The Physical vs. Mathematical Problem of Navier-Stokes Equations (NSE)

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MOTTO: "Where the turbulence of water is generated, where the turbulence of water maintains for long, where the turbulence of water comes to rest." Leonardo da Vinci 1452-1519

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Abstract: *The Navier-Stokes equations describing the motion of viscous/real fluids in* R^n *(*$n = 2$ *or 3) depend on a positive coefficient (the viscosity,* ν*) via the Reynolds number. The key of NSE problem is the Reynolds number, mathematically considered a simple small perturbation parameter without any physical explanation, or a vague physical Newtonian ratio of inertial to viscous forces,* $Re = \frac{UL}{\nu}$*, in spite of its quantic physical meaning as the initial excitation to response ratio, at the beginning of motion (IC at t = 0). The paper deals with the thixotropic property of real viscosity which softens (*$\nu \downarrow$*) when strained (*$Re \uparrow$*), but it doesn't tend to zero (*$\nu \to 0$*) as much as the Reynolds number increases, holding a finite value, corresponding to the new thermodynamic equilibrium state. The (*$\nu \to 0$ *for* $Re \to \infty$*) false physical condition renders the NSE problem to a unique solution less one beyond a critical Reynolds number,* Re_{cr}*. The understanding of the wall-bounded viscous flows, at both small-scales (slow motion, small Re) and larger scale (turbulent motion, large Re) must be in conjunction with the more-subtle torsional buckling effect of the "wall" lag concept that the wall has on the inherent fluid dynamics during the starting phase. The limitations of the diathermal wall associated with the starting accelerations at the onset of motion, of the order of* $a_{cr}/g \geq 2/3$*, create the physical conditions (thermomolecular changes) for the loss of the mathematical uniqueness of the NSE solutions. The physical limitations in conjunction with the validity area of NSE model are considered in the sequel. Because of the nonlinearity of the PDE differential equations, the variation of geometrical and physical properties can lead to bifurcations in the solution and thus, to multiple solutions. Considerations relative to laminar-turbulent transition as the main bifurcation source for the more complex structure of a solution, engendered by molecular structure changes of a flowing fluid in more or less contact* *with the walls, are given and illustrated for the canonical flows on flat plates and viscous decay of a starting/contact vortex ("vortex eye").*

Key Words: *Navier-Stokes equations, laminar-turbulent transition, solitary waves (solitons)*

1. THE NAVIER-STOKES EQUATIONS

The Navier-Stokes equations describing the motion of a real fluid are physically valid providing a number of physical restrictions are met: the fluid is Newtonian, continuously isotropic, the stress tensor is symmetric and the Stokes hypothesis holds, (i.e. the bulk viscosity vanishes: $3\lambda + 2\mu = 0$), along with neglecting heat sources (radiation and chemical reactions). Additionally, the boundary and initial conditions (BC and IC) are used with the equations of motion: no-slip condition (zero tangential velocity) and zero mass and heat transfer processes for the pure NES problem (real fluid with $\nu \neq 0$) and a slip condition for Euler's equations using the same equations with $\nu = 0$ (ideal fluid). The standard NES problem [1] is not physically satisfied for the full range of the Reynolds number, Re = 10 – 10^{10} ($Re_{max} \equiv c_0 = 10^{10}$ m/s – the light wave speed, the natural quantic scale [2]), for the canonical/simplest wall flows on smooth plates at zero incidence (or zero pressure gradient) and smooth flows on pipes (Hagen and Poiseuille flows) [4], even more so for the variation of geometrical and/or fluid mechanical parameters of the fluids, frequent sources of the multiple solutions. For the canonical wall flows exist asymptotic NSE-like solutions and numerical results corresponding to different initial excitation or starting impact Re_x (t = 0) or IC, and its late response, $C_{f,D}$ (Rex) (friction/drag coefficient):

- Stokes (Re < 1, Cd = 24/Re) and Oseen (Re < 10, CD = 24/Re(1 + 3/16Re) theories;
- laminar asymptotic Blasius theory ($10 \leq Re \leq 10^5$, $C_{fxlocal} = \frac{0.64}{\sqrt{Re_x}}$, $C_{Dglobal} = \frac{1.38}{\sqrt{Re}}$);
- turbulent asymptotic Prandtl-Schlichting theories ($10^5 < Re \leq 10^9$, $C_{fxlocal} = 2\left[\frac{k}{lnRe_x} \cdot G(lnRe_x)\right]^2$, $C_{Dglobal} = \left[\frac{k}{lnRe} \cdot G(lnRe)\right]^2$, where $k = 0.41$ - the Karman constant and $G(lnRe)$ - the empirical gauge function [4];
- numerical solution of Navier-Stokes equations [5], ($10 \leq Re \leq 10^5$).

Evidently, the theoretical and numerical NES results above show that the standard NSE solutions along with the mentioned physical limiting conditions are unique only for the Reynolds number range of $Re_{cr} = 10^5$ for air and $Re_{cr} = 10^6$ for water; afterward, the disembodying/ destruction process of the fluid structure, better known as the wavy laminar – turbulent transition process, begins being analyzed in detail in the following section for the technically important fluids, air and water.

The uniqueness of NSE solutions only for laminar like flows, Re = 10 - 10^5, has two causes: a) the mathematical one, due to Dirac-like initial condition [1]

$$U(x, 0) = U^0(x) \ \ (x \in \mathbb{R}^n), \tag{1}$$

where $U^0(x)$ is given on C^∞ divergent-free vector field, $\mathbb{R}^n$ and b) the second restrictive condition of physical nature, given by the Stokes hypothesis of Newtonian/ linear fluids (i.e. the bulk viscosity vanishes)

$$3\lambda + 2\mu = 0, \text{or } \lambda = -2/3\mu\,, \tag{2}$$

where to secure mathematical isotropy of equations no relaxation process may occur, i.e. the radiation and chemical reactions must be neglected, counter-physical nature of thermomolecular structure of fluids following near a “solid surface”-like impact point, i.e. the “wall” concept.

