Mathematical Modelling and Structural Stability in Aerospace

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Abstract: *In the present paper, the meaning of the structural stability is presented and examples are given of airplanes, UAVs and space vehicles models whose theoretical model is not structurally stable.*

Key Words: *Mathematical Modelling, Structural Stability, Aerospace*

1. INTRODUCTION

According to [1] there is no set rules, and an understanding of the "right" way to model real word phenomena. One learns it by practising. The model can be reached by familiarizing oneself with a variety of examples. A model is a mental representation of a process.

Usually, a mathematical model is a mental construction and takes the form of a set of equations describing a number of variables. We distinguish between continuous models, in which the variables vary continuously in space and time and discrete models whose variables varies discontinuously.

Applied mathematicians have a procedure, almost a philosophy, that they apply when building models, for a process of interest that they want to describe or, more importantly, explain. Observations of the process lead, sometimes after a great deal of effort, to a hypothetical mechanism that can explain the phenomenon.

The purpose of a model is then to formulate a description of the mechanism in quantitative terms. The analysis of the resulting model leads to results that can be tested against observations. Ideally, the model also leads to predictions, which if verified, lend authenticity to the model.

It is important to realize that all models are idealizations and limited in their applicability. In fact, one usually aims to simplify. The idea is that if a model is correct, then it can be subsequently complicated, but the analysis its is facilitated by by the fact that a more simplified version has been treated first.

Simplifications appear in the case of the differential systems that describe the movement of existing airplanes, UAVs and space vehicles, and even during the development period of new prototypes [2], [3], [4], [5], and [6]. Because of the undesirable consequences due to simplifications, as well as the cost of new prototypes, it is useful to have a theoretical tool that establishes the necessary condition that has to be satisfied by the simplified system of differential equation.

The tools could be the structural stability in S. Smale sense [7], [8] and bifurcation in sense of [9].

This paper presents structural stability and bifurcation and gives examples of airplane, UAV and spacecraft models whose theoretical model is not structurally stable or does not exhibit bifurcation.

2. STRUCTURAL STABILITY

According to [9], [10] the continuous-time dynamical system

$$\dot{x} = f(x, \alpha) \quad x \in U \subset R^n \ \alpha \in V \subset R^m \tag{1}$$

is topologically equivalent in $U \subset R^n$ to the dynamical system

$$\dot{y} = f(y, \beta) \quad y \in U \subset R^n \ \beta \in V' \subset R^m \tag{2}$$

if there is

-a homeomorphism of the parameter space $p: V \to V'$;

-a parameter-dependent homeomorphism of the phase-space $h_\alpha: U \to U$

such that for all $\alpha \in V,\ h_\alpha$ maps orbits of the first system onto orbits of the second system preserving the direction of time, i.e.

$$h_\alpha[x(t; \alpha, x^0)] = y[t; p(\alpha), h_\alpha(x^0)] \quad \text{for any } x^0 \in U \text{ and any } t \tag{3}$$

where $x(t; \alpha, x^0)$ is the solution of system (1) corresponding to the parameter α and to the initial condition x^0 and $y[t; p(\alpha), h_\alpha(x^0)]$ is the solution of system (2) corresponding to the parameter $p(\alpha)$ and initial condition $h_\alpha(x^0)$.

Let be a system $\dot{x} = f(x)$ defined in a region $U \subset R^n$ by the C^1 vector field $f: U \to R^n$. Consider a region $U_0 \subset U$ and assume that the scalar product $< f(x), x >$ is strictly negative for each $x \in \partial U_0 \subset U$.

According to [8] the system $\dot{x} = f(x)$ is structurally stable in $U_0 \subset U$ if there exist a neighborhood W of vector field f such that for any C^1 vector field $g: U \to R^n \quad g \in W$ the system $\dot{y} = g(y)$ is topologically equivalent in U_0 to the system $\dot{x} = f(x)$.

In [8] pg. 312-318 theorems and examples concerning structural stability and structural instability are given.

Remark that if the system $\dot{x} = f(x)$ is structurally stable in $U_0 \subset U$ and $g \in W$ then there exists a bijection between the steady states (equilibriums) of the system $\dot{x} = f(x)$ and equilibriums of the system $\dot{y} = g(y)$ located in U_0.

