An optimisation of rational fit for aircraft wings: improved version

Mihnea BUTIURCA*,[1], Valentin Ioan Remus NICULESCU[2], Dumitru POPESCU[3], Stefan STEBLEA[4]

*Corresponding author
[1]University of Cambridge, United Kingdom,
mb2619@cam.ac.uk*
[2]National Institute for Laser and Radiation Physics, Bucharest,
filo_niculescu@yahoo.com
[3]Department of Mathematical Modelling in Life Sciences,
"Gheorghe Mihoc-Caius Iacob" Institute for Mathematical,
Statistical and Applied Mathematics of the Romanian Academy,
dghpopescu@gmail.com
[4]"Ion Mincu" National University of Urban and Architecture,
stefansteblea@gmail.com

DOI: 10.13111/2066-8201.2026.18.2.1

Received: 21 April 2026/ Accepted: 27April 2026/ Published: June 2026
Copyright © 2026. Published by INCAS. This is an "open access" article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/)

Abstract: *In this work, we will present optimization techniques and improvements for airfoil parametrisation using a rational fit [1], [3]. Our optimization proposals include a refinement of the original fit to eliminate errors and discontinuities at the leading edge, and using complex root/pole analysis in order to optimize the search region for the solving algorithm. We have found that there is a significant jump in performance between an 8 parameter fit and a 9 parameter fit, and that representing the rational fit in terms of the complex roots and poles can significantly reduce the parameter search region.*

Key Words: *Wing geometric characteristics, lacunary polynomials, optimization techniques*

1. INTRODUCTION

Niculescu et. al. [1] previously came up with a potential fit for aircraft wings that represents them as a ratio of 2 polynomials:

$$f(x) = \frac{\Sigma_{i=0}^{i=n_1} a_i x^i}{\Sigma_{i=0}^{i=n_2} b_i x^i} \tag{1}$$

They then tested it against Class Shape Transformation (CST) [6] for 50 different airfoils [2] (their lower surfaces), and showed that in certain situations, a lower root-mean-squared error (rms error) and maximum error (max error) as percentages of chord could be achieved. We are going to use the same 50 airfoils, but we will analyse both the upper and lower surface.

In this work we will explore 2 potential improvements to the fit, including:

1. Parameter number optimization (diminishing returns analysis);
2. Complex roots/poles analysis;

We will then analyse the effect of these improvements on our fit's performance in comparison to Class Shape Transformation, using a CMA-ES fitting algorithm [4] [5].

2. THEORETICAL METHOD

a. Parameter number optimization

Since computational power is very valuable in CFD simulations, it is very important that our proposed fits have as few parameters as possible. Having a small number of parameters means a smaller search region for optimization algorithms, which means less iterations for the CFD simulation, and faster results.

We are going to analyze the effect of parameter number for our rational fit, and evaluate it in terms of root-mean-squared error as a % of chord, maximum error as a % of chord, and number of iterations until convergence is reached.

We are going to use a least-squares solver in order to evaluate the best fit in order to plot the iteration history.

We are going to evaluate multiple combinations of n_1 and n_2 parameters (see equation (1)), and see the evolution of the fit and iteration time with the number of parameters. We are going to go try multiple combinations of n_1 and n_2.

In order to facilitate the root/pole analysis, we will limit the maximum order of the denominator polynomial to 4.

$$(n_1, n_2) = \{(3,3);(3,4);(4,3);(4,4);(5,3);(5,4);(6,3);(6,4)\} \quad (2)$$

For our CST comparison, we choose to keep the index number to n=10, with variable basis exponents N_1 and N_2, (our class function $C_{(x)} = x^{N_1}(1-x)^{N_2}$) as this has shown to give very high precision for the 50 airfoils in [1]. We also add a pitch angle and trailing edge thickness.

b. Complex roots/poles analysis

It is possible to substitute the a_i and b_i parameters in our fit formula with the roots and poles of our formula, plus a proportionality factor.

$$f(x) = \frac{\Sigma_{i=0}^{i=n_1} a_i x^i}{\Sigma_{i=0}^{i=n_2} b_i x^i} = k \frac{\prod_{i=0}^{i=n_1}(x - r_i)}{\prod_{i=0}^{i=n_2}(x - p_i)} \quad (3)$$

where r_i and p_i are the roots and poles of our fit function.