Both mathematical and physical limitations, Eqs. 1, 2 render the NSE to a nonlinear differential operator of a Korteweg-de Vries (KdV) equation type, retaining their identity, carrying constant mass at continuum scales ($Re_{cr} \cong 10^{-6}$) as the momentum and kinetic energy flux/flow. At $t = 0$, there are two singular physical-mathematical problems in conjunction with the onset of a motion: the flow may evolve either smoothly (i.e. laminar flow or shearing mode) or, fast/impulsively (i.e. transitional flow or pumping mode) at the critical Reynolds number. At $Re > Re_{cr}$, the Reynolds averaged Navier-Stokes (RANS) equations constitute a NES hybrid model for mass dispersion at molecular scales/ nanoparticles ($\leq Re_{cr}^{-1}$), by means of the momentum and kinetic energy equations of transitional relativistic flows. Mathematically, the NSE (laminar flow or non-relativistic regime) model and the RANS (turbulent flow or relativistic regime) model, both derived from the fundamental conservation principles, are differentiated by the starting impact (IC), the former of a Gaussian-like collision and the latter of a chi-squared-like collision (ballistic shock).

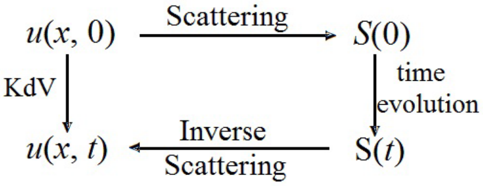
2. UNSTEADY STARTING FLOWS

At any starting impact, there are exact NSE solutions followed by exact solutions of the corresponding steady flows, but the early unsteady short stage/regime is frequently ignored and embodied in the dissipative or dispersive nonlinear terms ($\nu\Delta U_i$) and the motion equations get Burgers-like equations, where soliton characteristic of starting flows is hidden and/or lost to the detriment of mathematical formalism/model and its solving method (theoretical and numerical) (see the Stokes hypothesis). The Gaussian collision engenders the hydrodynamic flows, without relativistic effects ($Re < Re_{cr}$, constant mass and gravity), approachable by the unique NSE solutions, while for the ballistic collisions, the wavy flows with two or more solitons (crests) (water hammer, turbulent relativistic regimes) are produced, rather suitable to approach by means of Korteweg-de Vries equations.

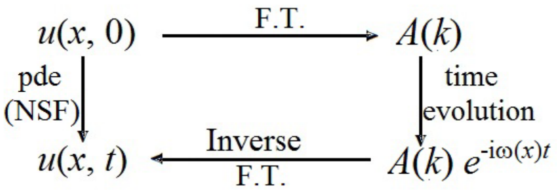
The Newtonian fluids (both air and water) in supercritical flows ($Re \geq Re_{cr}$) show relativistic effects by retaining the identity of two-solitons upon starting collision (by their mutual interaction). The two-soliton interaction in the laminar-turbulent transition flows of Newtonian fluids, is a complicated mechanism involving thermomolecular mutations with metastable equilibrium.

If the dissipative subcritical flows are immediately recognized as quiet smooth flows (“laminar” regimes), their opposite “turbulent” regimes (irregular space-time variations of velocity, pressure and temperature along with mass dispersion) are hard to define, their two-soliton mutual mechanism being unknown up to now. This bifocal interaction functions like the solar system [2].

The mathematical problem of NSE, modeling real fluid flows is a KdV-like initial-value problem where a post-starting impact/ collision wavy flow can be represented as the inverse scattering transform for the KdV equation-like NSE as,



and time evolution by means of the Fourier transform (F.T.) for the linear partial differential (NSE or NSF)



The origin of NSE is forgot and their solving along with the uniqueness of solutions arises as a pure NSE mathematical problem [1], physical meaningless, long term controversial and often misunderstood (see Stokes problem). However the NSE incorporate dissipative mechanisms and explain certain phenomena that the Euler equations cannot explain since they describe the so-called ideal fluids.

Features such as the drag on an object moving through a real (viscous) fluid, at any scale from molecules to cosmic bodies, probably cannot be properly explained by the Euler equations, referred to as d'Alembert's paradox (zero drag at U = 0) [6].

The technological importance of the hydrodynamic description of gravitized matter (or mass) justifies the scientific interest in NSE solutions, which must govern the frictional motions of terrestrial fluids, air and water, including in the early self-starting stage when an escape/detachment tendency of motion from gravity seems to occur, defined by a critical Reynolds number, Re_{cr} (singular Strokes' second problem).

The Re_{cr} is associated with the beginning of laminar-turbulent transition regimes of wavy, gravity-like flows, of gravitational nature, and concomitantly, the uniqueness loss of NSE solutions and Lissajous-type multiple solution appearance/ bifurcation. The solution bifurcation in conjunction with the solitary waves which retain their identity upon starting collision, the so-called **solitons**, is illustrated for the simplest dispersive wave equation

$$u_t + u_x + u_{xx} = 0, t > 0;\ -\infty < x < \infty, \tag{3}$$

with $u(x, 0)$ and scattering data $S(0)$.

The harmonic wave solution

$$u(x,t) = \int A(k) e^{i(kx-\omega t)} dk, \tag{4}$$

for the given $A(k)$ is the frequency-dispersion relation

$$\omega = k - k^3, \tag{5}$$

where k is the wave number of an oscillating motion which propagates at the velocity

$$c = \frac{\omega}{k} = 1 - k^2, \tag{6}$$

usually termed phase velocity c_φ and

$$c_g = \frac{d\omega}{dk} = 1 - 3k^2, k_{cr} \equiv c_g = 0, \tag{7}$$

is the group velocity of a wave packet.

For $c_g = 0$, $k \geq k_{cr} = \frac{\sqrt{3}}{3} = 0.577$ the wall established Navier-Stokes-Fourier (NSF) equations ceases to be valid, i.e. for $k > k_{cr}$ the NSE solution is non-unique solution less, and multiple solutions can be found using the NSE-like hybrid models (RANS, URANS, LES) usually referred to as higher order hydrodynamics or relativistic Navier-Stokes regimes, see Fig. 1.

The transition from laminar to turbulent flows ($k > k_{cr}$) cannot be explained within the framework of the well-established Navier-Stokes theory [7] and for the stronger disturbances/collisions associated with a two-solitons construction must be considered.