3. THE SYSTEM DESCRIBING THE DECUPLED LONGITUDINAL FLIGHT IN CASE OF ALFLEX SPACE SHUTTLE IS NOT STRUCTURALLY STABLE

Automatic-Landing Flight-Experiment (ALFLEX) is a model plane, developed by NASDAQ, Japan. This vehicle is a reduced-scale model of the H-II Orbiting Plane, an unmanned reusable orbiting spacecraft.

It has been built to study the flight of the spacecraft during its final approach and landing phases.

This flight is made possible due to complicated automatic-flight control systems, designed to perform quick responses to commands.

As the mass of this vehicle is concentrated in its fuselage, the phenomenon of inertial coupling may occur, i.e., a gyroscopic effect, causing small perturbations or small changes of the control surface angles that may lead to dramatic changes in roll rate

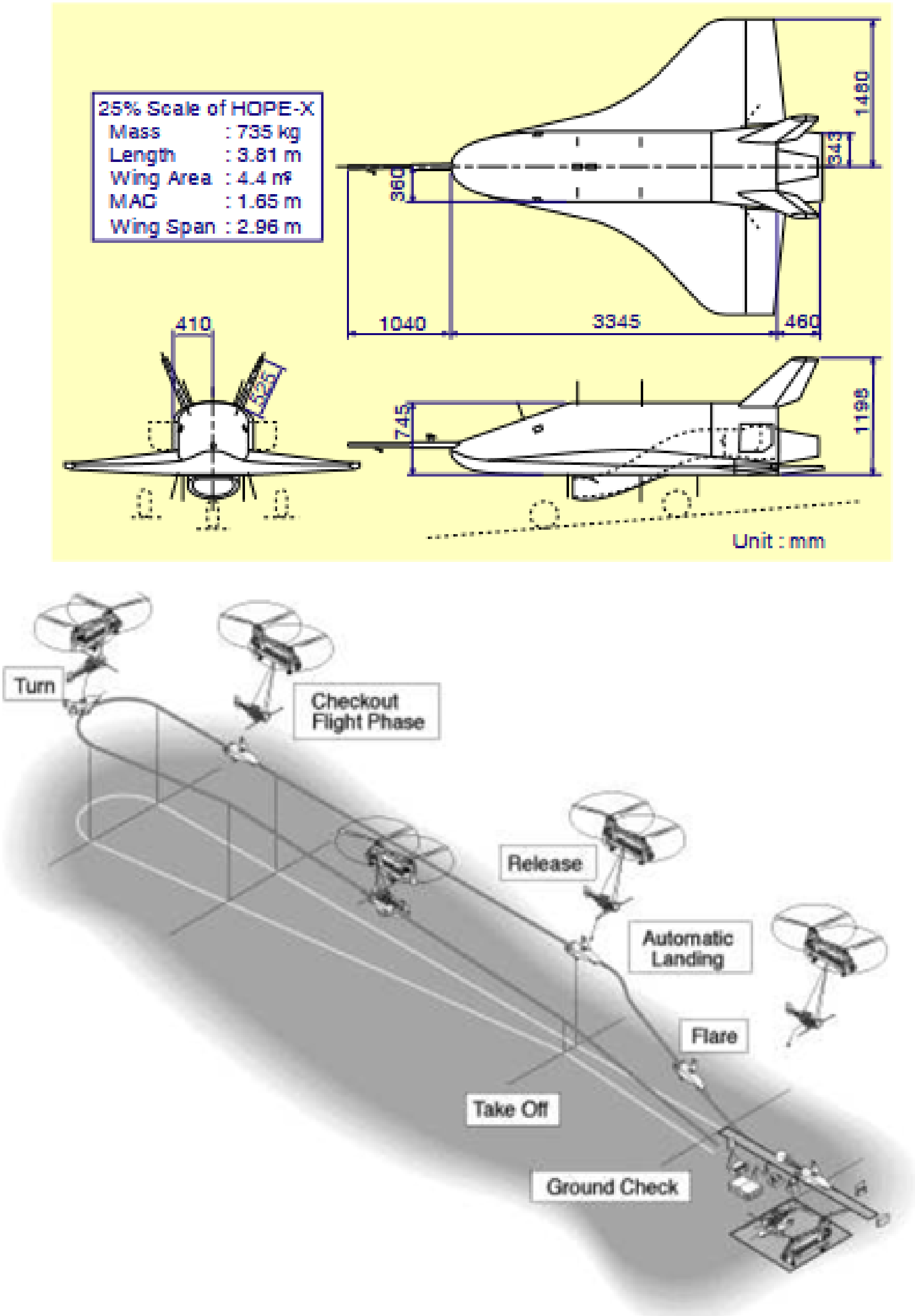


Fig. 1 ALFLEX (Figure 5.1 Landing experiment [1])