This way, we can plot the locations of the roots/poles on an Argand diagram, in order to map out the search regions for the poles/roots, and optimize the search algorithm by specifying limits for these search regions.

3. RESULTS AND DISCUSSIONS

a. Parameter number optimisation

We have measured the rms error and the maximum error as % of chord, and plotted them in terms of the airfoil analysed [1].

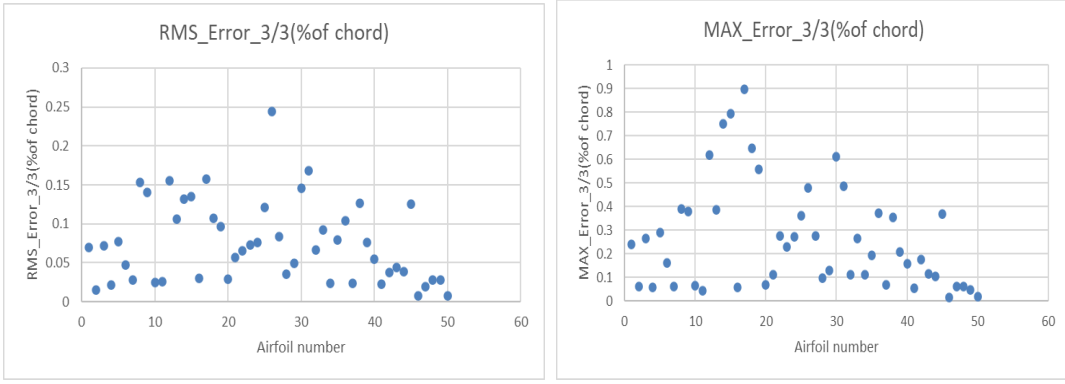


Figure 1: Fit analysis for 3/3 rational fit