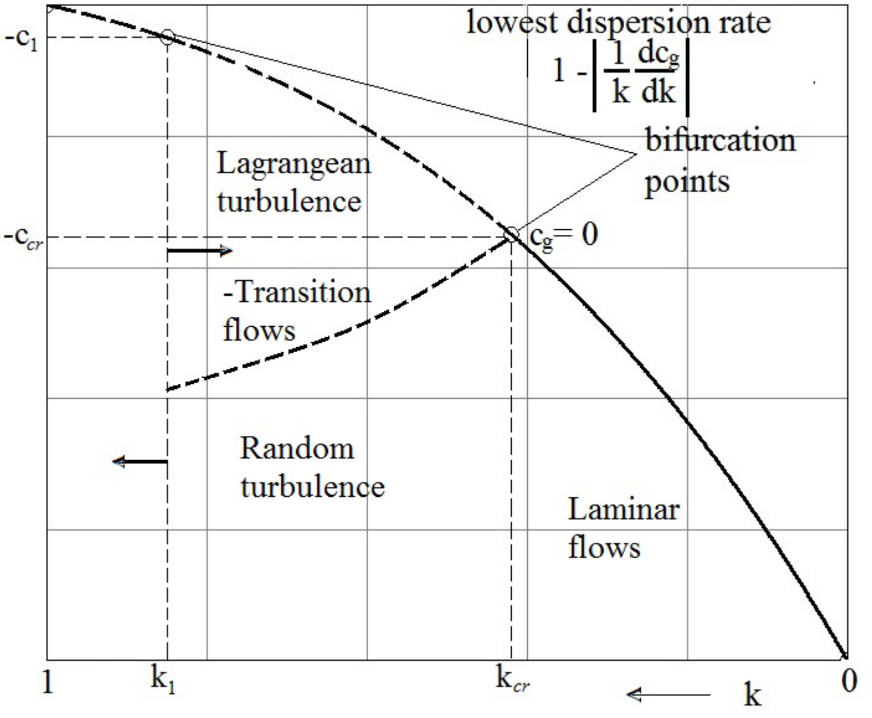


Figure 1 – The hydrodynamical polarization (Stokes's hypothesis): ——— unique NSE solutions; — — — relativistic multiple NSE-like solutions.

For $\left|\frac{1}{k}\frac{dc_g}{dk}\right| = \frac{1}{6}$ the lowest dispersion rate, i.e. the metastable equilibrium, of two-soliton configuration is reached, where

$$k_1 = 1 - \frac{1}{k}\left|\left(\frac{dc_g}{dk}\right)\right| = 0.833, \text{the second bifurcation,} \tag{8}$$

physically corresponding to a full turbulent flow and the end of self-sustained mechanism of turbulence (Lagrangean turbulence).

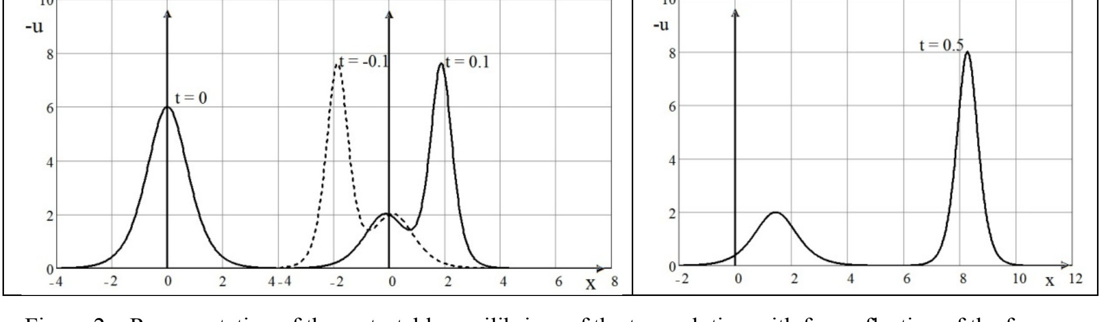


Figure 2 – Representation of the metastable equilibrium of the two solution with free reflection of the former.

Figure 2 illustrates the two-soliton solution of a KdV equation in the standard form

$$u_t - 6uu_x + u_{xxx} = 0, t > 0, -\infty < x < \infty,$$

coupled with the Sturm-Liuville equation

$$\psi_{xx} + (\lambda - u)\psi = 0, -\infty < x < \infty,$$

and IC, $u(x,0) = f(x)$, a sufficiently well-behaved function (herein $f(x) = 6\text{sech}^2(x)$) [8]. The figure depicts two waves near solitary, where the taller one catches the shorter, coalesces to form a single wave, the initial profile at $t = 0$ and then self regenerates to the right and moves away from the shorter one as t increases.

This interaction is not a linear process and the asymptotic two-soliton solution (reflexionless gravity potentials) shows the taller wave moving forward by an amount $x = \frac{1}{2}log3$, and the shorter moving back by $x = log3$.

But, the gravitational terrestrial motions through the thermomolecular field (antigravitational $-$g) balances the kinetic field (u $\perp$ $-$g) of opposite motions in the case of the gravitational soliton-bound.

The metastable equilibrium of the soliton-bound corresponding to the bifurcation points, Eqs. 7, 8 $\left(k_{cr} = \frac{\sqrt{3}}{3},\ k_1 = \frac{5}{6}\right)$ is the Reynolds number range of in continuum transition flows (Fig. 3), occurring in many phenomena in nature, characterized by a special instability state.