The general system of differential equations which describes the motion around the center of gravity of a rigid aircraft, with respect to an xyz body–axis system, where xz is the plane of symmetry, according to [2], [3] is:

$$\begin{gathered}\frac{\dot{V}}{V}\times cos\alpha\times cos\beta-\dot{\beta}\times cos\alpha\times sin\beta-\dot{\alpha}\times sin\alpha\times cos\beta\\=r\times sin\beta-q\times sin\alpha\times cos\beta+\frac{X}{m\times V}\\\frac{\dot{V}}{V}\times sin\beta+\dot{\beta}\times cos\beta=p\times sin\alpha\times cos\beta-r\times cos\alpha\times cos\beta+\frac{Y}{m\times V}\\\frac{\dot{V}}{V}\times sin\alpha\times cos\beta-\dot{\beta}\times sin\alpha\times sin\beta+\dot{\alpha}\times cos\alpha\times cos\beta\\=-p\times sin\beta+q\times cos\alpha\times cos\beta+\frac{Z}{m\times V}\\I_x\times\dot{p}-I_{xz}\times\dot{r}=\left(I_y-I_z\right)\times q\times r+I_{xz}\times p\times q+L\\I_y\times\dot{q}=(I_z-I_x)\times p\times r-I_{xz}\times(p^2-r^2)+M\\I_z\times\dot{r}-I_{xz}\times\dot{p}=\left(I_x-I_y\right)\times p\times q-I_{xz}\times q\times r+N\\\dot{\Phi}=p+q\times sin\phi\times tan\theta+r\times cos\phi\times tan\theta\\\dot{\theta}=q\times cos\phi-r\times sin\phi\end{gathered}\tag{4}$$

State parameters of this system are: forward velocity V, angle of attack α, sideslip angle β, roll rate p, pitch rate q, yaw rate r, Euler roll angle ϕ, and Euler pitch angle θ. The constants I_x, I_y, I_z are moments of inertia about the x-, y-, and z-axis, respectively; I_{xz} product of inertia, g gravitational acceleration; and m mass of the vehicle.

The external forces and moments X, Y, Z, L, M, N are, in general, functions of the state parameters and the control parameters: δ_a aileron angle; δ_e elevator angle; and δ_r rudder angle (the body flap and the speed break are available as additional controls but, for simplicity, they are set to 0 in the analysis to follow).

In particular, the following expressions for the external forces and moments are considered for ALFLEX.

$$\begin{gathered}X=-m\times g\times(sin\theta-sin\theta_0)+k\times V^2\times[C_{x\alpha}(\alpha-\alpha_0)+C_{x\delta_e}\times(\delta_e-\delta_{e0})]\\Y=m\times g\times sin\phi\times cos\theta+k\times V^2(C_{y\beta}\times\beta+C_{yr}\times r+C_{y\delta_r}\times\delta_r)\\Z=\ m\times g(cos\phi\times\ cos\theta-cos\theta_0)+k\times V^2\times[C_{z\alpha}(\alpha-\alpha_0)+C_{z\delta_e}\times(\delta_e-\delta_{e0})]\\L=b\times k\times V^2\times(C_{l\beta}\times\beta+C_{lp}\times p+C_{lr}\times r+C_{l\delta_a}\times\delta_a+C_{l\delta_r}\times\delta_r)\\M=c\times k\times V^2\times[C_{m\alpha}\times(\alpha-\alpha_0)+C_{mq}\times q+C_{m\delta_e}\times(\delta_e-\delta_0)]\\N=b\times k\times V^2\times(C_{n\beta}\times\beta+C_{np}\times p+C_{nr}\times r+C_{n\delta_a}\times\delta_a+C_{n\delta_r}\times\delta_r)\end{gathered}\tag{5}$$

A simplified version of the motion in case of the ALFLEX reentry vehicle has been presented in Goto and Matsumoto [11] and Goto and Kawakita [12].

This version was obtained from the general system presented in [2], [3], assuming that the forward velocity V is constant $V=V_0$ and α the angles of attack and sideslip β, respectively, are small.