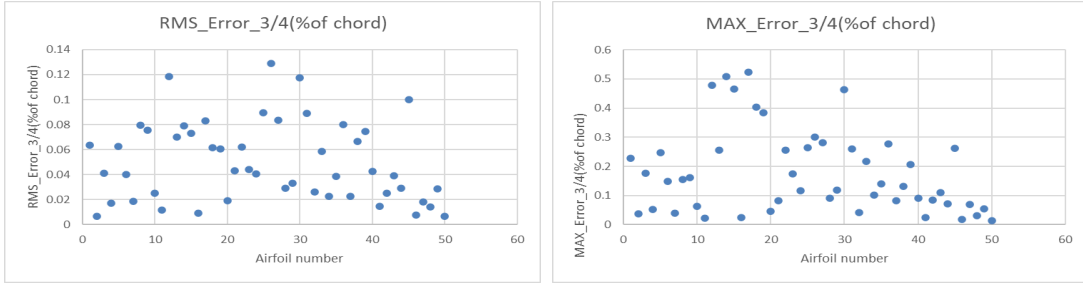


Figure 2: Fit analysis for 3/4 rational fit

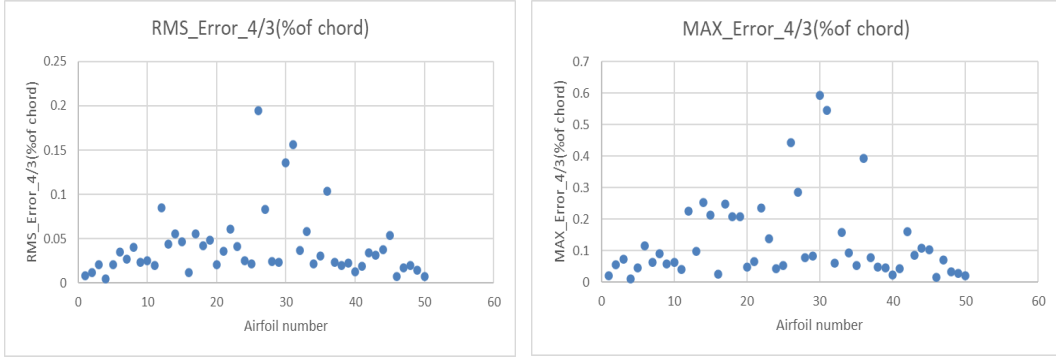


Figure 3: Fit analysis for 4/3rational fit

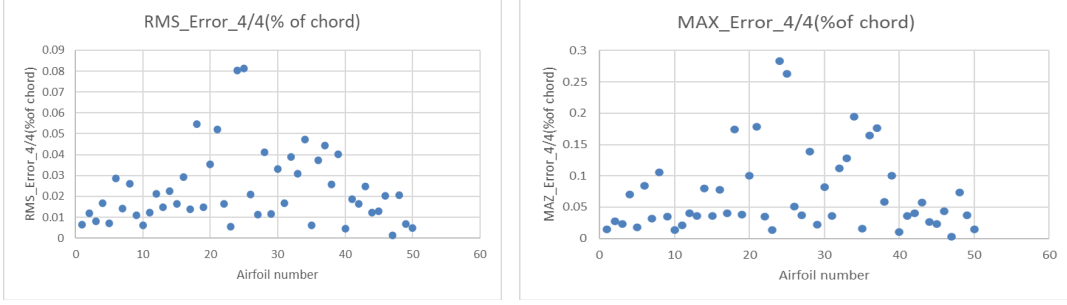


Figure 4: Fit analysis for 4/4 rational fit

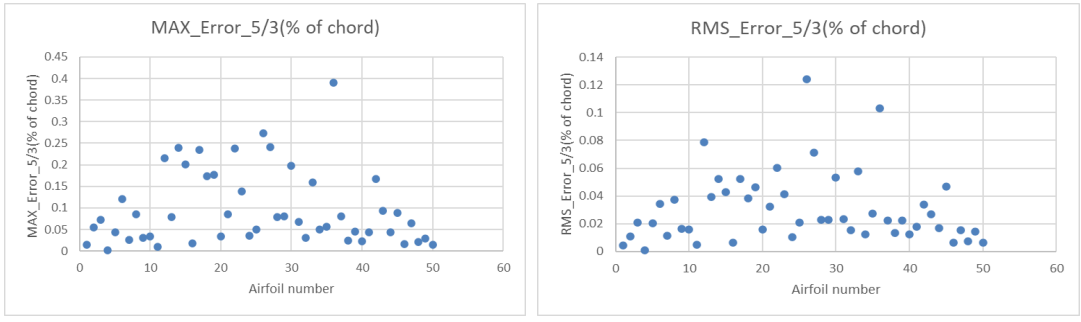


Figure 5: Fit analysis for 5/3 rational fit

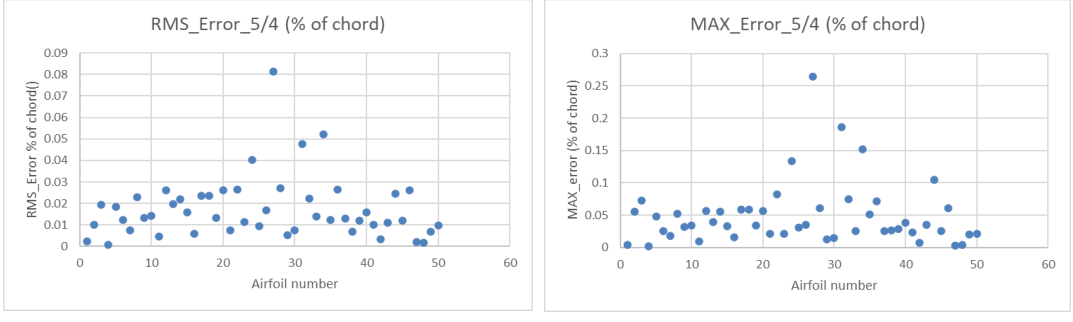


Figure 6: Fit analysis for 5/4 rational fit

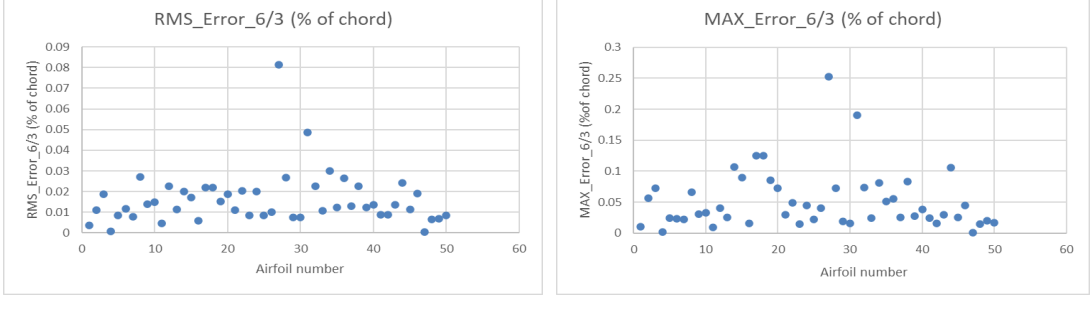