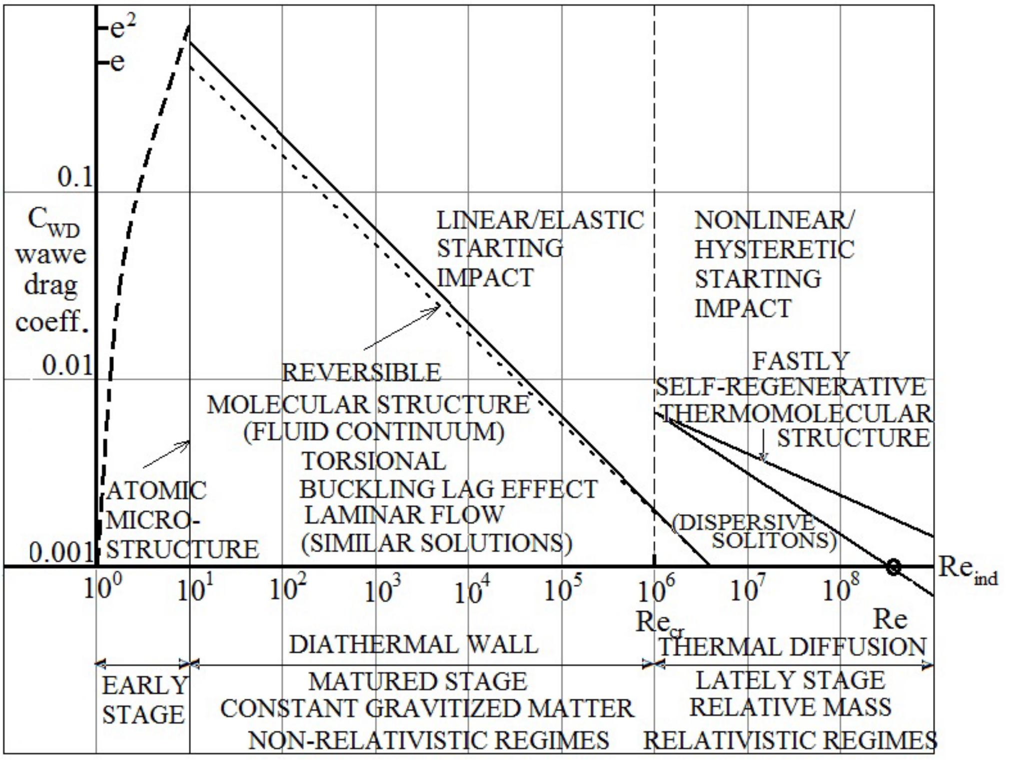


Figure 3 – The Stokes's hypothesis or hydrodynamical polarization: non-and relativistic regimes: o - $Re_{cr} = \frac{\sqrt{3}}{3} \cdot 10^6$; – $Re_{ind} = \frac{5}{6} \cdot 10^9$, the laminar-turbulent transition in continuum flows with Lissajous distributions

The existence and smoothness problem of NSE solution considered as an open mathematical problem [1], in fact is the limitation of the solutions "beyond" the first bifurcation point on account of the interchanges process of the pair of data ($u(x, t)$, $S(t)$) via the scattering method of the Fourier transform. Therefore, the mathematical problem is closed one by the Stokes's hypothesis, i.e. the eigenvalue of gravity (2/3g) without both thermomolecular and chemical changes. The relativistic Navier-Stokes regimes including the laminar-turbulent transition flows, in continuum transition flow with Lisajous-like velocity profiles between the bifurcation points are approached by different less or more precise hydrodynamical models.

The physical problem of NSE-like models, however, remains an open issue concerning the origin of gravity.

In the quantum theory of gravity, recently presented in [2], [3], the gravity field of self-consistent material systems engendered by light self-ignition/ simultaneous combustion ("light-hammer") in a bounded 4-DQuantic (e/π) space is a bifocal gravity field like interacting two-solitons associated with both thermal and regenerating matter covariant fields in the form of less or more warm relative mass (or gravitized matter). The two-soliton like bifocal gravity interaction is given by a quantum-gravity bundle as

$$e \equiv e^{2/3}e^{1/3} = 2 + ln\, e, \text{ the photon polarization} \tag{9a}$$

$$g_0 \equiv e + e^2, \text{ the first/warm focus (shorter soliton)} \tag{9b}$$

$$\frac{e+e^2}{e^3} = \frac{1}{2}, \text{ the regression rate,} \tag{9c}$$

$$g_1 \equiv \frac{e^4}{2}, \text{ the second/cold focus (taller soliton),} \tag{9d}$$

$$g_1 = eg_0 \cong \left(\frac{g_0}{2}\right)^2, \text{ the mutual interaction of foci/solitons,} \tag{9e}$$

The photon polarization (9a) is the start-up phase of the simultaneous photon-regeneration process, where a timeless quantic chain reaction in the light field is occurred.

The quantic reaction of light is an energy lossless cyclic process with the maximum efficiency (a perpetuum mobile-like state), the photon polarization Ker(2/3, 1/3), conserving both their size and energy according to (1/3 + 2/3) = $(1/3)^2$ + 2(2/3) ≡ 1, the photon regeneration rule.

The 4-DQ photon structure is a twisted structure featured in its inter-changeable scales (e, π) which turn into the torsion-free state after a complete rotation 2π,

$$e + \frac{\pi^2}{e} = e + \frac{g_0}{e} = 2\pi.$$

The photon regeneration is a two-stroke process executed per one revolution, a slowly varying evolution (9b, 9c) and a fast varying evolution (9e) given by

$$e^{3/2} \rightleftarrows g_0^{2/3}, \text{ the local quantum-gravity ponderomotive force/motion? (quantum variation) (photon reflexion)} \tag{10a}$$

$$(eg_0^2)^{1/4} \rightleftarrows (eg_1)^{1/3}, \text{ the global quantum- gravity ponderomotive force/ (gravity reflexion)} \tag{10b}$$

The in continuum quantic evolution, Eqs 10, describe all cooperative phenomena in nature, as the shearing determinist mode. (10a) given by $g_0^{2/3}$, and the relativistic/reflexive mode (10b) of the mutual interaction of gravitational potentials (g_0, g_1).

The gravitational mutual interaction (10b) is the so-called **the torsional buckling lag** where its eigenvalue Ker (2/3) from (10a) represents the Stokes's hypothesis.

3. WELL-KNOWN OLD RESULTS, BUT PHYSICALLY LESS UNDERSTOOD

The canonical flows or wall-bounded flows are some unsteady starting state subjected to the torsional buckling lag effect of gravity during the starting regimes which are dissimulated through different Lighthill-like diffusion of vorticity mechanisms (BVF- boundary vorticity flux [9], CBV-concentrated boundary vorticity [10], [11], their gravitational nature being unknown and/or ignored.

The flow examples considered to illustrate the torsional buckling lag of starting, distinguish clear two-flow regimes, the earlier NSE-like theories viewed rather as mathematical exercises than physical models.