Due to these assumptions, the first three equations of general system were simplified, so that the following system was obtained:

$$\begin{gathered}
\dot{V}=0 \\
\dot{\beta} = p \times sin\alpha - r \times cos\alpha + \frac{Y}{m \times V} \\
\dot{\alpha} = -p \times \beta + q + \frac{Z}{m \times V} \\
I_x \times \dot{p} - I_{xz} \times \dot{r} = \left(I_y - I_z\right) \times q \times r + I_{xz} \times p \times q + L \\
I_y \times \dot{q} = (I_z - I_x) \times p \times r - I_{xz} \times (p^2 - r^2) + M \\
I_z \times \dot{r} - I_{xz} \times \dot{p} = \left(I_x - I_y\right) \times p \times q - I_{xz} \times q \times r + N \\
\dot{\Phi} = p + q \times sin\phi \times tan\theta + r \times cos\phi \times tan\theta \\
\dot{\theta} = q \times cos\phi - r \times sin\phi
\end{gathered} \tag{6}$$

This system of differential equations has been used to determine the set of steady states corresponding to ALFLEX, to undertake a stability analysis along the existing paths of steady states by Goto and Matsumoto in [11] (2000); and Goto and Kawakita in [5] (2004).

According to Goto [4] (2004), the simplified system (6) can be useful for getting an idea about the behavior of the system, although quantitatively, the general system (4) should be taken into consideration. In the same paper, the author remarks that if the steady states of the simplified system (6) are used as an initial guess in the continuation method applied to determine the steady states of the general system (4), the results are not always satisfying, and the continuation method does not always converge.

If the general system (4) and system (6) are to accurately reflect reality, they must bear resemblance on some level. For example, one might hope that the behavior of the dynamical systems (4) and that defined by (6) is qualitatively the same, i.e., they are topological equivalent.

In [7] it is proven that the dynamical systems defined by (4) and (6) are not topologically equivalent. Furthermore, System (6) is not structurally stable.

Consequently, these systems offer quite different images about the real motion around the center of gravity of a rigid aircraft.

For example, for a certain combination of control angles, the Simplified System (6) has a steady state, while the General System (4) has no steady states.

Hence, it is not surprising that using the steady state of Eq. (6) as an initial guess in the continuation method applied in order to find the steady state of Eq. (4), the method does not converge (the limit does not exist). Even if the method converges, the limit cannot be a steady state of Eq. (4).

The longitudinally flight system decoupled from (4) is:

$$\begin{gathered}
V^{\cdot} = g \times [sin\ (\alpha - \theta) - sin\ (\alpha - \theta_0\ )\ ] + V\text{^}2 \times [(A_1 \times sin\Delta\alpha \\
+ A_2 \times cos\Delta\alpha) \times \Delta\alpha + (B_1 \times sin\Delta\alpha + B_2 \times cos\Delta\alpha) \times \Delta\delta_e \\
\dot{\alpha} = q + \frac{g}{V} \times [\cos(\alpha - \theta) - \cos(\alpha - \theta_0)] + V \times [(A_1 \times cos\Delta\alpha - A_2 \times sin\Delta\alpha) \times \Delta\alpha \\
+ (B_1 \times cos\Delta\alpha - B_2 \times sin\Delta\alpha) \times \Delta\delta_e \\
\dot{q} = \frac{c \times k}{I_y} \times V^2 \times (C_{m\alpha} \times \Delta\alpha + C_{mq} \times q + C_{m\delta_e} \times \delta_e) \\
\dot{\theta}=q
\end{gathered} \tag{7}$$

The longitudinally flight system decoupled from (6) is:

$$\dot{V} = 0$$
$$\dot{\alpha} = q + \frac{g}{V}(\cos\theta - cos\theta_0)$$
$$\dot{q} = \frac{c \times k}{I_y} \times V^2 \times (C_{m\alpha} \times \Delta\alpha + C_{mq} \times q + C_{m\delta_e} \times \delta_e) \qquad (8)$$
$$\dot{\theta}=q$$

In [7] it is proven that the dynamical systems defined by (7) and (8) are not topologically equivalent. Furthermore, System (8) is not structurally stable in the bounded region $X_0 = (V_{min}, V_{max}) \times (-\pi, \pi) \times (q_{min}, q_{max}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ with $0 < V_{min} \leq 1 < V_{max},\ q_{min} < 0 < q_{max}$.

Consequently, systems (7) and (8) offer quite different images about the real longitudinal flight of a rigid aircraft.

The main message of our findings is that in case of the ALFLEX reentry vehicle the simplified flight system and the simplified decupled longitudinal flight system are oversimplified. They are not appropriate to be used in the mathematical description of the real flight.