Figure 7: Fit analysis for 6/3 rational fit

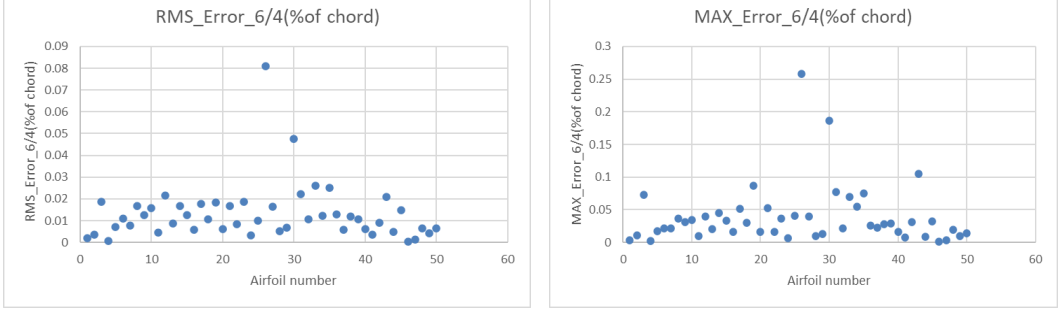


Figure 8: Fit analysis for 6/4 rational fit

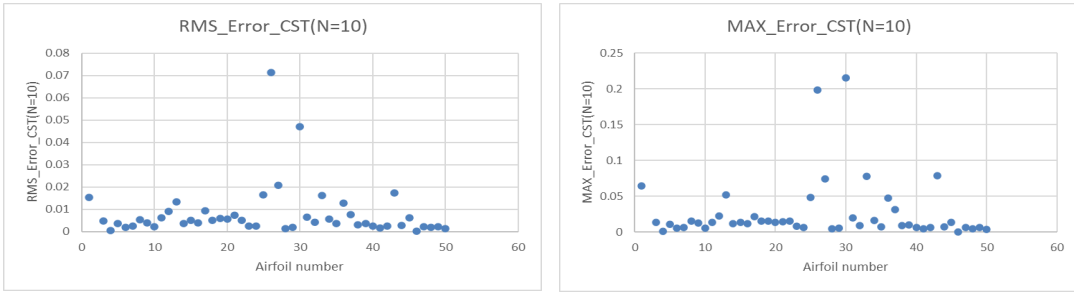


Figure 9: Fit analysis for CST fit with 11 parameters

In order to compare our fits with the CST, we plot the difference in rms and maximum errors as % of chords between the CST error and the different rational fits errors(with negative difference meaning the CST gives a lower error).