These are special two-dimensional exact solutions of the nonstationary Navier-Stokes equations for the 2-D flow near a stagnation point, where for $\mathrm{Re} \le \mathrm{Re}_{cr}$ a laminar-like flow regime is occurred, and for $\mathrm{Re} > \mathrm{Re}_{cr}$ in continuum laminar-turbulent transition regimes, beyond the bifurcation point (Fig. 1), follow until the second bifurcation is reached (the lowest dispersion rate) at $\mathrm{Re} = \mathrm{Re}_{ind(ifference)},$ the full/ saturated turbulent flow (Fig. 3).

The range of Reynolds number, (Re_{cr} – Re_{ind}), constitutes the self-sustained turbulence mechanism or synthetic jets-like turbulence generator [12], mathematically the interaction of bifurcation solitons.

The critical Reynolds numbers: $\mathrm{Re}_{cr} = (2/3)^{\pm}\bullet 10^6$ (+ air, - water) and $\mathrm{Re}_{ind} = 1/3\ p_0^2/g_0 \le 10^9$, $p_0 = 10^5$ Pa – the terrestrial pressure.

Stokes singular problems. The general solution in vorticity formulation in a unidimensional shear flow on the semi-infinite plane $y > 0$ the velocity and vorticity fields take the form

$$\mathrm{u} = (u(y,t),0,0), \omega = (0,0,\omega(y,t)), \omega(y,t) = -\frac{\partial u}{\partial t} \tag{11}$$

$$\frac{\partial u}{\partial t} = -P(t) + \nu\frac{\partial^2 u}{\partial y^2} \tag{12}$$

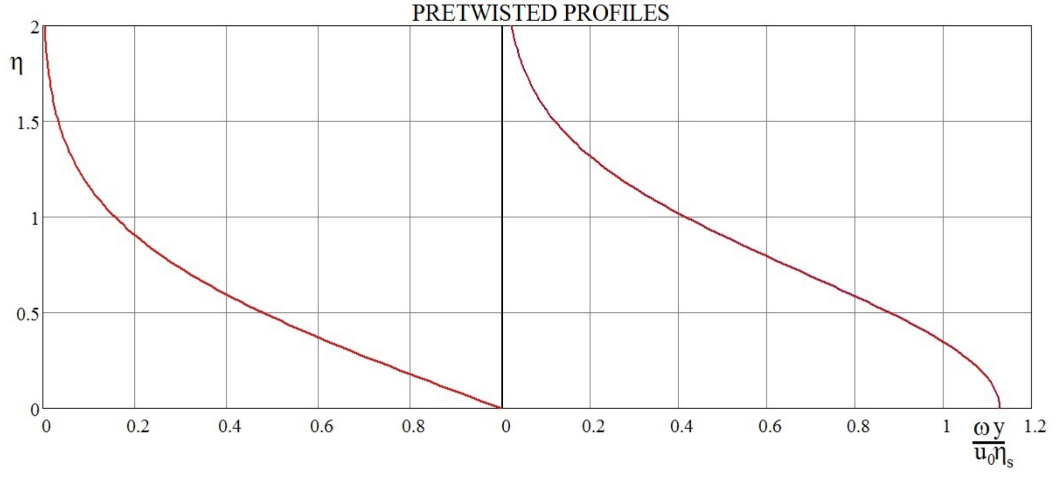
where the continuity equation is automatically satisfied $\left(\frac{\partial u}{\partial x} = 0\right)$. The momentum formulation is used as an alternative vorticity formulation

$$\frac{\partial \omega}{\partial t} = \nu\frac{\partial^2 \omega}{\partial y^2}, \text{ the vorticity transport equation,} \tag{12'}$$

along with a Dirac singular pressure gradient $P = -u_\infty\delta(t)$, independent of the Reynolds number [1].

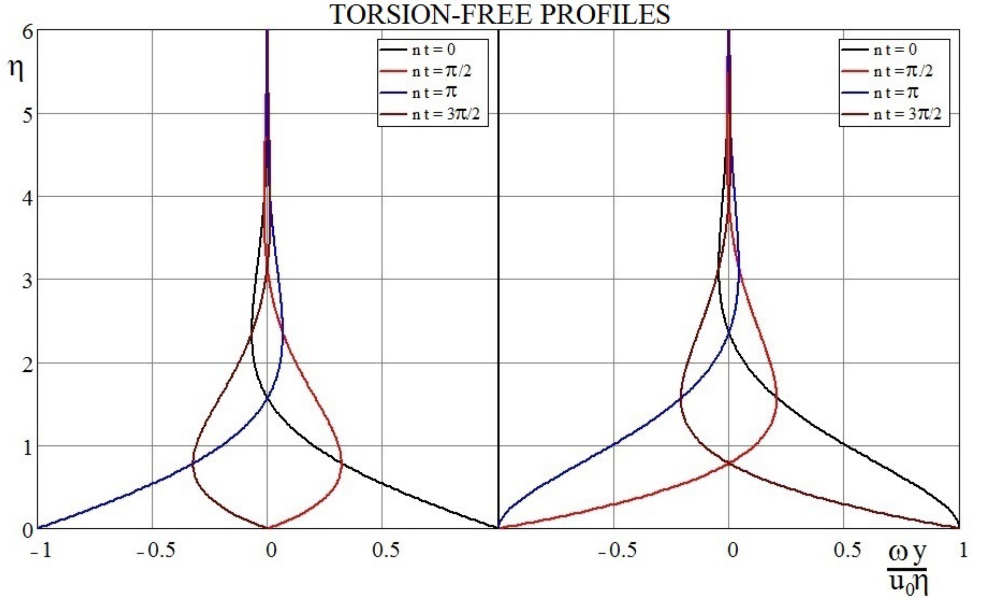
The interchange of the ($\mathrm{u}(t)$, $\omega(t)$) fields like the NSE problem introduces a bifurcation of solution and the flow is no longer remotely irrotational $y \ll (4\nu t)^{1/2}$. In the case of NSE problem, the solutions no longer remain unique beyond their bifurcation, i.e. the critical Reynolds number.