4. CRASHES OF HIGH-PERFORMANCE FIGHTER AIRPLANE SUCH AS YF-22A AND B-2, DUE TO OSCILLATIONS

Fig. 2 YF-22 A fighter airplane

Fig. 3 B2 fighter airplane

Interest in oscillation susceptibility of aircrafts has been generated by the crashes of high-performance fighter airplanes such as YF-22A and B-2, due to oscillations that were not predicted during the aircraft development process [12]. Flight quality criteria for oscillation prediction are based on linear analysis and quasi-linear extensions [13]. However, these criteria cannot, in general, predict the presence or the absence of oscillations, because of the large variety of non-linear phenomena that have been identified as factors contributing to oscillations and which are neglected in the linear approach. Sources of these factors include pilot behavioral transitions, actuator rate limiting [14–15] and changes in aircraft dynamics caused by transitions in operating conditions [17], gain scheduling and mode switching [18]. The analysis of nonlinear oscillations involves the computation of non-linear phenomena including Hopf bifurcation that led sometimes to large changes in the stability of the pilot-vehicle-system [19]. More recently, theoretical bifurcation studies have been undertaken for longitudinal flight dynamics, using the elevator deflection and mass of the vehicle as bifurcation parameters [20 - 22]. The occurrence of saddle-node and Hopf bifurcations has been pointed out in the case of the F-8 aircraft, and it has been emphasized that these bifurcations may result in jump behavior and pitch oscillations of flight dynamics. Moreover, system controllability with respect to the variation of the elevator deflection angle has been discussed in [10, 11]. However, these bifurcation studies can only explain locally the appearance of oscillatory behavior (associated with supercritical Hopf bifurcations), and they do not represent a tool for understanding the global nature of longitudinal flight dynamics. More precisely, a supercritical Hopf bifurcation that occurs at the critical value δ_e^* of the elevator deflection, can only explain the appearance of asymptotically stable limit cycles for values of δ_e close to the critical value δ_e^*, i.e. for δ_e in a neighborhood of the form $(\delta_e^* -$

$\varepsilon, \delta_e^*)$ or $(\delta_e^*, \delta_e^* + \varepsilon)$. Nevertheless, Hopf bifurcations are not the only type of bifurcation phenomena leading to oscillatory behavior. In [19, 24], it has been shown that in a longitudinal flight with constant forward velocity, equilibria exist for the ADMIRE aircraft and the ALFLEX reentry vehicle only if the elevator deflection δ_e belongs to a closed and bounded interval J. When the elevator deflection is at the boundary of the interval J, a countable infinity of saddle-node bifurcation points is present. When the elevator deflection exceeds these critical values and is outside the interval J, numerical simulations show that the angle of attack and pitch rate oscillate with the same period, while the pitch angle increases or decreases infinitely. Hence, the orbit of the system is spiraling.

In [25] the existence of oscillatory solutions of the simplified dynamical system which governs the motion around the center of gravity in a longitudinal flight with constant forward velocity of a rigid aircraft, when the automatic flight control system is decoupled and the elevator deflection exceeds the bifurcation values. Sufficient conditions are obtained for the existence of oscillatory solutions for any value of the elevator deflection outside the interval which corresponds to the existence of equilibria.

Oscillatory longitudinal flight of the ALFLEX reentry vehicle [25].