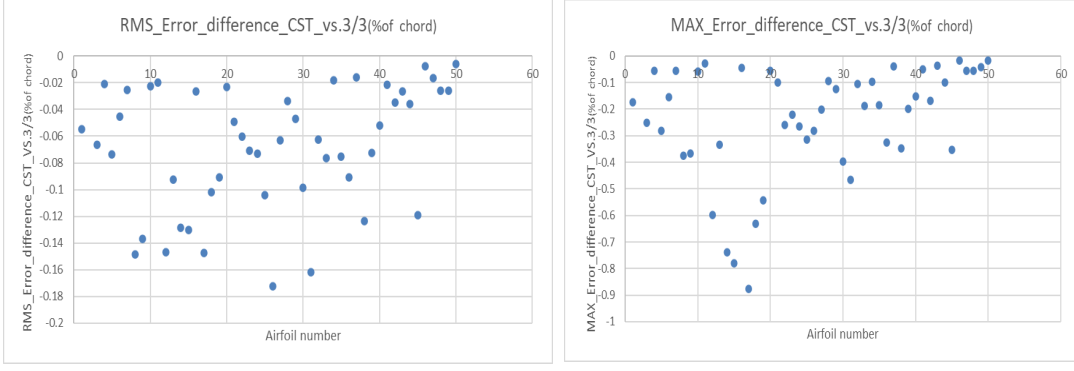


Figure 10: Error difference graph for 3/3, rational fits vs. CST (N=10)

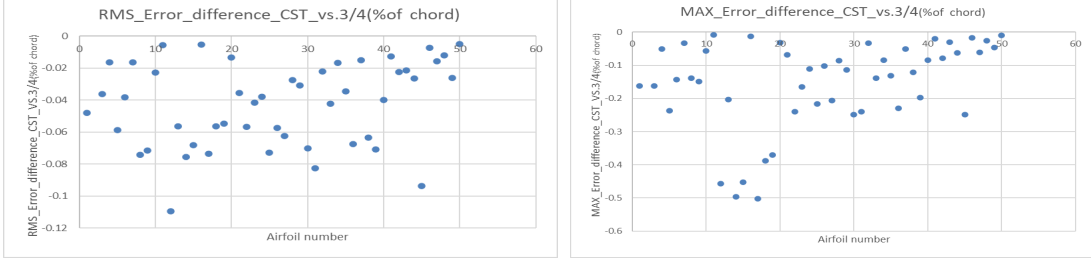


Figure 11: Error difference graph for 3/4, rational fits vs. CST (N=10)

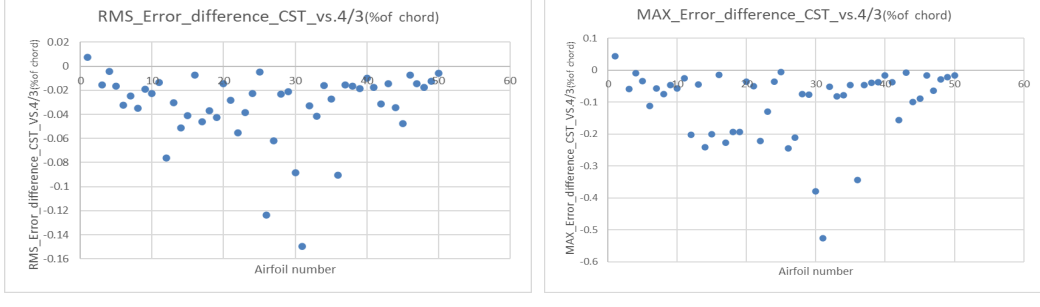


Figure 12: Error difference graph for 4/3, rational fits vs. CST (N=10)