Figure 4 shows the velocity and vorticity distributions in a dimensionless form using a similarity variable $\eta = \frac{y}{2\sqrt{\nu t}}$ and $\omega_b = \frac{u_0}{\sqrt{\pi\nu t}}$ for the classical Stokes first problem (or Rayleigh problem), with the unique and reciprocal solution for $\eta \le 2$, Fig. 4a, a priori first bifurcation, and for the Stokes second problem, beyond the bifurcation with multiple and hysteretic solutions for the profiles of velocity and vorticity, using the similarity variable (a phase lag) $\eta_s = y\sqrt{\frac{n}{2\nu}}$, Fig. 4b.



$$\frac{u}{u_0} = erfc(\eta) = 1 - \text{erf}\,(\eta) \qquad \frac{\omega y}{u_0 \eta} = \frac{2}{\sqrt{\pi}} e^{-\eta^2}$$

a)



$$\frac{u}{u_0} = e^{-\eta_s} \cos\,(nt - \eta_s) \qquad \frac{\omega y}{u_0 \eta_s} = \sqrt{2} e^{-\eta_s} \cos\left(nt - \eta_s + \frac{\pi}{4}\right)$$

b)

Figure 4 – The Stokes problems in conjunction with the Stokes' hypothesis as the NSE self-containing limit (the first bifurcation): a) the non-relativistic unique solutions in Stokes first problem; b) the relativistic multiple solutions in Stokes' second problem.

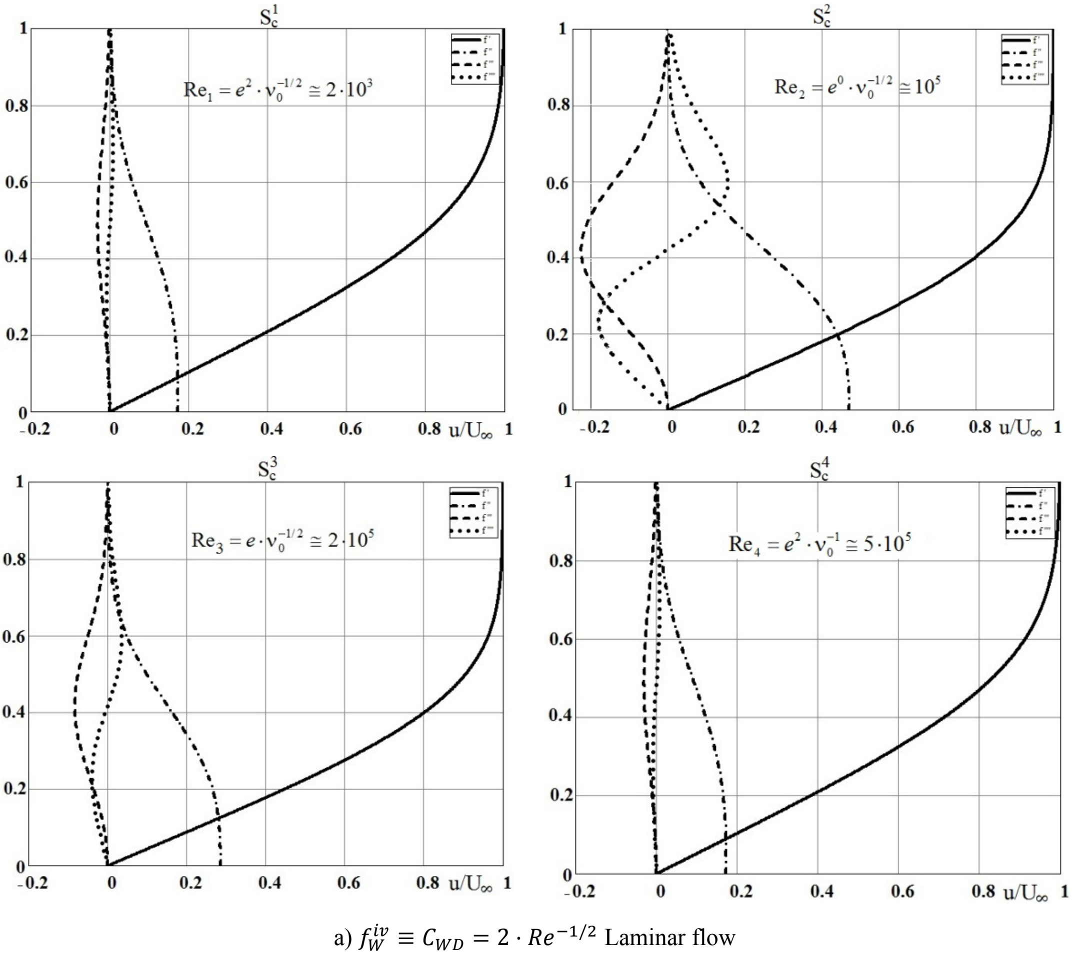
The Blasius and "vortex eye" problems both are special two-dimensional solutions of the nonstationary NSE equations and physical hydrodynamical models of wall-bounded/ surface flows in continuum shear thin layer (or boundary-layer and mixing layer), where diffusion mechanisms of a viscous Newtonian fluid are triggered off by starting collisions.

These “start up” flows engender surface soliton-like waves across a shear layer that travel with the wave speed U_∞ carrying transverse waves as packets of momentum (ρu), kinetic energy (1/2 ρu^2) and pressure (p) waves (or the kinetic triad of inertial waves), with the wave drag as skin friction coefficients (c_{WD}).

The surface/contact transverse waves, of gravitational nature, are torsional perturbations at a starting impact that are self-damped down by viscous diffusion and/or dispersion mechanisms, the less-known torsional buckling lag effect, but the easily recognized as self-sustained turbulence generator.