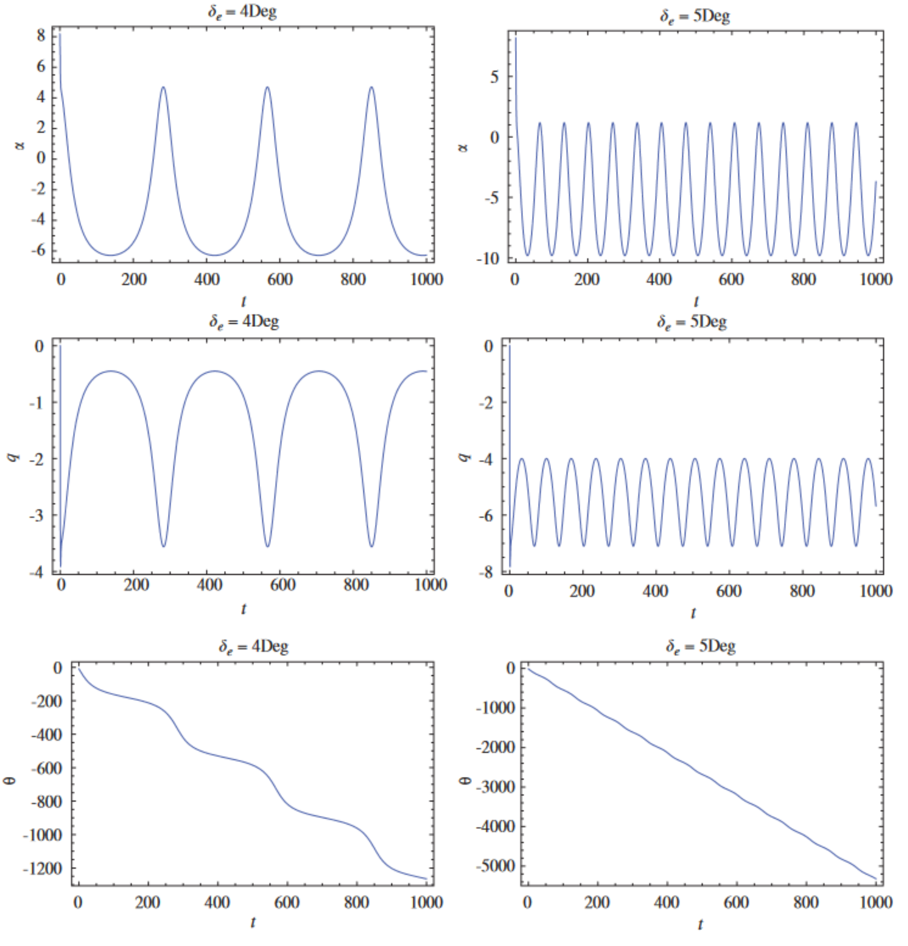
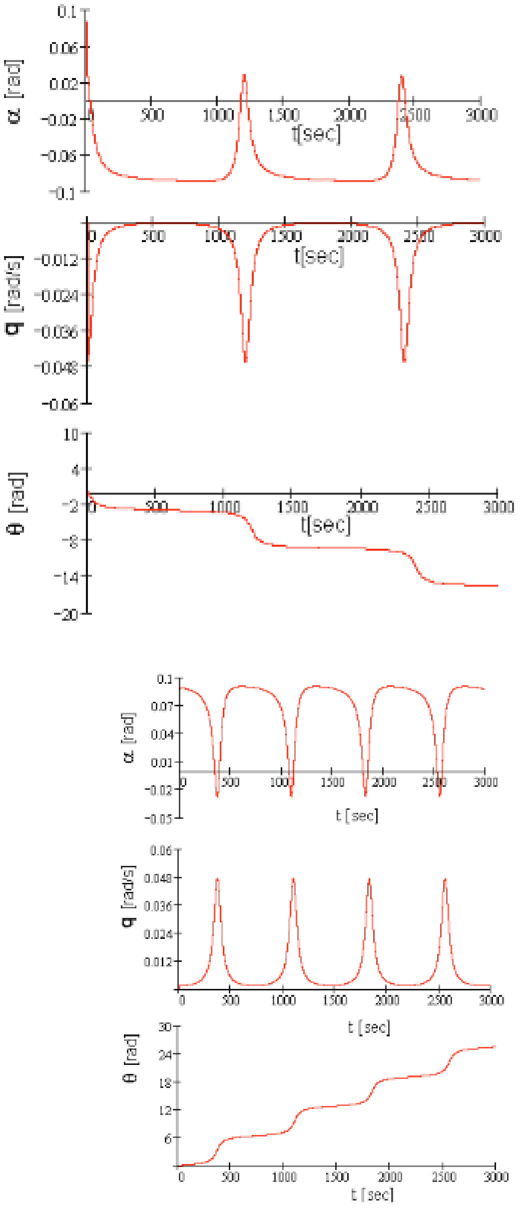


Fig. 4 Oscillatory longitudinal flight of ALFLEX

Evolution of the state parameters (α, q, θ) considering the initial condition $(8.18, 0, -9.16)$.
Oscillatory longitudinal flight of the ADMIER unmanned aircraft [26].



$\delta_e = 0.048\, rad$ $\delta_e = -0.05\, rad$

Fig. 5 Oscillatory longitudinal flight of ADMIRE

Evolution of the state parameters (α, q, θ) considering the initial condition $(0.08869, 0, 0.159329) rad$.

5. CONCLUSION

Mathematical tools such as structural stability and bifurcations can be used to analyze the system of differential equations governing aircraft dynamics, in terms of the fit between the calculated description of the motion and the actual motion.

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