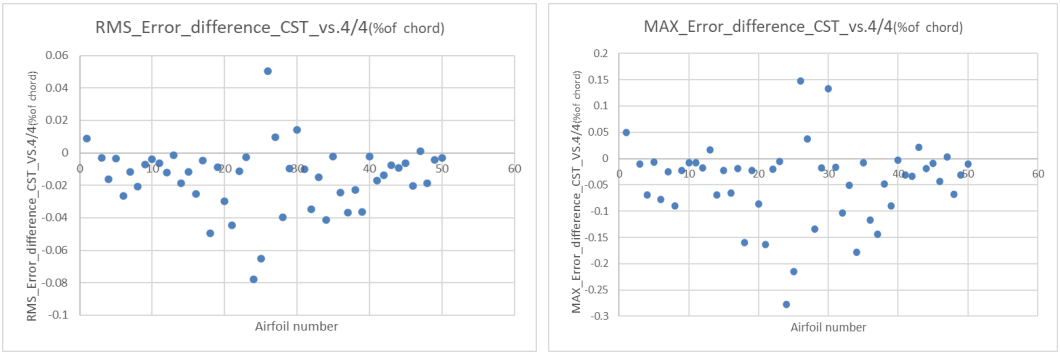


Figure 13: Error difference graph for 4/4, rational fits vs. CST (N=10)

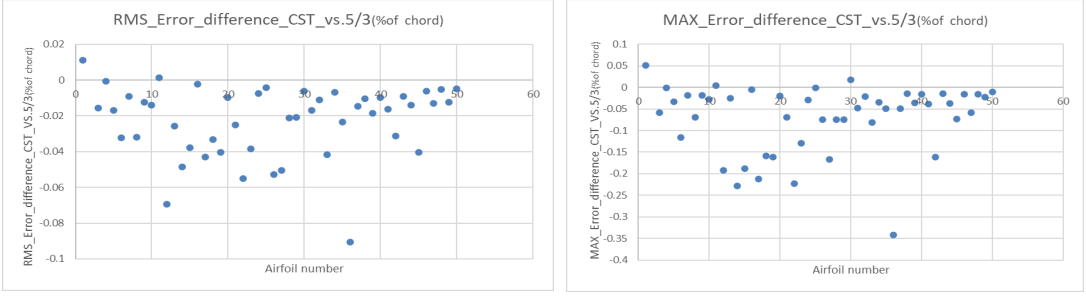


Figure 14: Error difference graph for 5/3, rational fits vs. CST (N=10)

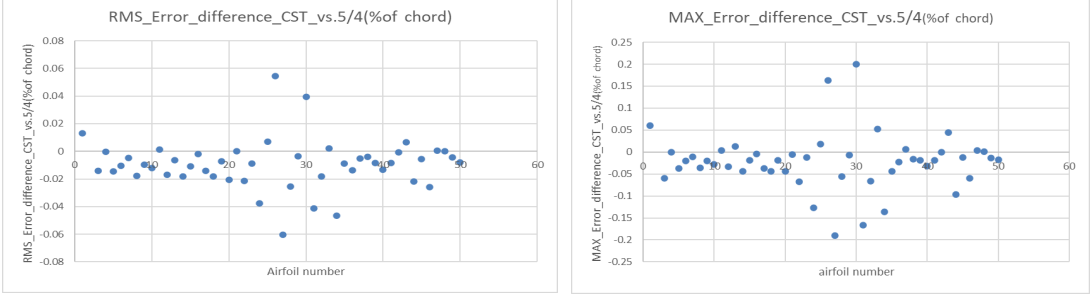


Figure 15: Error difference graph for 5/4, rational fits vs. CST (N=10)

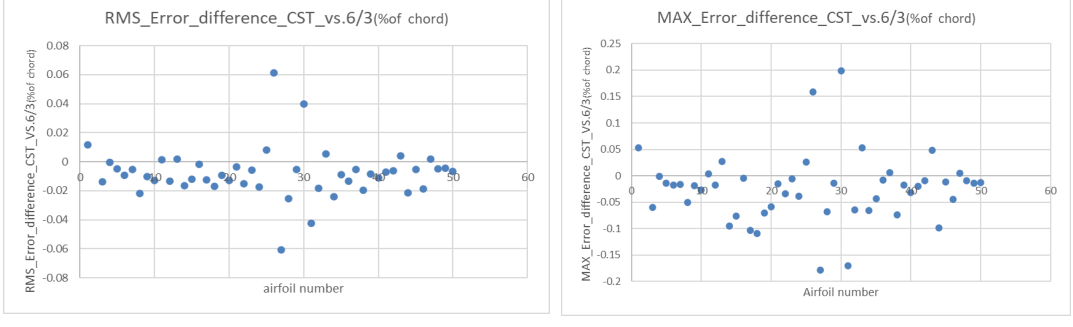


Figure 16: Error difference graph for 6/3, rational fits vs. CST (N=10)