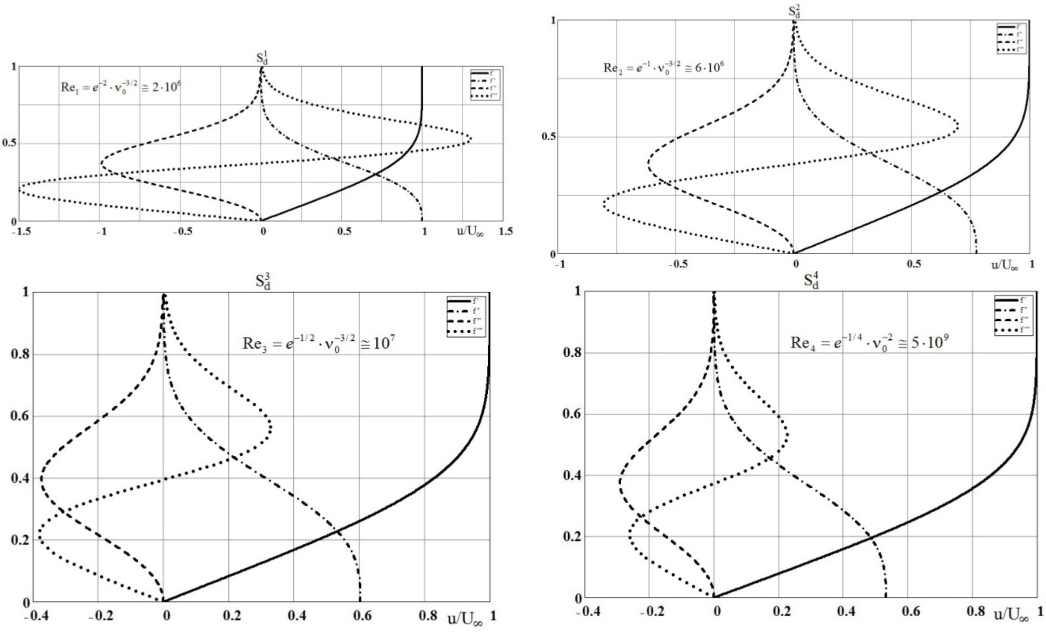
The torsional buckling lag, mathematically described by solitons: the Blasius-like soliton $\left(k_w f''' + f f'' = 0, \frac{u}{U_\infty} = f', \eta = y/\delta\right)$, and vortex-like soliton $\left(\frac{\partial \omega}{\partial t} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right), \nu_t(\mathrm{Re}) - \text{thixotropic kinetic viscosity}\right)$ [4] and associated with the torsional concentrated boundary vorticity-thixotropic fluid model (CBV) [11], [12] is illustrated through Figs. 5, 6.

Slow conservative solution, Sc (monochromatic reflexion)
TOLMIEN-SCHLICHTING WAVES



a) $f_W^{iv} \equiv C_{WD} = 2 \cdot Re^{-1/2}$ Laminar flow

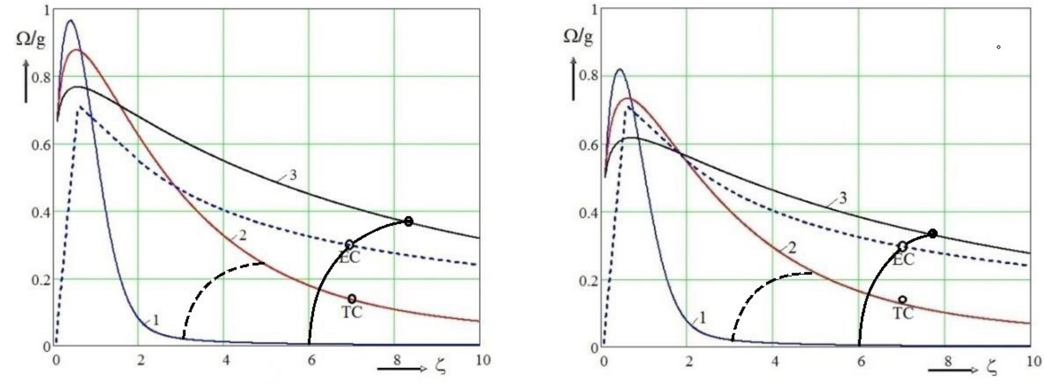
Fast dispersive soliton S_d (polychromatic reflexion)
THIXOTHROPIC FLUID RE ≡ $1/\nu_T$

