \quad t > 1, \tag{32}$$

4.2 First-order approximation

By using the obtained results (27), (28), (30) and (32) and the development (26), we get the solution of the first order of the problem as following equations:

$$q_1(t) \approx 1 - \frac{1}{We}\pi(t+1)\left(\frac{\partial\theta}{\partial t}\right)_1; \quad t > 1, \tag{33}$$

$$\theta_1(t) \approx \frac{2}{\pi}tan^{-1}\left(\frac{(1-\sqrt{1-t_c})\sqrt{1-t}}{t-1+\sqrt{1-t_c}}\right) - \frac{\sqrt{1-t}}{We}\left(p.v.\int_1^{+\infty}\frac{(s-1)\left(\frac{\partial\theta}{\partial s}\right)_1(s)}{(s-t)\sqrt{s-1}}ds\right); \quad t > 1, \tag{34}$$

$$x_1(t) \approx -\frac{\theta_1(t)}{We}; \quad t > 1, \tag{35}$$

and

$$y_1(t) \approx -\frac{1}{\pi}\int_t^{+\infty}\frac{\theta_1(s)}{1-s}ds; \quad t > 1. \tag{36}$$

For a very large value of the Weber number We, we may neglect the second term with respect to the first one of equation (34) and we get:

$$\theta_1(t) \approx \frac{2}{\pi}tan^{-1}\left(\frac{(1-\sqrt{1-t_c})\sqrt{1-t}}{t-1+\sqrt{1-t_c}}\right); \quad t > 1, \tag{37}$$

Substituting (37) into (335) and (36) and by integration, we find

$$x_1(t) \approx -\frac{2}{\pi We}tan^{-1}\left(\frac{(1-\sqrt{1-t_c})\sqrt{1-t}}{t-1+\sqrt{1-t_c}}\right); \quad t > 1, \tag{38}$$

and

$$y_1(t) \approx 4\frac{(1-\sqrt{1-t_c})}{\pi^2\sqrt[4]{1-t_c}}tan^{-1}\left(\frac{\sqrt[4]{1-t_c}}{\sqrt{t-1}}\right); \quad t > 1, \tag{39}$$

Finally, by using the results (30), (32), (38), (39) and (26) we obtain the free surface equations:

$$x(t) \approx -\frac{1}{\pi}ln(t-1) - \frac{2\alpha}{\pi We}tan^{-1}\left(\frac{(1-\sqrt{1-t_c})\sqrt{1-t}}{t-1+\sqrt{1-t_c}}\right); \quad t > 1, \tag{40}$$

and

$$y(t) \approx 1 + 4\alpha\frac{(1-\sqrt{1-t_c})}{\pi^2\sqrt[4]{1-t_c}}tan^{-1}\left(\frac{\sqrt[4]{1-t_c}}{\sqrt{t-1}}\right); \quad t > 1, \tag{41}$$

5. RESULTS AND CONCLUSIONS

The obtained approximate scheme is used to calculate the solutions and the free surface profiles for a fixed values of the inclination angle $\alpha = \frac{\pi}{6}$, the step height $r = 0.7$, the point $t_c = 0.9877$ and different values of Weber number.

Figure 4 presents the variation of the free surface shape with respect to the Weber number.

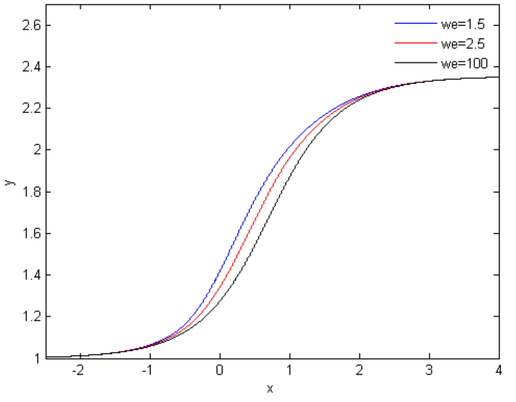

Figure 4: Effect of the weber number on the free-surface profile for the angle $\alpha = \frac{\pi}{6}$, and the step height $r = 0.7$

As shown in figure 4, if the Weber number decreases, the curvature of the free surface is decreased, because this is one of the most important characteristic property of the surface tension effects.

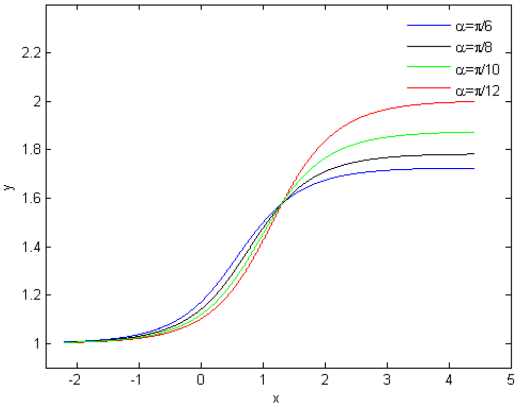

Figure 5: Effect of the inclination angle α on the free-surface profile for $We = 10^4$ and $r = 0.6$

Figure 5 illustrates the free-surface profiles for different angles α, for a fixed values of the step height $r = 0.6$ and the Weber number $We = 10^4$.
As we can see from Figure 5, if the angle value decreases, the elevation of the free surface at the downstream increases.
Table 1 presents some values of the positions of the point t_C and the length of the step L for different values of the angle.

Table 1: Some t_C values for different values of the of the angle α

α	$\frac{\pi}{6}$	$\frac{\pi}{8}$	$\frac{\pi}{10}$	$\frac{\pi}{12}$
L	1.200	1.5679	1.9416	2.3182
t_C	0.9769	0.9927	0.9978	0.9993

For a fixed values of the angle $\alpha = \frac{\pi}{8}$ and the Weber number $We = 100$, the results of the position of the point t_C for different values of the step height r are presented in table 2:

Table 2: Some values of t_C for different values of the step height r

r	0.2	0.3	0.4	0.5	0.7
L	0.5226	0.7839	1.0453	1.3066	1.8292
t_C	0.8064	0.9148	0.9625	0.9835	0.9968

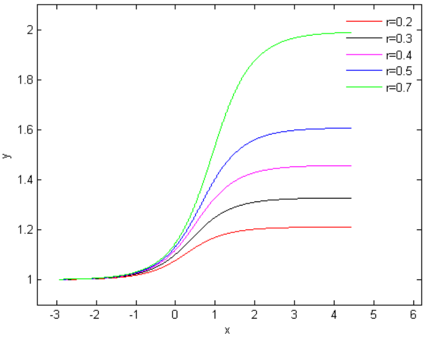

Figure 6: Effect of the step height r on the free-surface profile for $We = 100$ and $\alpha = \frac{\pi}{8}$

As the values of the step height increase, the flow velocity reduces, which leads to increasing also the elevation of the free surface at the downstream, as shown in Figure 6.

Finally, in this paper, the first order approximate method is employed to solve the nonlinear problem of steady free-surface flow over a step bottom. The results show that the used method is easy to implement, and provides solutions to these kinds of problems.

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