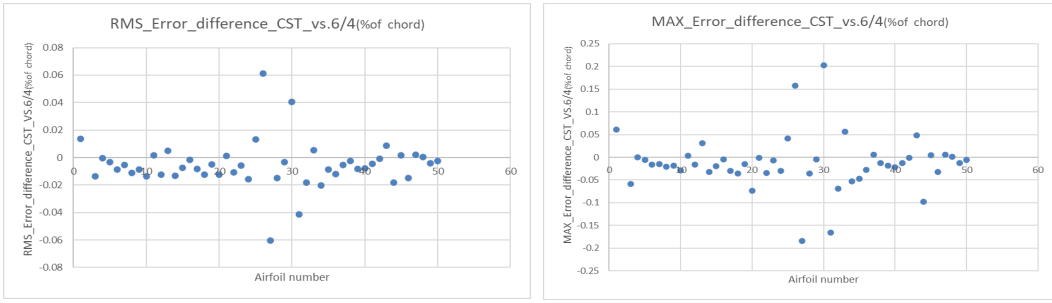


Figure 17: Error difference graph for 6/4 rational fits vs. CST (N=10)

As was expected, the higher the order of the polynomials, the lower the errors are going to be. Interestingly, there is a significant jump in relative performance compared to CST when going from 8 parameters to 9 parameters, with performance improvements diminishing significantly afterwards.

From the last 6 graphs from figure 4, we can see that the differences in both rms error and maximum error are almost symmetrical about the y=-0.01% horizontal axis, with our fit outperforming the CST in a significant amount of cases.

This is to be expected, as we can have used a CST fit with 10 polynomial indexes, while also allowing for a vertical offset and a pitch angle, and variable class function indexes N1,N2. If we were to count all these as parameters, the CST fit uses 15 parameters in total, while our fits only go up to 11 parameters.

If we are to compare our fits with a CST of order 6, so that there are the same number of parameters, we expect our rational fit to outperform it.

b. Complex roots/poles analysis

We shall focus our analysis on the best performing 3 rational fits (5/4, 6/3 and 6/4). Since we have limited the degree for the denominator polynomial to 4, we have an analytical solution for the poles.

We plot the Argand diagrams for the 3 fits below:

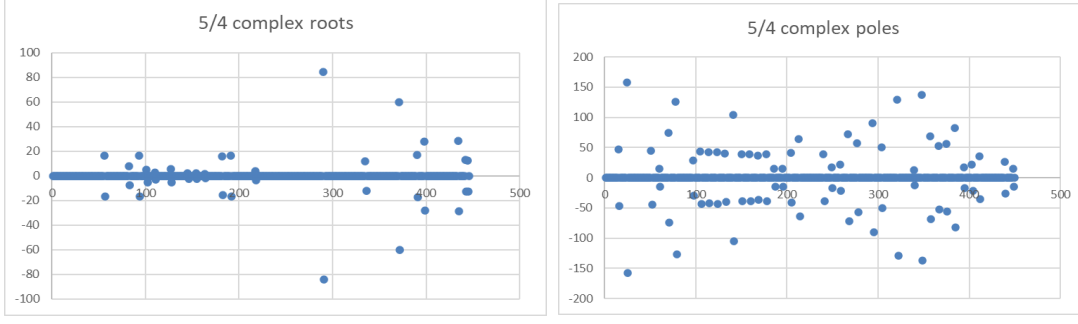


Figure 18: Argand diagrams for 5/4 rational fit roots and poles

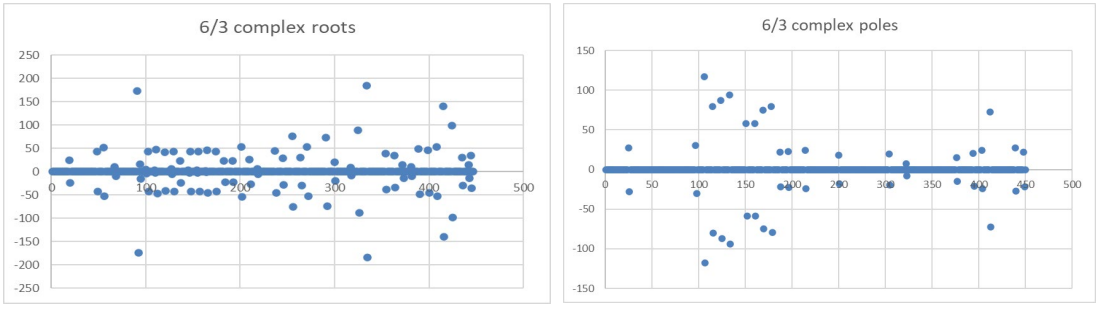


Figure 19: Argand diagrams for 6/3 rational fit roots and poles

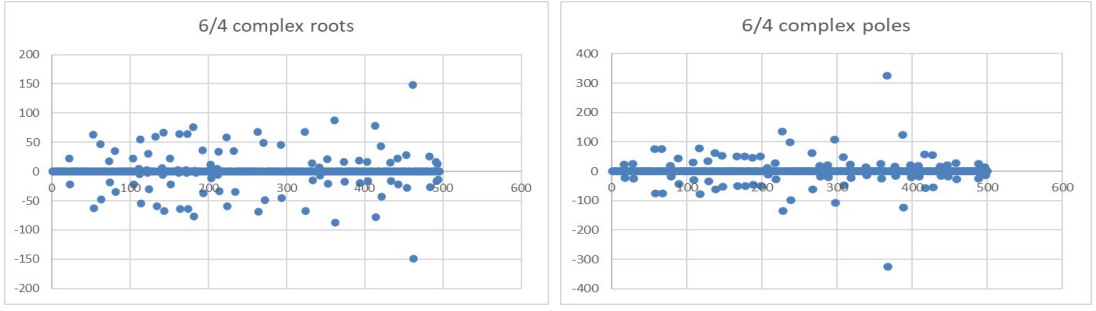


Figure 20: Argand diagrams for 6/4 rational fit roots and poles

As can be seen from the graphs, the vast majority of all roots and poles lie on the real axis. We further make the observation that for all our airfoils, there was at least 1 real root and 2 complex poles from our data.

As such, it is possible to restrict the search area for our parameters in our optimization algorithm by constraining a number of both the roots and poles to be strictly real.

These graphs also provide us with some good limits for our pole location values, which further restricts the search region for our algorithms (for example, no root/pole has a real value greater than 500).

These conditions could be used as a starting point in the optimization algorithm, which could greatly reduce computational cost in optimizing the airfoils.

4. CONCLUSIONS

In conclusion, we have found out that the rational fit performs very similar to CST algorithms, and is capable of highly accurate airfoil representations with less overall parameters for a similar precision, and that the search region for said parameters can be significantly reduced by writing the fit in terms of the complex roots and poles.

REFERENCES

[1] V. I. R. Niculescu, D. Popescu, A. G. Popescu, A versatile analytical representation of the different aircraft wings, *INCAS Bulletin*, Volume **16**, Issue 3/2024, pp. 61-65, https://doi.org/10.13111/2066-8201.2024.16.3.6

[2] * * * http://airfoiltools.com/ .

[3] UNPUBLISHED – Oral Lesson. Granada Seminar on Computational and Statistical Physics 2025 - *A simple analytical representation of aircraft wings: concave - convex case* - Mihnea Butiurca, Valentin Ioan Remus Niculescu, Dumitru Popescu, Stefan Steblea.

[4] N. Hasen and A. Ostermeier, Completely derandomized self-adaptation in evolution strategies, *Evolutionary Computation*, **9**(2):159–195, 2001.
[5] N. Hansen, pycma – A Python implementation of CMA-ES, https://github.com/CMA-ES/pycma.
[6] B. M. Kulfan, *Universal Parametric Geometry Representation Method*, AIAA Paper 2006-6948, 2006.