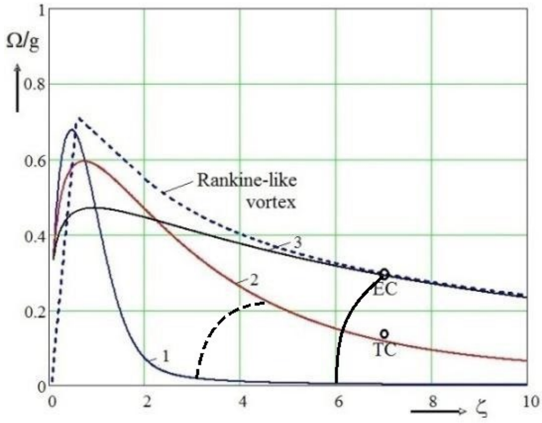


b) $f_W^{iv} \equiv C_{WD} = 2/3 \cdot Re^{-1/3}$ Lagrangean turbulence

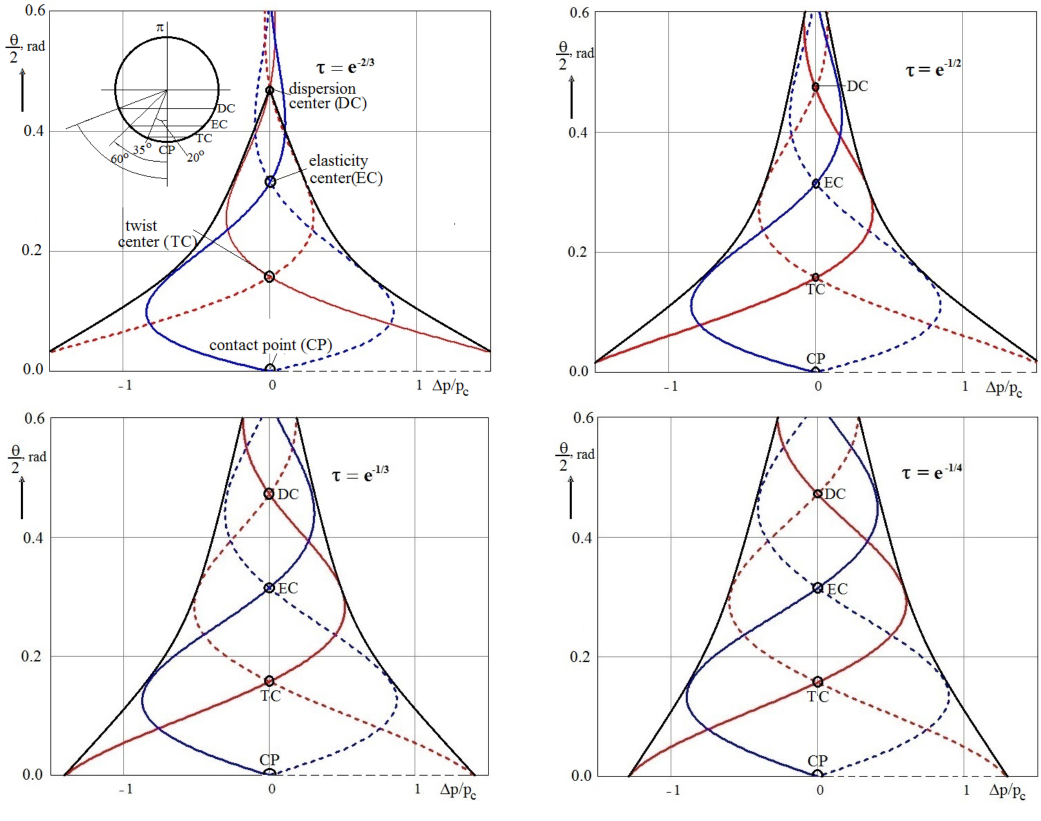
Figure 5 – The Blasius-like soliton solutions (Re_x): a) $Re_x \leq Re_{cr}$ ($= 2/3)^{\pm} \cdot 10^6$, + air and – water), the conservative energy soliton (f' – the group velocity, f'' – the across shear layer pression, f''' – the kinetic energy, f^{iv} – the dispersion rate); b) $Re_x > Re_{cr}$ dispersive soliton, the inertial wave packet $Re_{ind} = 2/3 \cdot 10^9$.

The mathematical problem of the vortex "eye" is of ($u_t + u_x - u_{xx} = 0$) type where the dispersion function $\omega(k) = k - ik^2$ is real only for real k and its solution describes a wave which propagates at a speed unit for all k and decays exponentially for any real k ($\neq 0$) as $t \to \infty$. The solution, $u(x, t) = \exp\{-kt + ik(x-t)\}$ is called dissipation. The physical problem $\left(\omega_t = \frac{\nu}{r}(\omega_r + r\omega_{rr})\right)$ describes the dissipation of a vortex that springs in a viscous fluid at the onset of the motion U_∞. The subcritical regimes a priori the critical Reynolds number ζ_{cr} (log Re_{cr}) = 3, 6, 9 correspond to some osmotic equilibrium pressure $\left(\frac{\Delta p}{\rho} = \frac{1}{2} U_\infty^2\right)$, followed immediately beyond ζ_{cr} a wave packet like the second Stokes problem, Fig. 4b. Figure 6 shows the evolution stages of a starting vortex, Ω/g (2/3, 1/2, 1/3), versus the space-time parameter ζ (log Re_x).





a)



b)

Figure 6 – The vortex-like soliton solutions (Re) cooperative solution: a) the evolution stages: 1-early stage, 2-matured stage, 3- lately stage • - the half-life of cooperative solitons; b) the structured turbulence dispersion Re ≤ Re_{ind} - Lagrangean turbulence.

4. CONCLUSIONS

The keynote of paper can be synthesized as follows:

1. The existence and uniqueness of NSE solutions is in conjunction with the origin of partial differential equations (PDE) as the inverse scattering transform of a KdV-like

equation via the Fourier transform (F.T.) associated with the hypothesis of interchange of pairs data: $u(x, t)$, $A(k, t)$, resulting in the isotropy of equations (Stokes's hypothesis, with the esoteric explanation of bulk viscosity vanishing). The device of NSE self-contains the limitation of the unique solutions beyond the first bifurcation with the critical phase velocity c_{cr} = -2/3 and $k_{cr} = \sqrt{3}/3$, wave number. From the natural quantic theory of gravity [2], [3], the Stokes's hypothesis is equivalent to the eigenvalue of the local quantum-gravity ponderomotive force, Eq. 10a.

2. The mathematical NES problem [1], in primitive variable formulation (velocity-pressure form) accepts a solution $p, u \in C^{\infty}(R^n \text{ x } (0, \infty))$ only if the Poisson equation is satisfied, i.e. the dilatation div u = 0, the limitation being self-contained by Stokes's hypothesis.
   Therefore, beyond the first **bifurcation the Navier-Stokes-Fourier equation ceases to be valid**; the NES problem is a self-consistent problem self-containing limit.
   The NSE problem self-contains the conservation condition of kinetic energy as
   $$\int_{R^n}|u(x,t)|^2dx \;\le \bar{C} = 1/2|u^0(x)|^2 \text{ (conserved energy)}$$
   The general bounded energy condition from [1]
   $$\int_{R^n}|u(x,t)|^2dx < C = 3/4\int_{R^n}|u(x,t)|^2dx \text{ for all } t \geq 0 \text{ (bounded energy)},$$
   expresses the lowest dispersion rate condition (Eq. 8) of second bifurcation, the other said the relativistic NSE field of all in continuum transition flows extremizes the energy functional. This is the outcome of gravitational potential interaction that physically marks the end of self-sustained turbulence generator/productions and regeneration process of fluid molecular structures. The random/free turbulence occurred beyond the second bifurcation distinguishes from the self-sustained turbulence or Lagrangean turbulence engendered by soliton-bound mutual interaction (Fig. 5b).
3. The $k_{cr}g_0 - 1 \equiv 4.774$ is consistent with the Feigenbaum's stability criterion for the first bifurcation.
4. The in continuum transition flows (or beyond Navier-Stokes non-relativistic regimes) can exist only for the stronger non-Gaussian collisions occurring two-soliton configurations at the onset of motion.
5. The physical problem is the gravity problem of quantic nature, according the local and global ponderomotive force, Eq. 10, where the gravity string has two torsional oscillating modes: two fixed-end mode preserving energy (the local effect) and one-free-end mode dissipating energy (the global effect). The main processional effects of gravity are the relative mass as gravitized matter and its periodical regeneration. The mass regeneration in hydrodynamics is represented by transitional flow regimes where chemical mutations/changes occur through oxidation (friction warming) of air (N_2O) and water (H_2O).

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